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Exactly Integrable Systems Connected to Semisimple Algebras of Second Rank A_2, B_2, C_2, G_2

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Abstract

Exactly integrable systems connected to semisimple algebras of second rank with an arbitrary choice of grading are presented in explicit form. General solutions of these systems are expressed in terms of matrix elements of two fundamental representations of the corresponding semisimple groups.

1 Introduction

The main goal of this paper is to demonstrate, on the examples of semisimple algebras of second order (A_2, B_2, C_2, G_2), the general construction connecting a semisimple algebra of a given grading to an exactly integrable system. The simplest example is the two-dimensional Toda lattice considered and integrated in the case of an arbitrary semisimple algebra almost 20 years ago [1, 2][1]. For main grading, exactly integrable systems were explicitly found and described in the recent papers of the author [3] (so called Abelian case).

In the present paper we follow three different and independent aims. The first is to relate unknown up to now integrable systems to nonabelian gradings[2] (see [5] in this connection). The second one is to get rid of the restriction of nonabelian Toda theory to use only subspaces with zero and ± 1 grading indices. The last, but not the least important one, is to provide the reader with a scheme of how the group representation theory (in the very restricted volume) can be applied to the theory of integrable systems.

For our purposes here it is not the shortest and simplest way to the result that is important, but the result by itself. Therefore, in concrete examples we tried to use calculations that can be followed and checked directly using only simplest algebra.

[1]For A_n series this problem was solved more then 150 years ago in Darbouxs papers!
[2]Note that the zero order subspace is a non-commutative algebra by itself.

The paper is organized as follows. Section 2 contains the background information on representation theory of semisimple algebras and groups (as a rule without proofs). Section 3 describes the general construction, mathematical tricks and methods used in main sections. In Section 4 concrete examples of semisimple algebras of second order are considered in details for all possible gradings. Concluding remarks and perspectives for further investigation are outlined in Section 5.

2 Semisimple algebras and groups

Let $\mathcal{G}$ be an arbitrary finite-dimensional graded Lie algebra[3]. Then $\mathcal{G}$ can be written as a direct sum of subspaces of different grading indices

$$\mathcal{G} = \left(\oplus_{k=1}^{N_-}\mathcal{G}_{-\frac{k}{2}}\right)\mathcal{G}_0\left(\oplus_{k=1}^{N_+}\mathcal{G}_{\frac{k}{2}}\right). \tag{2.1}$$

Generators with an integer grading index are called bosonic, while those with half-integer grading index are named fermionic. The positive (negative) grading corresponds to upper (lower) triangular matrices.

The grading operator H for an arbitrary semisimple algebra can be written as a linear combination of elements of commutative Cartan subalgebra taking unity or zero values on the generators of simple roots

$$H = \sum_{i=1}^{r}\left(K^{-1}c\right)_i h_i. \tag{2.2}$$

Here K^{-1} is the inverse Cartan matrix $K^{-1}K = KK^{-1} = I$ and c is a column of zeros and unites in an arbitrary order. Under the main grading all $c_i = 1$. In this case $(K^{-1}c)_i = \sum\limits_{j=1}^{r} K^{-1}_{i,j}$, where r is the rank of the algebra.

As usually, generators of simple roots $X^{\pm}_i$ (raising/lowering operators) and Cartan elements h_i satisfy the system of commutation relations:

$$[h_i, h_j] = 0, \qquad [h_i, X^{\pm}_j] = \pm K_{j,i}X^{\pm}_j, \qquad [X^+_i, X^-_j\} = \delta_{i,j}h_j, \quad (1 \le i, j \le r), \tag{2.3}$$

where K_{ij} are elements of Cartan matrix and brackets $[,\}$ stand for the graded commutator.

The highest vector $\mid j\rangle$ ($\langle j \mid \equiv \mid j\rangle^{\dagger}$) of the j–th fundamental representation has the following properties:

$$X^+_i \mid j\rangle = 0, \qquad h_i \mid j\rangle = \delta_{i,j} \mid j\rangle, \qquad \langle j \mid\mid j\rangle = 1. \tag{2.4}$$

The representation is exhibited by applying lowering operators X^-_i to the vector $\mid j\rangle$ repeatedly and extracting all linearly-independent vectors with non-zero norm. The first few basis vectors are

$$\mid j\rangle, \quad X^-_j \mid j\rangle, \quad X^-_i X^-_j \mid j\rangle \neq 0, \quad K_{i,j} \neq 0, \quad i \neq j. \tag{2.5}$$

[3]We make no distinction between algebras and super-algebras, just keeping in mind that even (odd) elements of super-algebras are always multiplied by even (odd) elements of the Grassman space.

In fundamental representations an important identity for matrix elements of a group element G holds[4] [2]

$$\operatorname{sdet}\begin{pmatrix} \langle j \mid X_j^+ G X_j^- \mid j\rangle, & \langle j \mid X_j^+ G \mid j\rangle \\ \langle j \mid G X_j^- \mid j\rangle, & \langle j \mid G \mid j\rangle \end{pmatrix} = \prod_{i=1}^{r} \langle i \mid G \mid i\rangle^{-K_{ji}}, \tag{2.6}$$

The identity (2.6) is in fact a generalization (to the case of an arbitrary semisimple Lee super-group) of the famous Jacobi identity that relates determinants of orders $(n-1)$, n and $(n+1)$ of some special matrices. As we will see in the next section, this identity is of such importance in deriving exactly integrable systems that one can even say that it is responsible for their existence. We will still refer to (2.6) as to "the first Jacobi identity". In addition to (2.6), there exists another independent identity of key importance [4]

$$\begin{aligned} &(-1)^P K_{i,j} \frac{\langle j \mid X_j^+ X_i^+ G \mid j\rangle}{\langle j \mid G \mid j\rangle} + K_{j,i} \frac{\langle i \mid X_i^+ X_j^+ G \mid i\rangle}{\langle i \mid G \mid i\rangle} \\ &\quad + K_{ij} K_{j,i} (-1)^{jP} \frac{\langle j \mid X_j^+ G \mid j\rangle}{\langle j \mid G \mid j\rangle} \frac{\langle i \mid X_i^+ G \mid i\rangle}{\langle i \mid G \mid i\rangle} = 0, \qquad i \neq j \end{aligned} \tag{2.7}$$

which will be called the second Jacobi identity. This identity is responsible (in the above sense) for the existence of hierarchies of integrable systems invariant with respect to integrable mappings that are connected to every exactly integrable system.

Either from (2.6) or from (2.7) it is possible to construct many usefull recurrent relations that are used in further consideration.

Taking into account the importance of Jacobi identities (2.6) and (2.7) for further consideration we present below a brief proof of (2.6).

Let us consider the left hand side of (2.6) as a function on the group. The action on an arbitrary group element G in the definite representation l of the operators of the right (left) regular representation is by definition

$$M_{\text{left}}(\tilde{M}_{\text{right}})G = M_l G(\tilde{M}_l), \tag{2.8}$$

where M_l, $\tilde{M}_l$ are the generators (the matrices of corresponding dimension) of shifts on the group in a given l representation. Now let us act with an arbitrary generator of the simple positive root $(X_s^+)_r$ on the left hand side of (2.6). This action is equivalent to differentiation and therefore should be applied consequently to the first and second columns of the matrix (2.6) adding the results. The action on the second column results in zero as a corollary of the definition of the higest state vector (2.4). Action on the first column is different from zero only in the case $s = j$. But in this case using the same definition of the highest state vector we conclude that as a result of differentiation of the first column it becomes equal to the second one with the zero final result. Thus considered as a function on the group the left hand side of (2.6) is also proportional to the highest vector (or a linear combination of such vectors) of some other representation. The higest vector of the irreducible representation is uniquely defined by the values that Cartan

[4] Recall that a superdeterminant is defined as $\operatorname{sdet}\begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \equiv \det(A - BD^{-1}C)(\det D)^{-1}$.

generators take on it. If Cartan generators take on the highest vector values $V(h_i) = l_i$, the last can be uniquely represented in the form

$$\langle l \mid G \mid l\rangle = C\prod_{i=1}^{r}(\langle i \mid G \mid i\rangle)^{l_i}. \tag{2.9}$$

Calculating the values of Cartan generators on the left hand side of equation (2.6) (both left and right with the same result) and using the last comment about the form of the highest vector, we prove (2.6) ($C = 1$, as can be seen by putting $G = 1$ and comparing both sides).

The second Jacobi identity can be proven by similar argument [4].

The following generalization of the first Jacobi identity will be very important in calculations dealing with nonabelian gradings.

Let $\mid \alpha\rangle$ be basis vectors of some representation in the strict order of increasing the number of lowering generators (see (2.4) and (2.5)). We also assume that the action of a generator of an arbitrary positive simple root on each basis vector results in a linear combination of the previous ones.

Then the principal minors of an arbitrary order of the matrix (G is an arbitrary element of the group):

$$G_\alpha = \langle \alpha \mid G \mid \alpha\rangle$$

are annihilated from the right (from the left) by generators of positive (negative) roots.

Indeed this is equivalent to differentiation and therefore it is necessary to act on each column (line) of the minors matrix and add the results. But the action of the generator of a positive simple root on the state vector with a given number of lowering operators transforms it into a state vector with a number of lowering operators on unity less, which according to our assumption is a linear combination of previous columns (lines). Thus in all cases the lines or columns of the resulting determinant are linearly dependent with zero result.

The generators of Cartan subalgebra obviously take the definite values on minors of these kind and if the corresponding values are l^s_i, it is possible to write the equality in correspondence with (2.9)

$$Min_s = C_s\prod_{i=1}^{r}\langle i \mid G \mid i\rangle^{l^s_i}, \tag{2.10}$$

where constants C_s can be determined as described above.

3 General construction and technique of computation

The grading of a semisimple algebra is defined by the values that the grading operator H takes on the simple roots of the algebra. As it was mentioned above, this values can be only zeros and unites in an arbitrary order.

$$[H, X^{\pm}_i] = \pm X^{\pm}_i, \qquad H = \sum_1^r (K^{-1}c)_i h_i, \qquad c_i = 1, 0.$$

On the level of Dynkin's diagrams the grading can be introduced by using two colors for its dots: black for simple roots with $c_i = 1$ and red for roots with $c_i = 0$. To each consequent sequence of the red (simple) roots the corresponding semisimple algebra (subalgebra of the initial one) is connected. All these algebras are obviously mutually commutative and belong to the zero graded subspace. Cartan elements of the black roots also belong to the zero graded subspace. We will use the usual numeration of the dots of Dynkin diagrams and all red algebras will be distinguished by an index of their first root m_s. The rank of m_s-th red algebra will be denoted as R_s. Thus $X^{\pm}_{m_s}, X^{\pm}_{m_s+1}, \ldots, X^{\pm}_{m_s+R_s-1}$ is the system of simple roots of m_s red algebra.

After these preliminary comments turn to the general construction [6].

Let two group valued functions $M^+(y)$, $M^-(x)$ be solutions of S-matrix type equations

$$
\begin{aligned}
M^+_y &= \left(\sum_0^{m_2} B^{(+s}(y)\right) M^+ \equiv \left(B^{(0} + L^+\right) M^+, \\
M^-_x &= M^- \left(\sum_0^{m_1} A^{(-s}(x)\right) \equiv M^- \left(A^{(0} + L^-\right),
\end{aligned}
\tag{3.1}
$$

where $B^{(+s}(y)$, $A^{(-s}(x)$ take values in $\pm s$ graded subspaces correspondingly and $s = 0, 1, 2, \ldots, m_{1,2}$. In each finite-dimensional representation $B^{(+s}(y)$, $A^{(-s}(x)$ are upper (lower) triangular matrices and therefore equations (3.1) are integrated in quadratures.

The composite group valued function K plays the key role in our construction

$$K = M^+M^-. \tag{3.2}$$

It turns out that matrix elements of K in various fundamental representations are related by closed systems of equivalent relations, which can be interpreted as exactly integrable system with known general solution.

Bellow we describe calculation methods to prove this proposition.

First of all let us calculate the second mixed derivative $(\ln\langle i \mid K \mid i\rangle)_{x,y}$, where index i belongs to the black dot of Dynkin diagram. We have

$$(\ln\langle i \mid K \mid i\rangle)_x = \frac{\langle i \mid K(A^0 + L^-) \mid i\rangle}{\langle i \mid K \mid i\rangle} = A^0_i(x) + \frac{\langle i \mid KL^- \mid i\rangle}{\langle i \mid K \mid i\rangle}. \tag{3.3}$$

Indeed, $K_x = M^+(y)M^-_x(x) = K(A^0 + L^-)$ as a corollary of equation for M^-. All red components of A^0 under the action on the black highest vector state $\mid i\rangle$ lead to zero result in connection with (2.5). The action of Cartan elements of the black roots state vector satisfies the condition $h_j \mid i\rangle = \delta_{i,j} \mid i\rangle$ and thus only coefficient on h_i remains in the final result (3.3).

Further differentiation (3.3) with respect to y, with the help of arguments above, leads to following result:

$$(\ln\langle i \mid K \mid i\rangle)_{x,y} = \langle i \mid K \mid i\rangle)^{-2} \begin{pmatrix} \langle i \mid K \mid i\rangle, & \langle i \mid KL^- \mid i\rangle \\ \langle i \mid L^+K \mid i\rangle, & \langle i \mid L^+KL^- \mid i\rangle \end{pmatrix}. \tag{3.4}$$

Applying (2.8) of the previous section to the left hand side of (3.4), we finally obtain

$$(\ln\langle i \mid K \mid i\rangle)_{x,y} = L^-_r L^+_l \ln\langle i \mid K \mid i\rangle)^{-1}. \tag{3.5}$$

Thus the problem of calculating the mixed second derivative is reduced to purely algebraic manipulations on the level of representation theory of semisimple algebras and groups. Further evaluation of (3.5) is connected with repeated application of the first (2.6) and second (2.7) Jacobi identities as it will be clear from the material of the next section.

As it was mentioned above, the red algebras of zero order graded subspace in general case are not commutative. This leads to additional computational difficulties. Let us denote by $\mid m_i\rangle$ the highest vector of m_ith fundamental representation of the initial algebra. Of course, $\mid m_i\rangle$ is simultaneously the highest vector of the first fundamental representation of the m_i red algebra. Let $\langle\alpha_i\mid, \mid\beta_i\rangle$ be basis vectors of the first fundamental representation (this restriction is not essential) of m_i-th red algebra and let us consider the matrix elements of element K in this basis. $R_i+1\times R_i+1$ matrix (R_i+1 is the dimension of the first fundamental representation) with matrix elements $\langle\alpha_i\mid K\mid\beta_i\rangle$ will be denoted by a single symbol u_i (index i takes values from one to the number of the red algebras, which is the function of the choosen grading).

For derivatives of matrix elements of so constructed matrix we have consequently (index i we omite for a moment):

$$\langle\alpha\mid u_x\mid\beta\rangle=\langle\alpha\mid K(A^0+L^-)\mid\beta\rangle=\sum_{\gamma}\langle\alpha\mid K\mid\gamma\rangle\langle\gamma\mid IA^0\mid\beta\rangle+\langle\alpha\mid KL^-\mid\beta\rangle. \tag{3.6}$$

Or equivalently

$$u^{-1}u_x=A^0(x)+u^{-1}\langle\mid KL^-\mid\rangle.$$

Further differentiation with respect to y variable leads to

$$\begin{aligned}\langle\mid((u^{-1}u_x)_y\mid\rangle&=u^{-1}\langle\mid(B^0+L^+)KL^-\mid\rangle-u^{-1}\langle\mid(B^0+L^+)K\mid\rangle u^{-1}\langle\mid KL^-\mid\rangle\\&=u^{-1}(\langle\mid L^+KL^-\mid\rangle-\langle\mid L^+K\mid\rangle u^{-1}\langle\mid KL^-\mid\rangle).\end{aligned} \tag{3.7}$$

The last expression may be brought to the form of the ratio of two determinants of R_i+2 and R_i+1 orders respectively with the help of standard transformations:

$$\langle\mid u(u^{-1}u_x)_y\mid\rangle=\frac{\mathrm{Det}_{N_i+1}\begin{pmatrix}u & KL^-\mid\rangle\\ \langle\mid L^+K & \langle\mid L^+KL^-\mid\rangle\end{pmatrix}}{\mathrm{Det}_{N_i}(u)}. \tag{3.8}$$

The generalised Jacobi identity (2.10) of the previous section plays the key role for discovery of the last expression and will be exploited many times.

4 The algebras of second rank A_2, B_2, C_2, G_2

All elements of these algebras may be constructed by consequent multi-commutation of generators of four simple roots $X^{\pm}_{1,2}$ with the basic system of commutation relations

$$\begin{aligned}&[X^+_1,X^-_1]=h_1, \qquad [X^+_1,X^-_2]=[X^+_2,X^-_1]=0, \qquad [X^+_2,X^-_2]=h_2,\\&[h_1,X^{\pm}_1]=\pm 2X^{\pm}_1, \qquad [h_2,X^{\pm}_2]=\pm 2X^{\pm}_2,\\&[h_1,X^{\pm}_2]=\mp pX^{\pm}_2, \qquad [h_2,X^{\pm}_1]=\mp X^{\pm}_1, \qquad p=1,2,3.\end{aligned} \tag{4.1}$$

In all cases there are three possible nontrivial gradings: $(1,1)$ – the principle one (Abelian case), $(1,0)$ – the grading of the first simple root and $(0,1)$ – of the second simple one. In the case of the principle grading corresponding integrable systems for arbitrary semisimple algebras were found and described in [3]. Each further subsections will be devoted to detail consideration of nonabelian gradings $(1,0)$, $(0,1)$, which are equivalent to each other only in the case of A_2 algebra.

In the end of this mini-introduction we present the second Jacobi identity as applied to the algebras of second rank:

$$\frac{\langle 2\mid X_2^+X_1^+K\mid 2\rangle}{\langle 2\mid K\mid 2\rangle}+p\frac{\langle 1\mid X_1^+X_2^+K\mid 1\rangle}{\langle 1\mid K\mid 1\rangle}=p\frac{\langle 2\mid X_2^+K\mid 2\rangle}{\langle 2\mid K\mid 2\rangle}\frac{\langle 1\mid X_1^+K\mid 1\rangle}{\langle 1\mid K\mid 1\rangle} \tag{4.2}$$

or in notation, which will be introduced by the way of consideration

$$\bar{\alpha}_{21}+p\bar{\alpha}_{12}=p\bar{\alpha}_1\bar{\alpha}_2, \qquad \alpha_{12}+p\alpha_{21}=p\alpha_1\alpha_2.$$

4.1 Unitary A_2 serie

The root system of this algebra consists of three elements with the generators $X_1^\pm, X_1^\pm, X_{12}^\pm \equiv \pm[X_1^\pm, X_2^\pm]$. This case corresponds to $p=1$ in (4.1). For definiteness we restrict ourselves by $(1,0)$ grading $[H,X_1^\pm]=\mp X_1^\pm$, $[H,X_2^\pm]=0$.

$L^\pm$ operators belong to ±1 graded subspaces and have the form:

$$L^+=\bar{c}_1X_1^++\bar{c}_2[X_2^+,X_1^+], \qquad L^-=c_1X_1^-+c_2[X_1^-,X_2^-],$$

where $c_{1,2}\equiv c_{1,2}(x)$, $\bar{c}_{1,2}\equiv\bar{c}_{1,2}(y)$.

The object of investigation is 2×2 matrix u in the basis of the second fundamental representation of A_2 algebra[5]:

$$u=\begin{pmatrix}\langle 2\mid K\mid 2\rangle, & \langle 2\mid KX_2^-\mid 2\rangle\\ \langle 2\mid X_2^+K\mid 2\rangle, & \langle 2\mid X_2^+KX_2^-\mid 2\rangle\end{pmatrix}. \tag{4.3}$$

In correspondence with (3.8) we have:

$$\langle\mid u(u^{-1}u_x)_y\mid\rangle=\frac{\mathrm{Det}_3\begin{pmatrix}u & IKL^-\mid\rangle\\ \langle\mid L^+KI & \langle\mid L^+KL^-\mid\rangle\end{pmatrix}}{\mathrm{Det}_2(u)}. \tag{4.4}$$

The action of operators $L^\pm$ on basis vectors $\mid 2\rangle, X_2^-\mid 2\rangle$ ($\langle 2\mid, \langle 2\mid X_2^+$) is the following:

$$L^-\mid 2\rangle=c_2X_1^-X_2^-\mid 2\rangle, \qquad L^-X_2^-\mid 2\rangle=c_1X_1^-X_2^-\mid 2\rangle,$$
$$\langle 2\mid L^+=\bar{c}_2\langle 2\mid X_2^+X_1^+, \qquad \langle 2\mid X_2^+L^+=\bar{c}_1\langle 2\mid X_2^+X_1^+.$$

So in this case the following sequence of basis vectors from generalized Jacobi identity (2.10) takes places:

$$\langle 2\mid, \quad \langle 2\mid X_2^+, \quad \langle 2\mid X_2^+X_1^+.$$

[5]The (bra) basis vectors of the three dimensional ("qwark") second fundamental representation of A_2 algebra are the $\langle 2\mid$, $\langle 2\mid X_2^+$, $\langle 2\mid X_2^+X_1^+$.

The summed values of Cartan generators h_1, h_2 on this basis take zero values and so Det_3 from (4.4) equal to unity (with correct account of the constant). This is a really highest vector of scalar, one-dimensional representation of A_2 algebra.

Finally (4.4) leads to the system, which matrix function u satisfy:

$$(u^{-1}u_x)_y = (\mathrm{Det}\ u)^{-1}u^{-1}\begin{pmatrix} c_2\bar{c}_2, & c_1\bar{c}_2 \\ c_2\bar{c}_1, & c_2\bar{c}_2 \end{pmatrix}. \tag{4.5}$$

In usual notations the system (4.5) is nonabelian $A_2(1,0)$ Toda chain. The system (4.5) is obviously form-invariant with respect to transformation:

$$u \to \bar{g}(y)u\bar{g}(x).$$

With the help of this transformation the arbitrary up to now functions c, $\bar{c}$ may be evaluated to a constant values.

4.2 Orthogonal B_2 serie equivalent to simplectic one C_2

This case corresponds to the choise $p = 2$ in (4.1). Both gradings are not equivalent to each other and must be considered separately. First fundamental representation for B_2 algebra is the second one for C_2 serie and vice versa.

4.2.1 (1,0) grading

Generators $L^{\pm}$ may contain components with ± 1, ± 2 graded indexes and have the form:

$$L^+ = \bar{c}_1X_1^+ + \bar{c}_2[X_2^+, X_1^+] + \bar{c}^2[[X_2^+, X_1^+]X_1^+],$$
$$L^- = c_1X_1^- + c_2[X_1^-, X_2^-] + c^2[X_1^-[X_1^-, X_2^-]].$$

The object of investigation is two dimensional matrix u in the basis of the second fundamental representation of B_2 algebra. The main equation (4.4) also does not change. The action of $L^{\pm}$ operators on the basis vectors have now the form[6]:

$$L^- \mid 2\rangle = (c_2 + c^2X_1^-)X_1^-X_2^- \mid 2\rangle, \qquad L^-X_2^- \mid 2\rangle = (c_1 + c^2X_2^-X_1^-)X_1^-X_2^- \mid 2\rangle,$$
$$\langle 2 \mid L^+ = \langle 2 \mid X_2^+X_1^+(\bar{c}_2 + \bar{c}^2X_1^+), \qquad \langle 2 \mid X_2^+L^+ = \langle 2 \mid X_2^+X_1^+(\bar{c}_1 + \bar{c}^2X_1^+X_2^+).$$

Substituting this expression into (4.4) after some trivial evaluations we come to the following relation:

$$\begin{aligned} u(u^{-1}u_x)_y = (\mathrm{Det}\ u)^{-1}&\begin{pmatrix} \bar{c}_2 + \bar{c}^2(X_1^+)_l, & 0 \\ \bar{c}_1 + \bar{c}^2(X_1^+X_2^+)_l, & 0 \end{pmatrix} \\ &\times\begin{pmatrix} c_2 + c^2(X_1^-)_r, & c_1 + c^2(X_2^-X_1^-)_r \\ 0, & 0 \end{pmatrix}\mathrm{Det}_3. \end{aligned} \tag{4.6}$$

In the last expression Det_3 satisfy all conditions of (2.10), with the sequence of bases vectors:

$$\langle 2 \mid, \quad \langle 2 \mid X_2^+, \quad \langle 2 \mid X_2^+X_1^+.$$

[6]Five basis vectors of the first fundamental representation of the B_2 algebra are the following: $\mid 2\rangle$, $X_2^- \mid 2\rangle$, $X_1^-X_2^- \mid 2\rangle$, $X_1^-X_1^-X_2^- \mid 2\rangle$, $X_2^-X_1^-X_1^-X_2^- \mid 2\rangle$.

In this case the summed value of Cartan element h_1 is equal to 2, of h_2 – to 0. So with the correct value of numerical factor we obtain $\mathrm{Det}_3 = 2\langle 1 \mid K \mid 1\rangle^2$.

The action of the first line operator in (4.6) on $(\langle 1 \mid K \mid 1\rangle)^2$ leads to the line of the form:

$$2(\langle 1 \mid K \mid 1\rangle)^2(c_2 + 2c^2\alpha_1, c_1 + 2c^2\alpha_{21}), \tag{4.7}$$

where following abbreviations are used:

$$\bar\alpha_1 = \frac{\langle i \mid X_i^+ K \mid i\rangle}{\langle i \mid K \mid i\rangle}, \qquad \bar\alpha_{12} = \frac{\langle 1 \mid X_1^+ X_2^+ K \mid 1\rangle}{\langle 1 \mid K \mid 1\rangle},$$
$$\bar\alpha_{21} = \frac{\langle 2 \mid X_2^+ X_1^+ K \mid 2\rangle}{\langle 2 \mid K \mid 2\rangle}, \qquad \alpha_i = \frac{\langle i \mid K X_i^- \mid i\rangle}{\langle i \mid K \mid i\rangle}, \qquad i = 1, 2, \tag{4.8}$$
$$\alpha_{21} = \frac{\langle 1 \mid K X_2^- X_1^- \mid 1\rangle}{\langle 1 \mid K \mid 1\rangle}, \qquad \alpha_{12} = \frac{\langle 2 \mid K X_1^- X_2^- \mid 2\rangle}{\langle 2 \mid K \mid 2\rangle}.$$

Now it is necessary to act with the help of the column operator (4.6) on the line (4.7). The result of this action on scalar factor may be presented in the form ($\mathrm{Det}_2 u = \langle 1 \mid K \mid 1\rangle^2$):

$$2\begin{pmatrix} \bar c_2 + 2\bar c^2\bar\alpha_1, & 0 \\ \bar c_1 + 2\bar c^2\bar\alpha_{12}, & 0 \end{pmatrix}\begin{pmatrix} c_2 + 2c^2\alpha_1, & c_1 + 2c^2\alpha_{21} \\ 0, & 0 \end{pmatrix}.$$

The action of the column operator (4.6) on the line (4.7) leads to additional matrix:

$$4c^2\bar c^2\begin{pmatrix} (X_1^+)_l\alpha_1 & (X_1^+)_l\alpha_{21} \\ (X_2^+X_1^+)_l\alpha_1 & (X_2^+X_1^+)_l\alpha_{21} \end{pmatrix}.$$

With the help of formulae of Appendix I the last matrix may be evaluated to the form:

$$4c^2\bar c^2(\mathrm{Det}\ u)^{-1}u.$$

Gathering all results together, we obtain finally:

$$u(u^{-1}u_x)_y = 2\begin{pmatrix} p_1\bar p_1, & p_2\bar p_1 \\ p_1\bar p_2, & p_2\bar p_2 \end{pmatrix} + 4c^2\bar c^2(\mathrm{Det}\ u)^{-1}u, \tag{4.9}$$

where

$$p_1 = c_2 + 2c^2\alpha_1, \qquad \bar p_1 = \bar c_2 + 2\bar c^2\bar\alpha_1, \qquad p_2 = c_1 + 2c^2\alpha_{21}, \qquad \bar p_2 = \bar c_1 + 2\bar c^2\bar\alpha_{12}.$$

Now we would like to show that the derivatives $(p_\alpha)_y$ and $(\bar p_\alpha)_x$ are functionally dependent on matrix u and themselves, closing in this way the system of equations of equivalence and representing it in the form of closed system of equations for 8 unknown functions: 4 matrix elements of u and 4 components of 2 two-dimensional spinors p, $\bar p$.

Let us follow now the main steps of the necessary calculations. Using the introduced above technique we have subsequently:

$$(p_1)_y = 2c^2(\alpha_1)_y = \frac{2c^2}{\mathrm{Det}\,(u)}\,\mathrm{Det}\begin{pmatrix} \langle 1 \mid K \mid 1\rangle & \langle 1 \mid K X_1^- \mid 1\rangle \\ \langle 1 \mid L^+ K \mid 1\rangle & \langle 1 \mid L^+ K X_1^- \mid 1\rangle \end{pmatrix}.$$

The action of L^+ on the state vector $\langle 1 \mid$ is the following:

$$\langle 1 \mid L^+ = \langle 1 \mid X_1^+(\bar c_1 - \bar c_2 X_2^+ - 2\bar c^2 X_2^+ X_1^+).$$

Substituting the last expression in the previous equation and using the first Jacobi identity for its two first terms (linear in $\bar{c}_1$, $\bar{c}_2$) we obtain:

$$
\begin{aligned}
(p_1)_y = {} & \frac{2c^2}{\mathrm{Det}\,(u)}(\bar{c}_1\langle 2 \mid K \mid 2\rangle - \bar{c}_2\langle 2 \mid X_2^+ K \mid 2\rangle) \\
& - \frac{2c^2}{\mathrm{Det}\,(u)}\,\mathrm{Det}\begin{pmatrix} \langle 1 \mid K \mid 1\rangle & \langle 1 \mid K X_1^- \mid 1\rangle \\ \langle 1 \mid X_1^+ X_2^+ X_1^+ K \mid 1\rangle & \langle 1 \mid X_1^+ X_2^+ X_1^+ K X_1^- \mid 1\rangle \end{pmatrix}.
\end{aligned}
\tag{4.10}
$$

Substituting into the second Jacobi identity (4.2) ($p = 2$) the first one in the form:

$$
\langle 2 \mid K \mid 2\rangle = \mathrm{Det}\begin{pmatrix} \langle 1 \mid K \mid 1\rangle & \langle 1 \mid K X_1^- \mid 1\rangle \\ \langle 1 \mid X_1^+ K \mid 1\rangle & \langle 1 \mid X_1^+ K X_1^- \mid 1\rangle \end{pmatrix}
$$

we obtain after some trivial transformations equality for two second order determinants:

$$
\begin{aligned}
& \begin{pmatrix} \langle 1 \mid X_1^+ K \mid 1\rangle & \langle 1 \mid X_1^+ K X_1^- \mid 1\rangle \\ \langle 1 \mid X_1^+ X_2^+ K \mid 1\rangle & \langle 1 \mid X_1^+ X_2^+ K X_1^- \mid 1\rangle \end{pmatrix} \\
& \qquad = \begin{pmatrix} \langle 1 \mid K \mid 1\rangle & \langle 1 \mid K X_1^- \mid 1\rangle \\ \langle 1 \mid X_1^+ X_2^+ X_1^+ K \mid 1\rangle & \langle 1 \mid X_1^+ X_2^+ X_1^+ K X_1^- \mid 1\rangle \end{pmatrix}.
\end{aligned}
$$

Evaluating the last column of the first determinant with the help of the first Jacobi identity:

$$
\begin{aligned}
\langle 1 \mid X_1^+ K X_1^- \mid 1\rangle &= \frac{\langle 2 \mid K \mid 2\rangle + \langle 1 \mid X_1^+ K \mid 1\rangle\langle 1 \mid K X_1^- \mid 1\rangle}{\langle 1 \mid K \mid 1\rangle}, \\
\langle 1 \mid X_1^+ X_2^+ K X_1^- \mid 1\rangle &= \frac{\langle 2 \mid X_2^+ K \mid 2\rangle + \langle 1 \mid X_1^+ X_2^+ K \mid 1\rangle\langle 1 \mid K X_1^- \mid 1\rangle}{\langle 1 \mid K \mid 1\rangle}
\end{aligned}
$$

we obtain for it:

$$
\bar{\alpha}_1\langle 2 \mid X_2^+ K \mid 2\rangle - \bar{\alpha}_{12}\langle 2 \mid K \mid 2\rangle.
$$

Finally we have:

$$
(p_1)_y = \frac{2c^2}{\mathrm{Det}\,(u)}(u_{11}\bar{p}_2 - u_{21}\bar{p}_1), \qquad (p_2)_y = \frac{2c^2}{\mathrm{Det}\,(u)}(u_{12}\bar{p}_2 - u_{22}\bar{p}_1).). \tag{4.11}
$$

So (4.9), (4.11) and the same system for derivatives of $(\bar{p})_x$ is the closed system of identities or $B_2(1,0;2,2;c^2,\bar{c}^2)$ exactly integrable system connected with the B_2 semisimple serie. To the best of our knowledge this system was not mentioned in literature before.

¿From the physical point of view the exactly integrable system (4.9), (4.11) may be considered as a model of interacting charge $\frac{1}{2}$ particle $(\bar{p}, p)$ with scalar-vector neutral field u.

Putting $c^2 = \bar{c}^2 = 0$, we come back to nonabelian Toda lattice system for single matrix valued unknown function u.

4.2.2 (0,1) grading

Generators $L^{\pm}$ contain only the components with ± 1 graded indexes and have the form:

$$L^+ = \bar{d}_1 X_2^+ + \bar{d}_2 [X_1^+, X_2^+] + \frac{1}{2}\bar{d}_3 [X_1^+ [X_1^+, X_2^+]],$$

$$L^- = d_1 X_2^- + d_2 [X_2^-, X_1^-] + \frac{1}{2} d_3 [X_1^- [X_1^-, X_2^-]].$$

With respect to transformation of $1-$red group A_1 functions $d_i(x)$, $(\bar{d}_i(y))$ are components of three dimensional A_1 vectors.

The object of investigation is two dimensional matrix u in the basis of the second fundamental representation of B_2 algebra. The main equation (4.4) conserves its form. The action of $L^{\pm}$ operators on the basis vectors have now the form[7]:

$$L^- \mid 1\rangle = (d_2 - d_3 X_1^-), \qquad L^- X_1^- \mid 1\rangle = (d_1 - d_2 X_1^-) X_2^- X_1^- \mid 1\rangle,$$
$$\langle 2 \mid L^+ = \langle 2 \mid X_2^+ X_1^+ (\bar{d}_2 - \bar{d}_3 X_1^+), \qquad \langle 2 \mid X_2^+ L^+ = \langle 2 \mid X_2^+ X_1^+ (\bar{d}_1 - \bar{d}_2 X_1^+ X_2^+).$$

Substituting this expression into (4.4), keeping in mind that Det_3 satisfy all conditions of (2.10), after some trivial evaluations we come to the following relation ($\mathrm{Det}_3 = \langle 1 \mid K \mid 1\rangle$):

$$\begin{aligned} u(u^{-1}u_x)_y = (\mathrm{Det}\, u)^{-1} &\begin{pmatrix} \bar{d}_2 - \bar{d}_3 (X_1^+)_l, & 0 \\ \bar{d}_1 - \bar{d}_2 (X_1^+)_l, & 0 \end{pmatrix} \\ &\times \begin{pmatrix} d_2 - d_3 (X_1^-)_r, & d_1 - d_2 (X_1^-)_r \\ 0, & 0 \end{pmatrix} \langle 1 \mid K \mid 1\rangle. \end{aligned} \tag{4.12}$$

Not cumbersome transformation leads the last expression to the finally form:

$$(u^{-1}u_x)_y = (\mathrm{Det}\, u)^{-1} u^{-1} \begin{pmatrix} \bar{d}_2, & -\bar{d}_3 \\ \bar{d}_1, & -\bar{d}_2 \end{pmatrix} u \begin{pmatrix} d_2, & d_1 \\ -d_3, & -d_2 \end{pmatrix}. \tag{4.13}$$

(4.13) is nonabelian Toda chain for B_2 algebra with $(0,1)$ grading. To the best of our knowledge it was not considered before.

System (4.13) is form-invariant with respect to transformation $u \to \bar{g}(y) u g(x)$, with the help of which it is possible to evaluate matrices depending on x, y arguments to constant values. We omit here the question about the possible canonical forms of the system $B_2(0,1;1,1;\bar{d},d)$ (4.13).

4.3 The case of G_2 algebra

As it is possible to expect, this case is the most cumbersome. It corresponds to the choise $p = 3$ in (4.1). Firstly, we will consider the case of $(0,1)$ grading as the most simple one. It is connected with the 7-th dimensional first fundamental representation of G_2 algebra (group). The second one connected with $(1,0)$ grading is 14-th dimensional.

[7] Four basis vectors of the first fundamental of the C_2 algebra are the following: $\mid 1\rangle$, $X_1^- \mid 1\rangle$, $X_2^- X_1^- \mid 1\rangle$, $X_1^- X_2^- X_1^- \mid 1\rangle$.

4.3.1 (0,1) grading

In this case $L^{\pm}$ may contain the components ± 1, ± 2 graded subspaces and have the form:

$$L^+ = \bar{d}_1 X_2^+ + \bar{d}_2 [X_1^+, X_2^+] + \frac{1}{2}\bar{d}_3 [X_1^+[X_1^+, X_2^+]]$$
$$+\frac{1}{6}\bar{d}_4 [X_1^+[X_1^+[X_1^+, X_2^+]]] + \frac{1}{3}\bar{d}^2 [X_2^+[X_1^+[X_1^+[X_1^+, X_2^+]]]],$$

$L^- = (L^+)^T$, where T sign of transposition; with simultaneously exchanging all coefficients $\bar{d} \to d$. This operation we will call as “hermitian conjugation”.

Four coefficient functions d_i, $\bar{d}_i$ on the generators of the ± 1 graded subspaces in $L^{\pm}$ are united to the $\frac{3}{2}$ multiplate, with respect to gauge transformation initiated by group elements $g_0(x)$, $\bar{g}_0(y)$ belonging to the first red group.

The first fundamental representation of G_2 algebra is 7-th dimensional with the basis vectors:

$$\mid 1\rangle, \quad X_1^- \mid 1\rangle, \quad X_2^- X_1^- \mid 1\rangle, \quad X_1^- X_2^- X_1^- \mid 1\rangle, \quad X_1^- X_1^- X_2^- X_1^- \mid 1\rangle,$$
$$X_2^- X_1^- X_1^- X_2^- X_1^- \mid 1\rangle, \quad X_1^- X_2^- X_1^- X_1^- X_2^+ X_1^- \mid 1\rangle.$$

The action of the operators $L^{\pm}$ on A_1 basis of u matrix is as follows:

$$\langle 1 \mid L^+ = \langle 1 \mid X_1^+ X_2^+ (\bar{d}_2 - \bar{d}_3 X_1^+ + \frac{1}{2}\bar{d}_4 X_1^+ X_1^+ - \bar{d}^2 X_1^+ X_1^+ X_2^+),$$

$$\langle 1 \mid X_1^+ L^+ = \langle 1 \mid X_1^+ X_2^+ (\bar{d}_1 - \bar{d}_2 X_1^+ + \frac{1}{2}\bar{d}_3 X_1^+ X_1^+ - \bar{d}^2 X_1^+ X_1^+ X_2^+ X_1^+).$$

The action of the operator L^- on A_1 basis from the left may be obtained from the last formulae with the help of “hermitian conjugation”:

$$L^- \mid 1\rangle = (\langle 1 \mid L^+)^T, \qquad L^- X_1^- \mid 1\rangle = (\langle 1 \mid X_1^+ L^+)^T, \qquad \bar{d} \to d.$$

As in the previous sections the result of calculation of the main determinant (4.4) it is possible to present in the form of the product of column operator on the line one applied to the highest vector $\langle 1 \mid K \mid 1\rangle^2$ of the $(2,0)$ representation of G_2 algebra (see Appendix II). The line operator form is the following

$$\Big[d_2 - d_3 X_1^- + \frac{1}{2} d_4 (X_1^-)^2 - \frac{1}{4} d^2 (2X_1^- X_2^- - 3X_2^- X_1^-) X_1^-,$$
$$d_1 - d_2 X_1^- + \frac{1}{2} d_3 (X_1^-)^2 - \frac{1}{4} d^2 X_1^- (2X_1^- X_2^- - 3X_2^- X_1^-) X_1^-\Big],$$

where $X_i^- \equiv (X_i^-)_r$. Nonusual (compared with the previous examples) form of the coefficient on d^2 term is the prise for $p = 3$ in (4.1) in the case of G_2 algebra. The column operator is obtained from the line one with the help introduced above rules of the “hermitian conjugation”.

For rediscovering of last symbolical expression up to the form of usual 2×2 matrix let us introduce two dimensional column vector $\bar{q}$ the result of the action of the column operator on the highest vector $\langle 1 \mid K \mid 1\rangle^2$ divided by itself. The same in the line case will be denoted as q. Explicit expressions for the line components of q have the form:

$$\begin{aligned} q_1 &= \left(d_2 + \frac{1}{3}d^2\alpha_{112}\right) - 2\left(d_3 + \frac{2}{3}d^2\alpha_{12}\right)\alpha_1 + \left(d_4 + 2\alpha_2 d^2\right)\alpha_1^2, \\ q_2 &= \left(d_1 + \frac{1}{3}d^2\alpha_{1112}\right) - 2\left(d_2 + \frac{1}{3}d^2\alpha_{112}\right)\alpha_1 + \left(d_3 + \frac{2}{3}d^2\alpha_{12}\right)\alpha_1^2 \end{aligned} \tag{4.14}$$

and with the help of "hermitian conjugation" corresponding expressions for the components for the column $\bar{q}$.

The result of the action of line operator on the highest vector in connection with all said above is equal to the numerical line vector $\langle 1 \mid K \mid 1\rangle^2(q_1, q_2)$. The action of the column operator on it may be devided on two steps: the action on the scalar factor $\langle 1 \mid K \mid 1\rangle^2$, with the finally matrix $\langle 1 \mid K \mid 1\rangle^2\bar{q}q$ (multiplication by the law the column on the line) and the terms with partial mutual differentiation of the scalar and the lines factors. All formulae for concrete calculation of such kind the reader can find in Appendix II. It is necessary to pay attention to the fact, that $X_2^+ q_i = X_2^- \bar{q}_i = 0$, which one can check without any difficulties with the help of formulae of the Appendix I.

Gathering all these results we obtain the equation of equivalence for u function:

$$u(u^{-1}u_x)_y = \det{}^{-1}(u) \sum_{i,j,k,l} u_{ij}u_{kl}\epsilon_{ik}\epsilon_{jl}\bar{p}^{ik}p^{jl} + 4d^2\bar{d}^2(\mathrm{Det}\,(u))^{-1}u, \tag{4.15}$$

where u_{ij} elements of the matrix u, ϵ_{ij} symmetrical tensor of the second rank with the components $\epsilon_{12} = \epsilon_{21} = -1$, $\epsilon_{11} = \epsilon_{22} = 1$, $\bar{p}^{ij}$, p^{ij} are two-dimensional column and line vectors correspondingly with the components (the law of multiplication is the column on the line):

$$p^{11} = \left(d_2 + \frac{1}{3}d^2\alpha_{112}, d_1 + \frac{1}{3}d^2\alpha_{1112}\right), \qquad p^{22} = \left(d_4 + 2d^2\alpha_2, d_3 + \frac{2}{3}d^2\alpha_{12}\right),$$

$$p^{12} = p^{21} = \left(d_2 + \frac{1}{3}d^2\alpha_{112}, d_3 + \frac{2}{3}d^2\alpha_{12}\right).$$

It remains only to find the derivatives $(\bar{p}_{ij})_x$, $(p_{kl})_y$ and convince ourselves that together with the (4.15) they compose the closed system of equations of equivalence or exactly integrable $G_2(0,1;2,2;\bar{d}^2,d^2)$ system.

Four components of p^{22}, p^{11} with respect to transformation of the first red algebra compose the $\frac{3}{2}$ spin-multiplet. So it will be suitable to redenote them by single four-dimensional symbol p_i. And the same for "hermitian conjugating" values $\bar{p}_i$.

Let us follow the calculation of $(\bar{p}_4)_x = 2\bar{d}^2(\bar{\alpha}_2)_x$. The calculation of this the derivative do not different from the corresponding computations of Section 3 (see (3.4) and (3.5)). We have consequently:

$$(\bar{\alpha}_2)_x = \langle 2 \mid K \mid 2\rangle)^{-2} \begin{pmatrix} \langle 2 \mid K \mid 2\rangle, & \langle 2 \mid KL^- \mid 2\rangle \\ \langle 2 \mid X_2^+ K \mid 2\rangle, & \langle 2 \mid X_2^+ KL^- \mid 2\rangle \end{pmatrix}. \tag{4.16}$$

With the help of the technique used many times before we evaluate the last expression to:

$$\begin{aligned} (\bar{\alpha}_2)_x = L_r^-(X_2^+)_l \ln\langle 2 \mid K \mid 2\rangle = \Big[& d_1 - d_2X_1^- + \tfrac{1}{3}d_3(X_1^-)^2 \\ & -\frac{1}{6}d_4(X_1^-)^3 + d^2([[[X_2^-, X_1^-]X_1^-]X_1^-] - X_2^-(X_1^-)^3)\Big]\,\theta_2. \end{aligned}$$

Using with respect to the last expression formulae of Appendix I we come to the system of equations of equivalence for $\bar{p}$ components of $\frac{3}{2}$ multiplet:

$$\begin{aligned}
(\bar{p}_4)_x &= \frac{2\bar{d}^2}{\mathrm{Det}^2(u)}\left(p_1u_{11}^3 - 3p_2u_{11}^2u_{12} + 3p_3u_{11}u_{12}^2 - p_4u_{12}^3\right),\\
(\bar{p}_3)_x &= \frac{2\bar{d}^2}{\mathrm{Det}^2(u)}\left(p_1u_{11}^2u_{21} - p_2\left(u_{11}^2u_{22} + 2u_{11}u_{21}u_{12}\right)\right.\\
&\quad \left.+p_3\left(2u_{11}u_{12}u_{21} + u_{12}^2u_{21}\right) - p_4u_{12}^2u_{22}\right),\\
(\bar{p}_2)_x &= \frac{2\bar{d}^2}{\mathrm{Det}^2(u)}\left(p_1u_{11}u_{21}^2 - p_2\left(u_{21}^2u_{12} + 2u_{11}u_{21}u_{22}\right)\right.\\
&\quad \left.+p_3\left(2u_{22}u_{12}u_{21} + u_{22}^2u_{11}\right) - p_4u_{22}^2u_{12}\right),\\
(\bar{p}_1)_x &= \frac{2\bar{d}^2}{\mathrm{Det}^2(u)}\left(p_1u_{21}^3 - 3p_2u_{21}^2u_{22} + 3p_3u_{21}u_{22}^2 - p_4u_{22}^3\right).
\end{aligned} \tag{4.17}$$

And corresponding system for derivatives p_y, which can be obtained from (4.17) with the help of "hermitian conjugation".

The symmetry of the constructed exactly integrable $G_2(0,1;2,2;\bar{d}^2,d^2)$ system (4.15), (4.17) is higher than any possible espectations.

¿From the physical point of view this system may be considered as the interuction of charge $\frac{3}{2}$ spin particle $(p,\bar{p})$ with neutral scalar-vector field u.

4.3.2 (1,0) grading

In this case $L^{\pm}$ may contain the components ± 1, ± 2, ± 3 graded subspaces and have the form:

$$\begin{aligned}
L^+ &= \bar{c}_1X_1^+ + \bar{c}_2[X_1^+,X_2^+] + \bar{c}^2[X_1^+[X_1^+,X_2^+]]\\
&\quad +\bar{c}_1^3[X_1^+[X_1^+[X_1^+,X_2^+]]] + \bar{c}_2^3[X_2^+[X_1^+[X_1^+[X_1^+,X_2^+]]]],
\end{aligned}$$

$L^- = (L^+)^T$, where T sign of transposition $((X_i^+)^T = X_i^-)$; with simultaneously exchange of all coefficients $\bar{c} \to c$. This operation was called as "hermitian conjugation" in the previous subsection and we conserve here this notation.

As always we begin from the equation of equivalence for two dimensional matrix u connected with the second simple root of G_2 algebra. For the decoded of universal equation (4.4) it is necessary the knowledge of the action of $L^{\pm}$ on the basis. We represent below only part of basis vectors of the second fundamental (14-th dimensional) representation of G_2 algebra:

$$\begin{aligned}
&\mid 2\rangle,\ X_2^- \mid 2\rangle,\ X_1^-X_2^- \mid 2\rangle,\ X_1^-X_2^1X_2^- \mid 2\rangle,\ X_1^-X_1^-X_1^-X_2^- \mid 2\rangle,\ X_2^-X_1^-X_1^-X_2^- \mid 2\rangle,\\
&X_2^-X_1^-X_1^-X_1^-X_2^- \mid 2\rangle,\ X_1^-X_2^-X_1^-X_1^-X_2^- \mid 2\rangle,\ X_1^-X_2^-X_1^-X_1^-X_1^-X_2^- \mid 2\rangle.
\end{aligned}$$

The main equations (3.1) are obviously invariant with the respect to the gauge transformation iniciated by $g_0(x)$, $\bar{g}_0(y)$ elements of the red algebra of the second simple root. With respect to this transformations two coefficients of zero $(c^1,\bar{c}^1)$ and third $(c^3,\bar{c}^3)$ order graded subspaces are transformed as spinor (anti-) multiplets; c^2, $\bar{c}^2$ are the scalar ones. With the help of such transformation it is always possible to satisfy the condition $c_2^3 = \bar{c}_2^3 = 0$ (what is essential simplified the calculation) and reconstruct the general case at the final step using invariance condition.

The action of the $L^{\pm}$ operators on the basis states of the second red algebra has the form:

$$\langle 2 \mid L^+ = \langle 2 \mid X_2^+ X_1^+ (-\bar{c}_2^1 + \bar{c}^2 X_1^+ - \bar{c}_1^3 X_1^+ X_1^+ + \bar{c}_2^3 (2X_1^+ X_1^+ X_2^+ - 3X_1^+ X_2^+ X_1^+),$$
$$\langle 2 \mid X_2^+ L^+ = \langle 2 \mid X_2^+ X_1^+ (\bar{c}_1^1 + \bar{c}^2 X_1^+ X_2^+ + (X_1^+ X_1^+ X_2^+ - 3X_1^+ X_2^+ X_1^+)(\bar{c}_1^3 - \bar{c}_2^3 X_2^+).$$

The action of the operator L^- on A_1 basis from the left may be obtained from the last formulae with the help of "hermitian conjugation":

$$L^- \mid 2\rangle = (\langle 2 \mid L^+)^T, \qquad L^- X_1^- \mid 2\rangle = (\langle 2 \mid X_1^+ L^+)^T, \qquad \bar{c} \to c.$$

Taking into account arguments of the Appendix II the result of the calculation of determinant of the third order (4.4) may be presented in the operator column on line form, acting on the highest vector of $(4,0)$ representation $(3\langle 1 \mid K \mid 1\rangle^4)$ of G_2 algebra.

The line ("hermitian conjugating" column) operators has the form (in this expression we put $c_2^3 = \bar{c}_2^3 = 0$):

$$\left(-c_2^1 + c^2 X_1^- - \frac{1}{2}\bar{c}_1^3 (X_1^-)^2, \quad c_1^1 + c^2 X_2^- X_1^- + \frac{1}{8} c_1^3 (X_2^- X_1^- X_1^- - 6X_1^- X_2^- X_1^-)\right).$$

Further calculations are on the level of accurate application of differentiation rules and combination terms of the same nature. Equation of equivalence for u function have the final form:

$$u(u^{-1}u_x)y = 3\det^{\frac{1}{3}}(u)\bar{p}^1 p^1 + 12\det^{-\frac{1}{3}}(u)\bar{p}^2 p^2 u + 18\det^{-1}(u)(u\bar{c}^3)(c^3 u), \tag{4.18}$$

where p^1 is the spinor with the components $p^1 = (-c_2^1 + 4c^2\alpha_1 - 6c_1^3\alpha_1^2, c_1^1 + 4c^2\alpha_{21} - c_1^3(\alpha_{121} + 2\alpha_1\alpha_{21}))$; scalar $p^2 = c^2 - 3c_1^3\alpha_1$ and corresponding expressions for bar values.

We present the system of equivalence equations without any further comments:

$$(\bar{p}^2)_x = -3\det^{-\frac{2}{3}} \sum_{i,j,k,l} \bar{c}_i^3 u_{ij}\epsilon_{kl} p_l^1, \qquad (\bar{p}_i^1)_x = \det^{-\frac{2}{3}} \bar{p}^2 \sum_{j,k,l} u_{ij}\epsilon_{k,l} p_l^1, \tag{4.19}$$

where $\epsilon_{k,l} = -\epsilon_{l,k}$ antisymmetrical tensor of the second rank $\epsilon_{1,2} = -\epsilon_{2,1} = 1$. And, of, course the corresponding system with the derivatives p_y^1, p_y^2.

Physical interpretation of the last system may be connected with spinor particle interacting with charged scalar $(p^2, \bar{p}^2)$ and neutral scalar-vector fields in two dimensions.

5 Concluding remarks

In some sense in the present paper the initial idea of Sofus Lie to introduce continuous groups as powerful apparatus for solving the differential equations is realized.

On the examples of semisimple groups of second order we have decoded this idea and described explicitly exactly integrable systems whose general solutions can be obtained with the help and in the terms of group representation theory. We have no doubts (and partially can prove this) that the same construction is applicable to the case of arbitrary Lie groups and hope to prove this statement completely or to see the proof in the literature in the nearest future.

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Appendix I

The formulae below are the general ones and have in their foundation the first Jacobi identity only.

Let us define:

$$\theta_j = \prod_{i=1}^{r} (\langle i \mid G \mid i \rangle)^{-K_{ji}}.$$

As a result of differentiation of $\ln \theta_i$, we obtain:

$$(X_q^-)_r \theta_i = -\theta_i K_{iq} \alpha_q, \qquad (X_q^+)_l \theta_i = -\theta_i K_{iq} \bar{\alpha}_q, \tag{I.1}$$

$$(X_q^-)_r \bar{\alpha}_i = \delta_{q,i} \theta_i, \qquad (X_q^+)_l \alpha_i = \delta_{q,i} \theta_i. \tag{I.2}$$

In the case of the second order algebras:

$$\theta_1 = \frac{\langle 2 \mid G \mid 2 \rangle}{\langle 1 \mid G \mid 1 \rangle^2}, \qquad \theta_2 = \frac{\langle 1 \mid G \mid 1 \rangle^p}{\langle 1 \mid G \mid 1 \rangle^2}. \tag{I.3}$$

Appendix II

Let us consider the determinant of the third order the matrix entiries of which are coinsided with the matrix elements of G_2 group element K taken between the bra and the ket three dimensional bases:

$$\langle 1 \mid, \ \langle 1 \mid X_1^+, \ \langle 1 \mid X_2^+ X_1^+ X_1^+ X_2^+ X_1^+, \ \mid 1 \rangle, \ X_1^- \mid 1 \rangle, \ X_2^- X_1^- X_1^- X_2^- X_1^- \mid 1 \rangle. \tag{II.1}$$

Acting on such determinant by generator $(X_2^+)_r$ and taking $(X_1^- X_1^-)_r$ out of its sign we come to the following ket basis:

$$\mid 1 \rangle, X_1^- \mid 1 \rangle, X_2^- X_1^- \mid 1 \rangle$$

which in connection with the (2.10) tell us that the initial Det_3 (up to the terms anihilated by generators of the positive simple roots from right and negative ones from the left) belongs to $(2,0)$ ($Vh_1 = 2$, $Vh_2 = 0$) representation of G_2 group. For initial determinant $Vh_1 = 1$, $Vh_2 = 0$. Each basis vector (see (2.4)) may be obtained with consequent application of the lowering operators to the higest vector ($\langle 1 \mid K \mid 1 \rangle^2$ in the present case). There are two possibility to combination of the lowering operators:

$$((A X_2^- X_1^- + B X_1^- X_2^-) X_1^-)_r$$

and the same expression from the left combination of the raising generators. The condition that Det_3 is anihilated by generators $(X_1^+)_r(X_1^-)_l$, which is a direct corollary of the structure of the bra and ket basises, allow to find relation between the constants $3A+2B=0$ and obtain the expression used in the main text (4.14) and above. We obtain the following value for Det_3 in basis (II.1):

$$\mathrm{Det}_3 = \frac{1}{16}((2X_1^-X_2^- - 3X_2^-X_1^-)X_1^-)$$
$$\times (X_1^+(2X_2^+X_1^+ - 3X_1^+X_2^+))\langle 1 \mid K \mid 1\rangle^2 + \langle 1 \mid K \mid 1\rangle.$$

Below we present necessary formulae for calculation of (4.15). We restrict ourselves by (11) component of it. All "mixed" terms may be gothered in the following form:

$$\begin{aligned} &-\frac{\bar{d}^2}{4}\left[2(X_1^+X_2^+X_1^+q_1) - 3(X_2^+X_1^+X_1^+q_1) - 8\bar{\alpha}_1(X_2^+X_1^+q_1) - 8\bar{\alpha}_1(X_1^+q_1)\right] \\ &\quad -\bar{d}_3(X_1^+q_1) + 2\bar{d}_4\bar{\alpha}_1(X_1^+q_1) + \frac{\bar{d}_4}{2}((X_1^+)^2q_1). \end{aligned} \tag{II.2}$$

Using the definition of vector q (4.14) and formulae of Appendix I, we obtain:

$$(X_1^+q_1) = 2\theta_1(p_1^{22}\alpha_1 - p_2^{22}) \equiv 2\theta_1 P, \quad X_2^+P = 0, \quad (X_2^+X_1^+q_1) = 2\theta_1\bar{\alpha}_2 P,$$
$$(X_1^+X_1^+q_1) = 2\theta_1^2 p_1^{22} - 4\theta_1\bar{\alpha}_1 P, \quad (X_1^+X_2^+X_1^+q_1) = 2\theta_1(\bar{\alpha}_{21} - 2\bar{\alpha}_1\bar{\alpha}_2)P + 2\theta_1^2\bar{\alpha}_2 p_1^{22},$$
$$(X_2^+X_1^+X_1^+q_1) = 4\theta_1^2\bar{\alpha}_2 p_1^{22} + 4d^2\theta_1^2\theta_2 - 4\theta_1\bar{\alpha}_1\bar{\alpha}_2 P - 4\theta_1\bar{\alpha}_{12}P.$$

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