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Symmetries of a Class of Nonlinear Fourth Order Partial Differential Equations

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Abstract

In this paper we study symmetry reductions of a class of nonlinear fourth order partial differential equations

$$u_{tt} = \left(\kappa u + \gamma u^2\right)_{xx} + uu_{xxxx} + \mu u_{xxtt} + \alpha u_x u_{xxx} + \beta u_{xx}^2, \tag{1}$$

where α, β, γ, κ and μ are arbitrary constants. This equation may be thought of as a fourth order analogue of a generalization of the Camassa-Holm equation, about which there has been considerable recent interest. Further equation (1) is a "Boussinesq-type" equation which arises as a model of vibrations of an anharmonic mass-spring chain and admits both "compacton" and conventional solitons. A catalogue of symmetry reductions for equation (1) is obtained using the classical Lie method and the nonclassical method due to Bluman and Cole. In particular we obtain several reductions using the nonclassical method which are *not* obtainable through the classical method.

1 Introduction

In this paper we are concerned with symmetry reductions of the nonlinear fourth order partial differential equation given by

$$\Delta \equiv u_{tt} - \left(\kappa u + \gamma u^2\right)_{xx} - uu_{xxxx} - \mu u_{xxtt} - \alpha u_x u_{xxx} - \beta u_{xx}^2 = 0, \tag{1}$$

where α, β, γ, κ and μ are arbitrary constants. This equation may be thought of as an alternative to a generalized Camassa-Holm equation (cf. [24] and the references therein)

$$u_t - \epsilon u_{xxt} + 2\kappa u_x = uu_{xxx} + \alpha uu_x + \beta u_x u_{xx}. \tag{2}$$

This is analogous to the Boussinesq equation [9, 10]

$$u_{tt} = \left(u_{xx} + \tfrac{1}{2}u^2\right)_{xx} \tag{3}$$

which is a soliton equation solvable by inverse scattering [1, 13, 14, 30, 71], being an alternative to the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + 6uu_x \tag{4}$$

another soliton equation, the first to be solved by inverse scattering [39].

Two special cases of (1) have appeared recently in the literature both of which model the motion of a dense chain [62]. The first is obtainable via the transformation

$$(u, x, t) \mapsto (2\varepsilon\alpha_3 u + \varepsilon\alpha_2, x, t)$$

with the appropriate change of parameters, to yield

$$u_{tt} = \left(\alpha_2 u + \alpha_3 u^2\right)_{xx} + \varepsilon\alpha_2 u_{xxxx} + 2\varepsilon\alpha_3 \left[uu_{xxxx} + 2u_{xx}^2 + 3u_x u_{xxx}\right] \tag{5}$$

with $\varepsilon > 0$. This equation can be thought of as the Boussinesq equation (3) appended with a nonlinear dispersion. It admits both conventional solitons and compact solitons often called "compactons". Compactons are solitary waves with a compact support (cf. [62, 63, 64, 65]). The compact structures take the form

$$u(x,t) = \begin{cases} \dfrac{3c^2 - 2\alpha_2}{2\alpha_3} \cos^2\left\{(12\varepsilon)^{-1/2}(x - ct)\right\}, & \text{if} \quad |x - ct| \le 2\pi, \\ 0, & \text{if} \quad |x - ct| > 2\pi. \end{cases} \tag{6}$$

or

$$u(x,t) = \begin{cases} A\cos\left\{(3\varepsilon)^{-1/2}\left[x - \left(\tfrac{2}{3}\alpha_3\right)^{1/2} t\right]\right\}, & \text{if} \quad |x - ct| \le 2\pi, \\ 0, & \text{if} \quad |x - ct| > 2\pi. \end{cases} \tag{7}$$

These are "weak" solutions as they do not possess the necessary smoothness at the edges, however this would appear not to affect the robustness of a compacton [62]. Numerical experiments seem to show that compactons interact elastically, reemerging with exactly the same coherent shape [65]. See [48] for a recent study of non-analytic solutions, in particular compacton solutions, of nonlinear wave equations.

The second equation is obtained from the scaling transformation

$$(u, x, t) \mapsto \left(2\alpha_3 u/\varepsilon, \sqrt{\varepsilon}\, x, t\right),$$

again with appropriate parameterisation,

$$u_{tt} = \left(\alpha_2 u + \alpha_3 u^2\right)_{xx} + \varepsilon u_{xxtt} + 2\varepsilon\alpha_3 \left[uu_{xxxx} + 2u_{xx}^2 + 3u_x u_{xxx}\right] \tag{8}$$

with $\varepsilon > 0$. This equation, unlike (5) is well posed. It also admits conventional solitons and allows compactons like

$$u(x,t) = \begin{cases} \dfrac{4c^2 - 3\alpha_2}{2\alpha_3} \cos^2\left\{(12\varepsilon)^{-1/2}(x - ct)\right\}, & \text{if} \quad |x - ct| \le 2\pi, \\ 0, & \text{if} \quad |x - ct| > 2\pi, \end{cases} \tag{9}$$

or

$$u(x,t) = \begin{cases} A\cos\left\{(3\varepsilon)^{-1/2}\left[x - \left(\tfrac{3}{2}\alpha_2\right)^{1/2} t\right]\right\}, & \text{if} \quad |x - ct| \le 2\pi, \\ 0, & \text{if} \quad |x - ct| > 2\pi. \end{cases} \tag{10}$$

These again are weak solutions, and are very similar to the previous solutions: both (7) and (10) are solutions with a variable speed linked to the amplitude of the wave, whereas both (6) and (9) are solutions with arbitrary amplitudes, whilst the wave speed is fixed by the parameters of the equation.

The Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation

$$u_t - u_{xxt} + 2\kappa u_x = uu_{xxx} - 3uu_x + 2u_x u_{xx}, \tag{11}$$

first arose in the work of Fuchssteiner and Fokas [34, 36] using a bi-Hamiltonian approach; we remark that it is only implicitly written in [36] – see equations (26e) and (30) in this paper – though is explicitly written down in [34]. It has recently been rederived by Camassa and Holm [11] from physical considerations as a model for dispersive shallow water waves. In the case $\kappa = 0$, it admits an unusual solitary wave solution

$$u(x,t) = A \exp\left(-|x - ct|\right),$$

where A and c are arbitrary constants, which is called a "peakon". A Lax-pair [11] and bi-Hamiltonian structure [36] have been found for the FFCH equation (11) and so it appears to be completely integrable. Recently the FFCH equation (11) has attracted considerable attention. In addition to the aforementioned, other studies include [12, 25, 26, 27, 29, 32, 33, 35, 40, 42, 56, 66].

Symmetry reductions and exact solutions have several different important applications in the context of differential equations. Since solutions of partial differential equations asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. In particular, exact solutions arising from symmetry methods can often be used effectively to study properties such as asymptotics and "blow-up" (cf. [37, 38]). Furthermore, explicit solutions (such as those found by symmetry methods) can play an important role in the design and testing of numerical integrators; these solutions provide an important practical check on the accuracy and reliability of such integrators (cf. [5, 67]).

The classical method for finding symmetry reductions of partial differential equations is the Lie group method of infinitesimal transformations, which in practice is a two-step procedure (see § 2 for more details). The first step is entirely algorithmic, though often both tedious and virtually unmanageable manually. As a result, symbolic manipulation (SM) programs have been developed to aid the calculations; an excellent survey of the different packages available and a description of their strengths and applications is given by Hereman [41]. In this paper we use the MACSYMA package `symmgrp.max` [15] to calculate the determining equations. The second step involves heuristic integration procedures which have been implemented in some SM programs and are largely successful, though not infallible. Commonly, the overdetermined systems to be solved are simple, and heuristic integration is both fast and effective. However, there are occasions where heuristics can break down (cf. [50] for further details and examples). Of particular importance to this study, is if the classical method is applied to a partial differential equation which contains arbitrary parameters, such as (1) or more generally, arbitrary functions. Heuristics usually yield the general solution yet miss those special cases of the parameters and arbitrary functions where additional symmetries lie. In contrast the method of differential Gröbner bases (DGBs), which we describe below, has proved effective in coping with such difficulties (cf. [20, 24, 50, 51]).

In recent years the nonclassical method due to Bluman and Cole [7] (in the sequel referred to as the "nonclassical method"), sometimes referred to as the "method of partial symmetries of the first type" [68], or the "method of conditional symmetries" [45], and the direct method due to Clarkson and Kruskal [18] have been used, with much success, to generate many new symmetry reductions and exact solutions for several physically significant partial differential equations that are not obtainable using the classical Lie method (cf. [16, 19] and the references therein). The nonclassical method is a generalization of the classical Lie method, whereas the direct method is an ansatz-based approach which involves no group theoretic techniques. Nucci and Clarkson [53] showed that for the Fitzhugh-Nagumo equation the nonclassical method is more general than the direct method, since they demonstrated the existence of a solution of the Fitzhugh-Nagumo equation, obtainable using the nonclassical method but not using the direct method. Subsequently Olver [55] (see also [6, 57]) has proved the general result that for a scalar equation, every reduction obtainable using the direct method is also obtainable using the nonclassical method. Consequently we use the nonclassical method in this paper rather than the direct method.

The method used to find solutions of the determining equations in both the classical and nonclassical method is that of DGBs, defined to be a basis ß of the differential ideal generated by the system such that every member of the ideal pseudo-reduces to zero with respect to the basis ß. This method provides a systematic framework for finding integrability and compatibility conditions of an overdetermined system of partial differential equations. It avoids the problems of infinite loops in reduction processes and yields, as far as is currently possible, a "triangulation" of the system from which the solution set can be derived more easily [20, 52, 60, 61]. In a sense, a DGB provides the maximum amount of information possible using elementary differential and algebraic processes in finite time.

In pseudo-reduction, one must, if necessary, multiply the expression being reduced by differential (non-constant) coefficients of the highest derivative terms of the reducing equation, so that the algorithms used will terminate [52]. In practice, such coefficients are assumed to be non-zero, and one needs to deal with the possibility of them being zero separately. These are called singular cases.

The triangulations of the systems of determining equations for infinitesimals arising in the nonclassical method in this paper were all performed using the MAPLE package `diffgrob2` [49]. This package was written specifically to handle nonlinear equations of polynomial type. All calculations are strictly 'polynomial', that is, there is no division. Implemented there are the Kolchin-Ritt algorithm using pseudo-reduction instead of reduction, and extra algorithms needed to calculate a DGB (as far as possible using the current theory), for those cases where the Kolchin-Ritt algorithm is not sufficient [52]. The package was designed to be used interactively as well as algorithmically, and much use is made of this fact here. It has proved useful for solving many fully nonlinear systems (cf. [20, 21, 22, 23, 24]).

In the following sections we shall consider the cases $\mu = 0$ and $\mu \neq 0$, when we set $\mu = 1$ without loss of generality, separately because the presence or lack of the corresponding fourth order term is significant. In § 2 we find the classical Lie group of symmetries and associated reductions of (1). In § 3 we discuss the nonclassical symmetries and reductions of (1) in the generic case. In § 4 we consider special cases of the the nonclassical method in the so-called $\tau = 0$ case; in full generality this case is somewhat intractable. In § 5 we discuss our results.

2 Classical Symmetries

To apply the classical method we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + \mathcal{O}\left(\varepsilon^2\right), \\ t^* &= t + \varepsilon\tau(x,t,u) + \mathcal{O}\left(\varepsilon^2\right), \\ u^* &= u + \varepsilon\phi(x,t,u) + \mathcal{O}\left(\varepsilon^2\right), \end{aligned} \tag{12}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set

$$S_\Delta \equiv \{u(x,t) : \Delta = 0\} \tag{13}$$

of solutions of (1). This yields an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The associated Lie algebra is realised by vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{14}$$

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equation

$$\frac{\mathrm{d}x}{\xi(x,t,u)} = \frac{\mathrm{d}t}{\tau(x,t,u)} = \frac{\mathrm{d}u}{\phi(x,t,u)}, \tag{15}$$

which is equivalent to solving the invariant surface condition

$$\psi \equiv \xi(x,t,u)u_x + \tau(x,t,u)u_t - \phi(x,t,u) = 0. \tag{16}$$

The set S_Δ is invariant under the transformation (12) provided that

$$\mathrm{pr}^{(4)}\mathbf{v}(\Delta)|_{\Delta\equiv 0} = 0,$$

where $\mathrm{pr}^{(4)}\mathbf{v}$ is the fourth prolongation of the vector field (14), which is given explicitly in terms of ξ, τ and ϕ (cf. [54]). This procedure yields the determining equations. There are two cases to consider, (i) $\mu = 0$ and (ii) $\mu \neq 0$.

2.1 Case (i) $\mu = 0$.

In this case we generate 15 determining equations, using the MACSYMA package `symmgrp.max`.

$$\begin{gathered} \tau_u = 0, \qquad \tau_x = 0, \qquad \xi_u = 0, \qquad \phi_{uu} = 0, \qquad \xi_t = 0, \\ \alpha(\phi_u u - \phi) = 0, \qquad \beta(\phi_u u - \phi) = 0, \qquad 2\phi_{tu} - \tau_{tt} = 0, \\ 4\phi_{xu}u - 6\xi_{xx}u + \alpha\phi_x, \qquad 2\tau_t u - 4\xi_x u + \phi = 0, \\ 4\beta\phi_{xu} + 3\alpha\phi_{xu} - 2\beta\xi_{xx} - 3\alpha\xi_{xx} = 0, \\ \phi_{tt} - \phi_{xxxx}u - 2\gamma\phi_{xx}u - \kappa\phi_{xx} = 0, \\ 3\alpha\phi_{xxu}u + 2\gamma\phi_u u + 4\xi_x\gamma u - \alpha\xi_{xxx}u - 2\gamma\phi = 0, \\ 6\phi_{xxu}u^2 + 4\xi_x\gamma u^2 - 4\xi_{xxx}u^2 + 2\beta\phi_{xx}u + 2\xi_x\kappa u - \kappa\phi = 0, \\ 4\phi_{xxxu}u + 4\gamma\phi_{xu}u - 2\xi_{xx}\gamma u - \xi_{xxxx}u + \alpha\phi_{xxx} + 4\gamma\phi_x + 2\kappa\phi_{xu} - \xi_{xx}\kappa = 0, \end{gathered}$$

and then use `reduceall` in `diffgrob2` to simplify them to the following system

$$\xi_u = 0, \qquad \xi_t = 0, \qquad \gamma(14\beta + 9\alpha)\xi_x = 0, \qquad \tau_u = 0,$$
$$\gamma\kappa(14\beta + 9\alpha)\tau_t = 0, \qquad \tau_x = 0, \qquad \gamma\kappa(14\beta + 9\alpha)\phi = 0.$$

Thus we have special cases when $\gamma = 0$, $\kappa = 0$ and/or $14\beta + 9\alpha = 0$. The latter condition provides nothing different unless we specialize further and consider the special case when $\alpha = -\frac{5}{2}$ and $\beta = \frac{45}{28}$. We continue to use `reduceall` in `diffgrob2` for the various combinations and it transpires that there are only four combinations which yield different infinitesimals. Where a parameter is not included it is presumed to be arbitrary.

(a) $\kappa = 0$, $\quad \xi_u = 0, \quad \xi_t = 0, \quad \xi_x = 0, \quad \tau_u = 0,$
$\tau_{tt} = 0, \quad \tau_x = 0, \quad 2\tau_t u + \phi = 0.$

(b) $\gamma = 0$, $\quad \xi_u = 0, \quad \xi_t = 0, \quad \xi_{xx} = 0, \quad \tau_u = 0,$
$\xi_x - \tau_t = 0, \quad \tau_x = 0, \quad 2\xi_x u + \phi = 0.$

(c) $\gamma = \kappa = 0$, $\quad \xi_u = 0, \quad \xi_t = 0, \quad \xi_{xx} = 0, \quad \tau_u = 0,$
$\tau_{tt} = 0, \quad \tau_x = 0, \quad 2\tau_t u - 4\xi_x u - \phi = 0.$

(d) $\alpha = -\frac{5}{2}, \quad \beta = \frac{45}{28}, \quad \gamma = \kappa = 0$, $\quad \xi_u = 0, \quad \xi_t = 0, \quad \xi_{xxx} = 0, \quad \tau_u = 0,$
$\tau_{tt} = 0, \quad \tau_x = 0, \quad 2\tau_t u - 4\xi_x u - \phi = 0.$

Hence we obtain the following infinitesimals.

Table 2.1

Parameters	ξ	τ	ϕ	
	c_1	c_2	0	(2.1i)
$\kappa = 0$	c_1	$c_3 t + c_2$	$-2c_3 u$	(2.1ii)
$\gamma = 0$	$c_3 x + c_1$	$c_3 t + c_2$	$2c_3 u$	(2.1iii)
$\gamma = \kappa = 0$	$c_4 x + c_1$	$c_3 t + c_2$	$(4c_4 - 2c_3)u$	(2.1iv)
$\alpha = -\frac{5}{2}, \quad \beta = \frac{45}{28}$, $\gamma = \kappa = 0$	$c_5 x^2 + c_4 x + c_1$	$c_3 t + c_2$	$[4(2c_5 x + c_4) - 2c_3]u$	(2.1v)

where $c_1, c_2, \ldots, c_5$ are arbitrary constants.

Solving the invariant surface condition yields the following seven different canonical reductions:

Reduction 2.1 α, β, γ and κ arbitrary. If in (2.1i–2.1v) $c_3 = c_4 = c_5 = 0$, then we may set $c_2 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z), \qquad z = x - c_1 t$$

where $w(z)$ satisfies

$$(\kappa - c_1^2)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 2\gamma\left[w\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2\right] + w\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 = 0.$$

Reduction 2.2 α, β and γ arbitrary, $\kappa = 0$. If in (2.1ii), (2.1iv) and (2.1v) $c_4 = c_5 = 0$, $c_3 \neq 0$, then we may set $c_2 = 0$ and $c_3 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = t^{-2}w(z), \qquad z = x - c_1 \log(t),$$

where $w(z)$ satisfies

$$w\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 + 2\gamma\left[w\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2\right] - c_1^2\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - 5c_1\frac{\mathrm{d}w}{\mathrm{d}z} - 6w = 0.$$

Reduction 2.3 α, β and κ arbitrary, $\gamma = 0$. If in (2.1iii) $c_3 \neq 0$, then we may set $c_1 = c_2 = 0$ and $c_3 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = t^2 w(z), \qquad z = x/t,$$

where $w(z)$ satisfies

$$w\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 + \kappa\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - z^2\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 2z\frac{\mathrm{d}w}{\mathrm{d}z} - 2w = 0.$$

Reduction 2.4 α and β arbitrary, $\kappa = \gamma = 0$. If in (2.1iv) and (2.1v) $c_3 = c_5 = 0$ and $c_4 \neq 0$, then we may set $c_1 = 0$ and $c_2 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z)\exp(4c_4 t), \qquad z = x\exp(-c_4 t),$$

where $w(z)$ satisfies

$$w\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 - c_4^2 z^2\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 7c_4^2 z\frac{\mathrm{d}w}{\mathrm{d}z} - 16c_4^2 w = 0.$$

Reduction 2.5 α and β arbitrary, $\kappa = \gamma = 0$. If in (2.1iv) and (2.1v) $c_5 = 0$ and $c_3c_4 \neq 0$, then we may set $c_1 = c_2 = 0$ and $c_3 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x,t) = w(z)t^{4c_4-2}, \qquad z = xt^{-c_4},$$

where $w(z)$ satisfies

$$w\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 - c_4^2 z^2\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(7c_4^2 - 5c_4\right)z\frac{\mathrm{d}w}{\mathrm{d}z} - \left(16c_4^2 - 20c_4 + 6\right)w = 0.$$

Reduction 2.6 $\alpha = -\frac{5}{2}$, $\beta = \frac{45}{28}$, $\gamma = \kappa = 0$. If in (2.1v) $c_3 = 0$ and $c_5 \neq 0$, then we may set $c_1 = -mc$, $c_2 = 1$, $c_4 = 0$ and $c_5 = m/c$, without loss of generality. Thus we obtain the reduction

$$u(x,t) = \frac{w(z)\exp(-8mt)}{[z-\exp(-2mt)]^8}, \qquad z = \left(\frac{x-c}{x+c}\right)\exp(-2mt),$$

where $w(z)$ satisfies

$$28w\frac{\mathrm{d}^4w}{\mathrm{d}z^4} - 70\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3w}{\mathrm{d}z^3} + 45\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}\right)^2 - c^4m^2\left(1792z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} - 12544z\frac{\mathrm{d}w}{\mathrm{d}z} + 28672w\right) = 0.$$

Reduction 2.7 $\alpha = -\frac{5}{2}$, $\beta = \frac{45}{28}$, $\gamma = \kappa = 0$. If in (2.1v) $c_3c_5 \neq 0$, then we may set $c_1 = -mc$, $c_2 = c_4 = 0$, $c_3 = 1$ and $c_5 = m/c$, without loss of generality. Thus we obtain the reduction

$$u(x,t) = \frac{w(z)t^{-2(1+4m)}}{(z-t^{-2m})^8}, \qquad z = \left(\frac{x-c}{x+c}\right)t^{-2m},$$

where $w(z)$ satisfies

$$28w\frac{\mathrm{d}^4w}{\mathrm{d}z^4} - 70\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3w}{\mathrm{d}z^3} + 45\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}\right)^2 - 1792c^4m^2z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + \left(12544m^2 - 4480m\right)c^4z\frac{\mathrm{d}w}{\mathrm{d}z} - \left(28672m^2 - 17920m + 2688\right)c^4w = 0.$$

2.2 Case (ii) $\mu \neq 0$.

In this case we generate 18 determining equations,

$$\begin{aligned}
&\tau_u = 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \xi_t = 0, \quad \phi_{uu} = 0,\\
&\phi_{xtu} = 0, \quad \alpha(\phi_u u - \phi) = 0, \quad 2\phi_{tu} - \tau_{tt} = 0, \quad \beta(\phi_u u - \phi) = 0,\\
&2\phi_{xu} - \xi_{xx} = 0, \quad 4\phi_{xu}u - 6\xi_{xx}u + \alpha\phi_x = 0,\\
&2\tau_t u - 2\xi_x u + \phi = 0,\\
&4\beta\phi_{xu} + 3\alpha\phi_{xu} - 2\beta\xi_{xx} - 3\alpha\xi_{xx} = 0,\\
&\phi_{xxu}u + 2\tau_t u - 4\xi_x u + \phi = 0,\\
&3\alpha\phi_{xxu}u + 2\gamma\phi_u u + 4\xi_x\gamma u - \alpha\xi_{xxx}u - 2\gamma\phi = 0,\\
&\phi_{tt} - \phi_{xxxx}u - 2\gamma\phi_{xx}u - \kappa\phi_{xx} - \phi_{xxtt} = 0,\\
&6\phi_{xxu}u^2 + 4\xi_x\gamma u^2 - 4\xi_{xxx}u^2 + 2\beta\phi_{xx}u + \phi_{ttu}u + 2\xi_x\kappa u - \kappa\phi = 0,\\
&4\phi_{xxxu}u + 4\gamma\phi_{xu}u - 2\xi_{xx}\gamma u - \xi_{xxxx}u + \alpha\phi_{xxx} + 4\gamma\phi_x + 2\kappa\phi_{xu} - \xi_{xx}\kappa = 0,
\end{aligned}$$

and then use `reduceall` in `diffgrob2` to simplify them to the following system,

$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_x = 0, \quad \tau_u = 0, \quad \kappa\tau_t = 0, \quad \tau_x = 0, \quad \kappa\phi = 0.$$

Here $\kappa = 0$ is the only special case, yielding the slightly different system

$$\xi_u = 0, \quad \xi_t = 0, \quad \xi_x = 0, \quad \tau_u = 0, \quad \tau_{tt} = 0, \quad \tau_x = 0, \quad \phi + 2\tau_t u = 0.$$

Thus we have two different sets of infinitesimals, and in both cases α, β and γ remain arbitrary.

Table 2.2

Parameters	ξ	τ	ϕ	
	c_1	c_2	0	(2.2i)
$\kappa = 0$	c_1	$c_3 t + c_2$	$-2c_3 u$	(2.2ii)

From these we have the following two canonical reductions:

Reduction 2.8 α, β, γ and κ arbitrary. If in (2.2i) and (2.2ii) $c_3 = 0$, then we may set $c_2 = 1$ without loss of generality. Thus we obtain the following reduction

$$u(x,t) = w(z), \qquad z = x - c_1 t,$$

where $w(z)$ satisfies

$$\left(\kappa - c_1^2\right) \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 2\gamma \left[w \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}z} \right)^2 \right] + w \frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha \frac{\mathrm{d}w}{\mathrm{d}z} \frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta \left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} \right)^2 + c_1^2 \frac{\mathrm{d}^4 w}{\mathrm{d}z^4} = 0.$$

Reduction 2.9 α, β, γ arbitrary, $\kappa = 0$. If in (2.2ii) $c_3 \neq 0$, then we may set $c_2 = 0$, $c_3 = 1$ without loss of generality. Thus we obtain the following reduction

$$u(x,t) = t^{-2} w(z), \qquad z = x - c_1 \log(t),$$

where $w(z)$ satisfies

$$w \frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha \frac{\mathrm{d}w}{\mathrm{d}z} \frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta \left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} \right)^2 + c_1^2 \frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + 5c_1 \frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + 2\gamma \left[w \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}z} \right)^2 \right] + \left(6 - c_1^2\right) \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - 5c_1 \frac{\mathrm{d}w}{\mathrm{d}z} - 6w = 0.$$

2.3 Travelling wave reductions

As was seen in § 1, special cases of (1) admit interesting travelling wave solutions, namely compactons. In this subsection we look for such solitary waves and others, in the framework of (1). Starting with compacton-type solutions, we seek solutions of the form

$$u(x,t) = a_2 \cos^n\{a_3(x - a_1 t)\} + a_4, \tag{17}$$

where a_1, a_2, a_3, a_4 are constants to be determined. We include the (possibly non-zero) constant a_4 since u is open to translation. The specific form of the translation will put conditions on a_4, which may or may not put further conditions on the other parameters in (17) and those in (1) – see below. If $n = 1$ we have the solutions, where the absence of a parameter implies it is arbitrary,

$$\text{(i)}\quad \alpha = 0, \quad \beta = -1, \quad \gamma = 0, \quad a_4 = \frac{\kappa - a_1^2 - a_1^2 a_2^2 \mu}{a_2^2}.$$

$$\text{(ii)}\quad \alpha = 1, \quad \beta = 0, \quad \gamma > 0, \quad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \quad a_3^2 = 2\gamma.$$

$$\text{(iii)}\quad \beta = \alpha - 1 \neq 0, \quad \frac{\gamma}{\alpha} > 0, \quad a_3^2 = \frac{2\gamma}{\alpha}, \quad a_4 = \frac{\alpha(\kappa - a_1^2) - 2a_1^2\gamma\mu}{2\gamma(1-\alpha)}.$$

These become $n = 2$ solutions via the trigonometric identity $\cos 2\theta = 2\cos^2\theta - 1$. By earlier reasoning the associated compactons are weak solutions of (1). When considering more general n we restrict n to be either 3 or ≥ 4 else the fourth derivatives of $u(x,t)$ that we require in (1) would have singularities at the edges of the humps; we find

$$\alpha = \frac{2}{n}, \quad \beta = \frac{2-n}{n}, \quad \gamma > 0, \quad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \quad a_3^2 = \frac{\gamma}{n}, \quad a_4 = -\mu a_1^2.$$

When $n = 3$ or $n = 4$ our compacton would be a weak solution since not all the derivatives of $u(x,t)$ in (1) in these instances are continuous at the edges. For $n > 4$ the solutions are strong.

For more usual solitary waves we seek solutions of the form

$$u(x,t) = a_2 \operatorname{sech}^n\{a_3(x - a_1 t)\} + a_4,$$

where a_1, a_2, a_3, a_4 are constants to be determined. If $n = 2$ then $\alpha = -1$, $\beta = -2$ and we have solutions

$$\text{(i)}\quad \gamma < 0, \quad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \quad a_3^2 = -\tfrac{1}{2}\gamma, \quad a_4 = -\tfrac{1}{3}(2a_2 + 3a_1^2\mu),$$

$$\text{(ii)}\quad \gamma^2 \neq 4a_3^4, \quad a_2 = \frac{3a_3^2(\kappa - a_1^2 - 2a_1^2\gamma\mu)}{(2a_3^2 - \gamma)(2a_3^2 + \gamma)}, \quad a_4 = -\frac{\kappa - a_1^2 + 4a_1^2 a_3^2 \mu}{2(2a_3^2 + \gamma)},$$

and for general n, including $n = 2$ $(\gamma > 0)$

$$\alpha = -\frac{2}{n}, \qquad \beta = -\frac{n+2}{n}, \qquad a_1^2(1 + 2\gamma\mu) - \kappa = 0, \qquad a_3^2 = \frac{\gamma}{n}, \qquad a_4 = -\mu a_1^2.$$

Now consider the general travelling wave reduction, $u(x,t) = w(z)$, $z = x - ct$, where $w(z)$ satisfies

$$\left(\kappa - c^2\right)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 2\gamma\left[w\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2\right]$$
$$+\mu c^2 \frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + w\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \beta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 = 0.$$

In the special case $\beta = \alpha - 1$, we can integrate this twice with respect to z to give

$$\left(\kappa - c^2\right) w + \gamma w^2 + \mu c^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{1}{2}(\alpha - 2)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + Az + B = 0 \tag{18}$$

with A and B the constants of integration. If we assume $A = 0$, then we make the transformation $W(z) = w(z) + \mu c^2$, multiply (18) by $W^{\alpha-3}\dfrac{\mathrm{d}W}{\mathrm{d}z}$ and integrate with respect to z to yield

$$\frac{\gamma}{\alpha}W^{\alpha} + \frac{A_1}{\alpha-1}W^{\alpha-1} + \frac{A_2}{\alpha-2}W^{\alpha-2} + W^{\alpha-2}\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^2 + C = 0$$

for $\alpha \neq 0, 1, 2$, where $A_1 = \kappa - c^2 - 2\gamma\mu c^2$ and $A_2 = B - \mu c^2\left(\kappa - c^2 - \gamma\mu c^2\right)$. In the special cases $\alpha = 0, 1, 2$ we obtain respectively

$$\gamma \log W - \frac{A_1}{W} - \frac{2A_2}{W^2} + \frac{1}{W^2}\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^2 + C = 0, \tag{19}$$

$$\gamma W + A_1 \log W - \frac{A_2}{W} + \frac{1}{W}\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^2 + C = 0, \tag{20}$$

$$\frac{1}{2}\gamma W^2 + A_1 W + A_2 \log W + \left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^2 + C = 0, \tag{21}$$

where C is a constant of integration. For $\alpha \in \mathbb{Z}$, an integer, with $\alpha \geq 3$, (2.3) may be written as

$$\begin{aligned}
&\left(w + \mu c^2\right)^{\alpha-2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \\
&\quad + \frac{\gamma w^2}{\alpha}\left[\left(w + \mu c^2\right)^{\alpha-2} + \frac{\alpha\left(\kappa - c^2\right) - 2\mu\gamma c^2}{\gamma}\sum_{n=0}^{\alpha-3}\frac{(\mu c^2)^{\alpha-3-n}}{(n+2)}\binom{\alpha-3}{n}w^n\right] \\
&\quad + \frac{B}{\alpha-2}\left(w + \mu c^2\right)^{\alpha-2} + C - \frac{(\mu c^2)^{\alpha-1}\left[\alpha\left(\kappa - c^2\right) - 2\mu\gamma c^2\right]}{\alpha(\alpha-1)(\alpha-2)} = 0,
\end{aligned} \tag{22}$$

where C is a constant of integration. If we require that w and its derivatives tend to zero as $z \to \pm\infty$, then $B = D = 0$. If $\alpha = 3$ this equation induces so-called peakons (cf. [11]) as $\alpha\left(\kappa - c^2\right) - 2\mu\gamma c^2 \to 0$ (see [12, 40, 44, 62]). Similarly if $\alpha = 4$ this equation is of the form found in [40] which induces the 'wave of greatest height' found in [31]. Both solutions, in their limit, have a discontinuity in their first derivative at its peak. Note that if $\alpha\left(\kappa - c^2\right) - 2\mu\gamma c^2 = 0$, equation (22) becomes

$$\left(w + \mu c^2\right)^{\alpha-2}\left[\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{\gamma}{\alpha}w^2\right] = 0. \tag{23}$$

Since $\alpha > 0$ then we require $\gamma < 0$ to give a peakon of the form

$$u(x,t) = \frac{\alpha\left(c^2 - \kappa\right)}{2\gamma}\exp\left\{-\left(\frac{-\gamma}{\alpha}\right)^{1/2}|x - ct|\right\}. \tag{24}$$

The height of the wave, because of the form of (1), is dependent upon the square of the speed, whereas the peakons in [11] and [31] are proportional to the wave speed.

3 Nonclassical symmetries ($\tau \neq 0$)

In the nonclassical method one requires only the subset of S_Δ given by

$$S_{\Delta,\psi} = \{u(x,t) : \Delta(u) = 0, \psi(u) = 0\}, \tag{25}$$

where S_Δ is defined in (13) and $\psi = 0$ is the invariant surface condition (16), to be invariant under the transformation (12). The usual method of applying the nonclassical method (e.g. as described in [45]), involves applying the prolongation $\mathrm{pr}^{(4)}\mathbf{v}$ to the system composed of (1) and the invariant surface condition (16) and requiring that the resulting expressions vanish for $u \in S_{\Delta,\psi}$, i.e.

$$\mathrm{pr}^{(4)}\mathbf{v}(\Delta)|_{\{\Delta=0,\psi=0\}} = 0, \qquad \mathrm{pr}^{(1)}\mathbf{v}(\psi)|_{\{\Delta=0,\psi=0\}} = 0. \tag{26}$$

It is well known that the latter vanishes identically when $\psi = 0$ without imposing any conditions upon ξ, τ and ϕ. To apply the method in practice we advocate the algorithm described in [22] for calculating the determining equations, which avoids difficulties arising from using differential consequences of the invariant surface condition (16).

In the canonical case when $\tau \neq 0$ we set $\tau = 1$ without loss of generality. We proceed by eliminating u_{tt} and u_{xxtt} in (1) using the invariant surface condition (16) which yields

$$\begin{aligned}
&\xi\xi_x u_x + 2u_x^2\xi\xi_u - 2\phi_u\xi u_x + \xi^2 u_{xx} - \phi_x\xi - \xi_t u_x + \phi\phi_u - \phi\xi_u u_x + \phi_t - \kappa u_{xx} \\
&\quad -2\gamma(uu_{xx} + u_x^2) - uu_{xxxx} - \alpha u_x u_{xxx} - \beta u_{xx}^2 + \mu\Big[2\phi_{xx}\xi_x - 2\phi_{xu}\phi_x - 4\xi_x^2 u_{xx} \\
&\quad -\phi_{tu}u_{xx} - \phi_u\phi_{xx} - \phi_{xxt} + \xi_{xxt}u_x - \phi_{tuu}u_x^2 + \phi_x\xi_{xx} + \xi_{tuu}u_x^3 + \xi_t u_{xxx} - \xi^2 u_{xxxx} \\
&\quad +\phi_{xxx}\xi - \phi_u^2 u_{xx} - \phi\phi_{xxu} - \xi\xi_{xxx}u_x + \phi\xi_{uuu}u_x^3 + \phi\xi_{xxu}u_x + \phi\xi_u u_{xxx} - \phi\phi_{uu}u_{xx} \\
&\quad -\phi\phi_{uuu}u_x^2 + 2\xi_{xt}u_{xx} + 2\xi_{xtu}u_x^2 - 2\phi_{xtu}u_x - 3\xi_x\xi_{xx}u_x - 4\xi_u\xi_{xx}u_x^2 - 4\xi\xi_{xx}u_{xx} \\
&\quad +2\phi_u\xi_{xx}u_x - 5\xi_{uu}\xi_x u_x^3 - 8\xi_{xu}\xi_x u_x^2 - 15\xi_u\xi_x u_x u_{xx} - 5\xi\xi_x u_{xxx} + 4\phi_u\xi_x u_{xx} \\
&\quad +4\phi_{uu}\xi_x u_x^2 + 6\phi_{xu}\xi_x u_x - 2\xi\xi_{uuu}u_x^4 - 5\xi\xi_{xuu}u_x^3 + 2\phi\xi_{xuu}u_x^2 - 6\xi_u\xi_{uu}u_x^4 \\
&\quad -12\xi\xi_{uu}u_x^2 u_{xx} + 3\phi\xi_{uu}u_x u_{xx} + 4\phi_u\xi_{uu}u_x^3 + 3\phi_x\xi_{uu}u_x^2 - 4\xi\xi_{xxu}u_x^2 - 10\xi_u\xi_{xu}u_x^3 \\
&\quad -15\xi\xi_{xu}u_x u_{xx} + 2\phi\xi_{xu}u_{xx} + 6\phi_u\xi_{xu}u_x^2 + 4\phi_x\xi_{xu}u_x - 12\xi_u^2 u_x^2 u_{xx} - 8\xi\xi_u u_x u_{xxx} \\
&\quad -6\xi\xi_u u_{xx}^2 + 9\phi_u\xi_u u_x u_{xx} + 3\phi_x\xi_u u_{xx} + 5\phi_{uu}\xi_u u_x^3 + 8\phi_{xu}\xi_u u_x^2 + 3\phi_{xx}\xi_u u_x \\
&\quad +3\xi_{tu}u_x u_{xx} + 2\phi_u\xi u_{xxx} + 6\phi_{uu}\xi u_x u_{xx} + 5\phi_{xu}\xi u_{xx} + 2\phi_{uuu}\xi u_x^3 + 5\phi_{xuu}\xi u_x^2 \\
&\quad +4\phi_{xxu}\xi u_x - 3\phi_u\phi_{uu}u_x^2 - 2\phi_{uu}\phi_x u_x - 2\phi\phi_{xuu}u_x - 4\phi_u\phi_{xu}u_x\Big] = 0.
\end{aligned} \tag{27}$$

We note that this equation now involves the infinitesimals ξ and ϕ that are to be determined. Then we apply the classical Lie algorithm to (27) using the fourth prolongation $\mathrm{pr}^{(4)}\mathbf{v}$ and eliminating u_{xxxx} using (27). It should be noted that the coefficient of u_{xxxx} is $(\xi^2 + \mu u)$. Therefore, if this is zero the removal of u_{xxxx} using (27) is invalid and so the next highest derivative term, u_{xxx}, should be used instead. We note again that this has a coefficient that may be zero so that in the case $\mu \neq 0$ and $\xi^2 + \mu u = 0$ one needs to calculate the determining equations for the subcases non-zero separately. Continuing in this fashion, there is a cascade of subcases to be considered. In the remainder of this section, we consider these subcases in turn. First, however, we discuss the case given by $\mu = 0$.

3.1 Case (i) $\mu = 0$.

In this case we generate the following 12 determining equations.

$$\begin{aligned}
&\xi_u = 0,\\
&\phi_{uuuu}u + \alpha\phi_{uuu} = 0,\\
&4\phi_{xuuu}u + 3\alpha\phi_{xuu} = 0,\\
&6\phi_{uuu}u + 2\beta\phi_{uu} + 3\alpha\phi_{uu} = 0,\\
&4\phi_{uu}u^2 + \alpha\phi_u u - \alpha\phi = 0,\\
&4\phi_{xu}u - 6\xi_{xx}u + \alpha\phi_x = 0,\\
&3\phi_{uu}u^2 + \beta\phi_u u - \beta\phi = 0,\\
&12\phi_{xuu}u + 4\beta\phi_{xu} + 3\alpha\phi_{xu} - 2\beta\xi_{xx} - 3\alpha\xi_{xx} = 0,\\
&6\phi_{xxuu}u^2 + 2\gamma\phi_{uu}u^2 + \kappa\phi_{uu}u - \xi^2\phi_{uu}u\\
&\qquad +3\alpha\phi_{xxu}u + 2\gamma\phi_u u + 4\xi_x\gamma u - \alpha\xi_{xxx}u - 2\gamma\phi = 0,\\
&6\phi_{xxu}u^2 + 4\xi_x\gamma u^2 - 4\xi_{xxx}u^2 + 2\beta\phi_{xx}u + 2\xi_x\kappa u - 4\xi^2\xi_x u - 2\xi\xi_t u - \kappa\phi + \xi^2\phi = 0,\\
&\phi_{tt}u - \phi_{xxxx}u^2 - 2\gamma\phi_{xx}u^2 - \kappa\phi_{xx}u - 4\xi\xi_x\phi_x u - 2\xi_t\phi_x u\\
&\qquad +\phi^2\phi_{uu}u + 4\xi_x\phi\phi_u u + 2\phi\phi_{tu}u + 4\xi_x\phi_t u + \xi\phi\phi_x - \phi^2\phi_u - \phi\phi_t = 0,\\
&4\phi_{xxxu}u^2 + 4\gamma\phi_{xu}u^2 - 2\xi_{xx}\gamma u^2 - \xi_{xxxx}u^2\\
&\qquad +\alpha\phi_{xxx}u + 4\gamma\phi_x u + 2\xi\phi\phi_{uu}u + 2\kappa\phi_{xu}u + 8\xi\xi_x\phi_u u + 2\xi_t\phi_u u\\
&\qquad +2\xi\phi_{tu}u - \xi_{xx}\kappa u - 4\xi\xi_x^2 u + 2\xi_t\xi_x u + \xi_{tt}u - 2\xi\phi\phi_u + \xi\xi_x\phi - \xi_t\phi = 0.
\end{aligned}$$

As guaranteed by the nonclassical method, we get all the classical reductions, but we also have some infinitesimals that lead to nonclassical reductions, namely

Table 3.1

Parameters	ξ	ϕ	
$\kappa = 0$	0	$g(t)u$ where $\dfrac{\mathrm{d}^2g}{\mathrm{d}t^2} + g\dfrac{\mathrm{d}g}{\mathrm{d}t} - g^3 = 0$	(3.1i)
$\alpha = \beta = \gamma = 0$	$\pm\sqrt{\kappa}$	$c_3y^3 + c_2y^2 + c_1y + c_0 \quad (y = x \pm \sqrt{\kappa}\,t)$	(3.1ii)
$\alpha = \beta = \gamma = \kappa = 0$	0	$-\dfrac{u}{g(t)}\dfrac{\mathrm{d}g}{\mathrm{d}t} + g(t)\left(c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0\right)$ where $g^2\dfrac{\mathrm{d}^3g}{\mathrm{d}t^3} - 4g\dfrac{\mathrm{d}g}{\mathrm{d}t}\dfrac{\mathrm{d}^2g}{\mathrm{d}t^2} + 2\left(\dfrac{\mathrm{d}g}{\mathrm{d}t}\right)^3 + 24c_4g^4 = 0$	(3.1iii)

From these we obtain three canonical reductions.

Reduction 3.1 α, β, γ arbitrary, $\kappa = 0$. In (3.1i) we solve the equation for $g(t)$ by writing $g(t) = [\ln(\psi(t))]_t$ then $\psi(t)$ satisfies

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = 4c_1\psi^3 + c_2, \tag{28}$$

where c_1 and c_2 are arbitrary constants; $c_1 = c_2 = 0$ is not allowed since $g(t) \not\equiv 0$. Hence we obtain the following reduction

$$u(x,t) = w(x)\psi(t),$$

where $w(x)$ satisfies

$$w\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \alpha\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + \beta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 + 2\gamma\left[w\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right] - 6c_1 w = 0.$$

There are three cases to consider in the solution of (28).

(i) If $c_1 = 0$, we may assume that $\psi(t) = t$ without loss of generality.

(ii) If $c_2 = 0$, then $\psi = \left[c_2(t+c_3)^2\right]^{-1}$ and we may set $c_2 = 1$, $c_3 = 0$ without loss of generality.

(iii) If $c_1 c_2 \neq 0$ we may set $c_1 = 1$, $c_2 = -g_3$ without loss of generality so that $\psi(t)$ is any solution of the Weierstrass elliptic function equation

$$\left[\frac{\mathrm{d}\wp}{\mathrm{d}t}(t;0,g_3)\right]^2 = 4\wp^3(t;0,g_3) - g_3. \tag{29}$$

Reduction 3.2 κ arbitrary, $\alpha = \beta = \gamma = 0$. From (3.1ii) we get the following reduction

$$u(x,t) = w(z) \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0 t, \qquad y = x \pm \sqrt{\kappa}\,t, \quad z = x \mp \sqrt{\kappa}\,t,$$

where $w(z)$ satisfies

$$\sqrt{\kappa}\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} \pm 3c_3 = 0.$$

This gives us the exact solution

$$u(x,t) = \mp\frac{c_3}{8\sqrt{\kappa}}z^4 + c_4 z^3 + c_5 z^2 + c_6 z + c_7 \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0 t.$$

Reduction 3.3 $\alpha = \beta = \gamma = \kappa = 0$. In (3.1iii) we integrate our equation for $g(t)$ up to an expression with quadratures

$$g\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} - 2\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + 24c_4 g\int^t g^2(s)\,\mathrm{d}s + 24c_5 g = 0. \tag{30}$$

We get the following reduction

$$u(x,t) = \frac{1}{g(t)}\left[w(x) + \left(c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0\right)\int^t g^2(s)\,\mathrm{d}s\right],$$

where $w(x)$ satisfies

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - 24c_5 = 0.$$

This is easily solved to give the solution

$$u(x,t) = \frac{1}{g(t)}\Bigg[c_5x^4 + c_6x^3 + c_7x^2 + c_8x + c_9 + \left(c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0\right)\int^t g^2(s)\,\mathrm{d}s\Bigg],$$

where $g(t)$ satisfies (30).

3.2 Case (ii) $\mu \neq 0$.

As discussed earlier in this section, we must consider, in addition to the general case of the determining equations, each of the singular cases of the determining equations.

3.2.1 $\xi^2 + u \neq 0$.

In this the generic case we generate 12 determining equations – see appendix A for details. As expected we have all the classical reductions, however we also have the following infinitesimals that lead to genuine nonclassical reductions (i.e. not a classical reduction).

Table 3.2

Parameters	ξ	ϕ	
$\kappa = 0$	0	$g(t)u$ where $\dfrac{\mathrm{d}^2g}{\mathrm{d}t^2} + g\dfrac{\mathrm{d}g}{\mathrm{d}t} - g^3 = 0$	(3.2i)
$1 + 2\gamma = 0$	$c_1t + c_2$	$-2c_1(c_1t + c_2)$	(3.2ii)
$\kappa = 1 + 2\gamma = 0$	$c_2(t + c_1)^2$	$u(t + c_1)^{-1} - 3c_2^2(t + c_1)^3$	(3.2iii)
$\alpha = \beta = \gamma = 0$	$\pm\sqrt{\kappa}$	$c_3y^3 + c_2y^2 + c_1y + c_0$ $(y = x \pm \sqrt{\kappa}t)$	(3.2iv)
$\alpha = -\frac{3}{2}, \beta = 2, \gamma = 0$	$\pm\frac{1}{2}\sqrt{\kappa}\,(x + c_1)$	$\pm 2\sqrt{\kappa}\,u \pm \frac{1}{4}\kappa^{3/2}(x + c_1)^2$	(3.2v)
$\alpha = \beta = \gamma = \kappa = 0$	0	$c_3x^3 + c_2x^2 + c_1x + c_0$	(3.2vi)
$\alpha = \beta = \gamma = \kappa = 0$	0	$(u + c_3x^3 + c_2x^2 + c_1x + c_0)(t + c_4)^{-1}$	(3.2vii)

From these infinitesimals we obtain six reductions.

Reduction 3.4 α, β, γ arbitrary, $\kappa = 0$. In (3.2i) we solve the equation for $g(t)$ by writing $g(t) = [\ln(\psi(t))]_t$ then $\psi(t)$ satisfies

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = 4c_1\psi^3 + c_2 \tag{31}$$

though $c_1 = c_2 = 0$ is not allowed to preserve the fact that $g(t) \not\equiv 0$. We obtain the following reduction

$$u(x,t) = w(x)\psi(t),$$

where $w(x)$ satisfies

$$w\frac{\mathrm{d}^4w}{\mathrm{d}x^4}+\alpha\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}^3w}{\mathrm{d}x^3}+\beta\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2+2\gamma\left[w\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]+6c_1\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-w\right)=0.$$

There are three cases to consider in the solution of (31).

(i) If $c_1=0$, we may assume that $\psi(t)=t$ without loss of generality.

(ii) If $c_2=0$, then $\psi=\left[c_2(t+c_3)^2\right]^{-1}$ and we may set $c_2=1$ and $c_3=0$ without loss of generality.

(iii) If $c_1c_2\neq 0$ we may set $c_1=1$ and $c_2=-g_3$ without loss of generality so that $\psi(t)$ is any solution of the Weierstrass elliptic function equation (29).

Note that in the special case

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-w=0,$$

we are able to lift the restrictions on $\psi(t)$ so that it is arbitrary, if $\beta+1+2\gamma=\alpha+2\gamma=0$. This yields the exact solution

$$u(x,t)=\psi(t)\left(c_2\mathrm{e}^x+c_3\mathrm{e}^{-x}\right),$$

where $\psi(t)$ is arbitrary, $\kappa=0$, $\alpha=-2\gamma$ and $\beta=-1-2\gamma$.

Reduction 3.5 α, β and κ are arbitrary and $\gamma=-\frac{1}{2}$. In (3.2ii) we assume $c_1\neq 0$ otherwise we get a classical reduction, and may set $c_2=0$ without loss of generality. Thus we obtain the following accelerating wave reduction

$$u(x,t)=w(z)-c_1^2t^2,\qquad z=x-\tfrac{1}{2}c_1t^2,$$

where $w(z)$ satisfies

$$w\frac{\mathrm{d}^4w}{\mathrm{d}z^4}+\alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3w}{\mathrm{d}z^3}+\beta\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}\right)^2$$
$$-c_1\frac{\mathrm{d}^3w}{\mathrm{d}z^3}-w\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\kappa\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2+c_1\frac{\mathrm{d}w}{\mathrm{d}z}+2c_1^2=0.$$

Reduction 3.6 α and β are arbitrary, $\gamma=-\frac{1}{2}$ and $\kappa=0$. From (3.2iii) the following holds for arbitrary c_2, and we may set $c_1=0$ without loss of generality. Thus we obtain the reduction

$$u(x,t)=w(z)t-c_2^2t^4,\qquad z=x-\tfrac{1}{3}c_2t^3,$$

where $w(z)$ satisfies

$$w\frac{\mathrm{d}^4w}{\mathrm{d}z^4}+\alpha\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3w}{\mathrm{d}z^3}+\beta\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}\right)^2-4c_2\frac{\mathrm{d}^3w}{\mathrm{d}z^3}-w\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2+4c_2\frac{\mathrm{d}w}{\mathrm{d}z}+12c_2^2=0.$$

Reduction 3.7 κ is arbitrary and $\alpha = \beta = \gamma = 0$. From (3.2iv) we get the following reduction

$$u(x,t) = w(z) \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0 t, \qquad y = x \pm \sqrt{\kappa}\,t, \quad z = x \mp \sqrt{\kappa}\,t,$$

where $w(z)$ satisfies

$$\sqrt{\kappa}\,\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} \pm 3c_3 = 0.$$

This gives us the exact solution

$$u(x,t) = \mp\frac{c_3}{8\sqrt{\kappa}}z^4 + c_4 z^3 + c_5 z^2 + c_6 z + c_7 \pm \frac{c_3}{8\sqrt{\kappa}}y^4 \pm \frac{c_2}{6\sqrt{\kappa}}y^3 \pm \frac{c_1}{4\sqrt{\kappa}}y^2 + c_0 t.$$

Reduction 3.8 κ is arbitrary, $\alpha = -\frac{3}{2}$, $\beta = 2$ and $\gamma = 0$. In (3.2v) we may set $c_1 = 0$ without loss of generality. Thus we obtain the following reduction

$$u(x,t) = w(z)x^4 - \tfrac{1}{4}\kappa x^2, \qquad z = \log(x) \mp \tfrac{1}{2}\sqrt{\kappa}\,t,$$

where $w(z)$ satisfies

$$4w\frac{\mathrm{d}^4 w}{\mathrm{d}z^4} - 6\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + 8\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right)^2 + 16w\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + 58\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}$$
$$+116w\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - \kappa\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 236\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + 776w\frac{\mathrm{d}w}{\mathrm{d}z} + 672w^2 = 0.$$

Reduction 3.9 $\alpha = \beta = \gamma = \kappa = 0$. From (3.2vi) and from (3.2vii) ($c_4 = 0$ without loss of generality) we get the following reductions

$$u(x,t) = w(x) + \left(c_3 x^3 + c_2 x^2 + c_1 x + c_0\right) t$$

and

$$u(x,t) = w(x)t - \left(c_3 x^3 + c_2 x^2 + c_1 x + c_0\right)$$

respectively. In both cases $w(x)$ satisfies

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0.$$

These reductions have a common exact solution, namely

$$u(x,t) = P_3(x)t + Q_3(x),$$

where P_3 and Q_3 are any third order polynomials in x.

3.2.2 $\xi^2 + u = 0$, not both $\alpha = 4$ and $2\xi\phi_u + \xi_u\phi = 0$.

The determining equations quickly lead us to require that both $\alpha = 4$ and $2\xi\phi_u + \xi_u\phi = 0$, which is a contradiction.

3.2.3 $\xi^2 + u = 0$, $\alpha = 4$, $2\xi\phi_u + \xi_u\phi = 0$ and $\beta \neq 3$.

The determining equations give us that $\gamma = -\frac{1}{2}$, $\kappa = 0$ and $\phi = 0$. The invariant surface condition is then

$$\pm\mathrm{i}\sqrt{u}\, u_x + u_t = 0$$

which may be solved implicitly to yield the solution

$$u(x,t) = w(z), \qquad z = x \mp \mathrm{i}\sqrt{u}\, t.$$

However, substituting into our original equation gives $\dfrac{\mathrm{d}w}{\mathrm{d}z} = 0$, i.e. $u(x,t)$ is a constant.

3.2.4 $\xi^2 + u = 0$, $\phi = H(x,t)u^{-1/4}$, $\alpha = 4$ and $\beta = 3$, not all of $H = 0$, $\kappa = 0$, $1 + 2\gamma = 0$.

For the determining equations to be satisfied, each of $H = 0$, $\kappa = 0$ and $1+2\gamma = 0$, which is in contradiction to our assumption.

3.2.5 $\xi^2 + u = 0$, $\phi = 0$, $\alpha = 4$, $\beta = 3$, $\gamma = -\frac{1}{2}$ and $\kappa = 0$.

Under these conditions equation (27) which we apply the classical method to is identically zero. Therefore any solution of the invariant surface condition is also a solution of (1). Hence we get the following reduction

Reduction 3.10 $\alpha = 4$, $\beta = 3$, $\gamma = -\frac{1}{2}$ and $\kappa = 0$. The invariant surface condition is

$$\pm\mathrm{i}\sqrt{u}\, u_x + u_t = 0$$

which may be solved implicitly to yield

$$u(x,t) = w(z), \qquad z = x \mp \mathrm{i}\sqrt{u}\, t,$$

where $w(z)$ is arbitrary.

4 Nonclassical symmetries ($\tau = 0$)

In the canonical case of the nonclassical method when $\tau = 0$ we set $\xi = 1$ without loss of generality. We proceed by eliminating u_x, u_{xx}, u_{xxx}, u_{xxxx} and u_{xxtt} in (1) using the invariant surface condition (16) which yields

$$\begin{aligned}
&u_{tt} - \kappa(\phi_x + \phi\phi_u) - 2\gamma(u\phi_x + u\phi\phi_u + \phi^2) - u(\phi_{xxx} + \phi_u\phi_{xx} + \phi_u^2\phi_x + \phi\phi_u^3 \\
&\quad +4\phi_u\phi^2\phi_{uu} + 5\phi_u\phi\phi_{xu} + 3\phi\phi_{uu}\phi_x + \phi^3\phi_{uuu} + 3\phi^2\phi_{xuu} + 3\phi\phi_{xxu} + 3\phi_{xu}\phi_x) \\
&\quad -\mu(\phi\phi_{uu}u_{tt} + \phi\phi_{uuu}u_t^2 + 2\phi\phi_{tuu}u_t + \phi\phi_{ttu} + \phi_u^2 u_{tt} + 3\phi_u\phi_{uu}u_t^2 + 4\phi_u\phi_{tu}u_t \\
&\quad +\phi_u\phi_{tt} + \phi_{xu}u_{tt} + \phi_{xuu}u_t^2 + 2\phi_{xtu}u_t + 2\phi_t\phi_{uu}u_t + 2\phi_t\phi_{tu} + \phi_{xtt}) \\
&\quad -\alpha\phi(\phi_{xx} + \phi_u\phi_x + \phi\phi_u^2 + \phi^2\phi_{uu} + 2\phi\phi_{xu}) - \beta(\phi_x + \phi\phi_u)^2 = 0
\end{aligned} \tag{32}$$

which involves the infinitesimal ϕ that is to be determined. As in the $\tau \neq 0$ case we apply the classical Lie algorithm to this equation using the second prolongation $\mathrm{pr}^{(2)}\mathbf{v}$ and eliminate u_{tt} using (32). Similar to the nonclassical method in the generic case $\tau \neq 0$, when $\mu \neq 0$ the coefficient of the highest derivative term, u_{tt} is not necessarily zero, thus singular cases are induced. As in the previous section we consider the cases (i) $\mu = 0$ and (ii) $\mu \neq 0$ separately.

4.1 Case (i) $\mu = 0$.

Generating the determining equations, again using `symmgrp.max`, yields three equations, the first two being $\phi_{uu} = 0$, $\phi_{tu} = 0$. Hence we look for solutions like $\phi = A(x)u + B(x,t)$ in the third. Taking coefficients of powers of u to be zero yields a system of three equations in A and B.

$$\begin{aligned}&\alpha A A_{xxx} + 2\beta A^2 A_{xx} + A_{xxxx} + \alpha A_x A_{xx} + 5\beta A A_x^2 + 6\alpha A A_x^2 + 10 A_{xx} A_x \\ &\quad +2\gamma A_{xx} + 5 A A_{xxx} + \alpha A^5 + 10 A^2 A_{xx} + 10 A^3 A_x + A^5 + \beta A^5 + 15 A A_x^2 \\ &\quad +4\gamma A^3 + 2\beta A_x A_{xx} + 10\gamma A A_x + 4\alpha A^2 A_{xx} + 6\beta A^3 A_x + 7\alpha A^3 A_x = 0,\end{aligned} \tag{33}$$

$$\begin{aligned}&5\alpha B A_x^2 + 2\beta A^2 B_{xx} + \alpha B A_{xxx} + 2\beta B A^4 + 13 B A A_{xx} + 2\alpha A^3 B_x + 10\gamma B A_x \\ &\quad +7 A A_x B_x + 2\beta A_x B_{xx} + \alpha A_x B_{xx} + 2\kappa A A_x + \alpha A^2 B_{xx} + 15 B A^2 A_x \\ &\quad +\alpha B_x A_{xx} + 6\gamma A B_x + 8\gamma B A^2 + 2\beta A^3 B_x + 2\beta B_x A_{xx} + 2\alpha B A^4 + \alpha A B_{xxx} \\ &\quad +B_{xxxx} + 6 A_{xx} B_x + 2\gamma B_{xx} + 5 B A_{xxx} + A B_{xxx} + \kappa A_{xx} + 2 B A^4 + A^2 B_{xx} \\ &\quad +A^3 B_x + 4 A_x B_{xx} + 11 B A_x^2 + 4\beta B A_x^2 + 6\beta A B_x A_x + 7\alpha A B_x A_x + 7\alpha B A A_{xx} \\ &\quad +2\beta B A A_{xx} + 10\beta B A^2 A_x + 12\alpha B A^2 A_x = 0,\end{aligned} \tag{34}$$

$$\begin{aligned}&\beta A B_x^2 + 4\gamma B^2 A + 2\beta B_x B_{xx} + 5 B^2 A A_x + \alpha A B_x^2 + 6\gamma B B_x + 2\kappa B A_x \\ &\quad +B A B_{xx} + \alpha B^2 A^3 + \beta B^2 A^3 + \alpha B_x B_{xx} + 3\alpha B^2 A_{xx} + \alpha B B_{xxx} + B A^2 B_x \\ &\quad +3 B A_x B_x - B_{tt} + B B_{xxx} + \kappa B_{xx} + B^2 A^3 + 3 B^2 A_{xx} + 2\beta B A B_{xx} \\ &\quad +\alpha B A B_{xx} + 4\beta B A_x B_x + 5\alpha B A_x B_x + 2\beta B A^2 B_x + 2\alpha B A^2 B_x \\ &\quad +4\beta B^2 A A_x + 5\alpha B^2 A A_x = 0,\end{aligned} \tag{35}$$

We try to solve this system using the `diffgrob2` package interactively, however the expression swell is too great to obtain meaningful output. Thus we proceed by making ansätze on the form of $A(x)$, solve (35) (a linear equation in $B(x,t)$) then finally (34) gives the full picture. Many solutions have been found as (33) lends itself to many ansätze through choices of parameter values. We present some in § 4.3.

4.2 Case (ii) $\mu \neq 0$.

The nonclassical method, when the coefficient of u_{tt} is non-zero, generates a system of three determining equations. However, far from being single-term equations the first two contain 41 and 57 terms respectively, and the third 329. The intractability of finding all solutions is obvious. To find some, we return to our previous case and look for $\phi = A(x)u + B(x,t)$. Three equations then remain, similar to (33,34,35) which we tackle in the same vein as previously. Some solutions are presented in § 4.3.

As mentioned in the start of this section, singular solutions may exist, when the coefficient of u_{tt} equals zero, i.e. when

$$1 - \phi\phi_{uu} - \phi_u^2 - \phi_{ux} = 0.$$

This may be integrated with respect to u to give

$$u - \phi\phi_u - \phi_x = H(x,t).$$

If ϕ satisfies (4.2) then the coefficients of u_t^2 and u_t in (32) are both zero. Since no u-derivatives now exist in (32) what is left must also be zero, i.e.

$$(2\gamma+\alpha)\phi^2 - \alpha H_x\phi + (2\gamma+\beta+1)u^2 \\ +[\kappa-(2\gamma+2\beta+1)H - H_{xx}]\,u + \beta H^2 - \kappa H - H_{tt} = 0.$$

Thus we need to solve (4.2) and (4.2). Note that once we have found $\phi(x,t,u)$, the related exact solution is given by solving the invariant surface condition, with no further restrictions on the solution. The following are distinct from each other and from solutions in § 4.3.

Case (a) $\gamma = -\frac{1}{2}$, $\alpha = 1$ and $\beta = \kappa = 0$. In this case $\phi(x,t,u)$ is given by the relation

$$u - \phi\phi_u - \phi_x = c_1 t + c_2.$$

For instance, if $\phi(x,t,u)$ is linear in u we have the exact solution

$$u(x,t) = w(t)\cosh[x + A(t)] + B(t)\sinh[x + A(t)] + c_1 t + c_2,$$

where $w(t)$, $A(t)$ and $B(t)$ are arbitrary functions.

Case (b) $\alpha = -2\gamma$, $\beta = -1-2\gamma$ and $\gamma \neq -\frac{1}{2}$. In this case $\phi(x,t,u)$ is given by the relation

$$u - \phi\phi_u - \phi_x = -\frac{\kappa}{1+2\gamma}.$$

For instance, if $\phi(x,t,u)$ is linear in u we have the exact solution

$$u(x,t) = w(t)\cosh[x + A(t)] + B(t)\sinh[x + A(t)] - \frac{\kappa}{1+2\gamma},$$

where $w(t)$, $A(t)$ and $B(t)$ are arbitrary functions.

Case (c) $\alpha = -2\gamma$ and $\beta = -1-2\gamma$. In this case

$$\phi(x,t,u) = \pm u - \kappa x \mp \left(-\tfrac{1}{2}\kappa t^2 + c_1 t + c_2\right) \pm \kappa,$$

and so

$$u(x,t) = w(t)\exp(\pm x) \pm \kappa x + \left(-\tfrac{1}{2}\kappa t^2 + c_1 t + c_2\right),$$

where $w(t)$ is an arbitrary function.

Case (d) $\gamma = -\frac{1}{2}$, $\alpha = 1$ and $\beta = 0$. In this case

$$\phi(x,t,u) = \frac{\kappa u - H_{xx}u - \kappa H - H_{tt}}{H_x},$$

where $H_x(x,t) \neq 0$ and also $H(x,t)$ satisfies the system

$$\kappa H_{xx} + H_x^2 - \kappa^2 = 0 \qquad \left(\kappa^2 - H_x^2\right)H_{tt} - 2\kappa H_{xt}^2 + \kappa^2\left(\kappa^2 - H_x^2\right) = 0.$$

We have assumed that $\kappa^2 - H_x^2 \neq 0$, for a different solution to (c). This yields

$$u(x,t) = [w(t) - 2\kappa x]\sinh z\cosh z \\ + \cosh^2 z\left[4\kappa\log(\cosh z) - \kappa^2 t^2 + 2c_3 t + 2c_4 - 2\kappa + 2c_1^2\right] - c_1^2,$$

where $z = \frac{1}{2}(x + c_1 t + c_2)$ and $w(t)$ is an arbitrary function.

Case (e) $\beta = -1 - \alpha - 4\gamma$, $\gamma \neq \frac{1}{2}$ and $\alpha + 2\gamma \neq 0$. In this case

$$\phi(x,t,u) = \pm\left(u + \frac{\kappa}{1+2\gamma}\right),$$

and so

$$u(x,t) = w(t)\exp(\pm x) - \frac{\kappa}{1+2\gamma},$$

where $w(t)$ is an arbitrary function.

Case (f) $\gamma = 0$, $\alpha = -2$ and $\beta = 1$. In this case $\phi(x,t,u)$ satisfies

$$-2\phi^2 + 2\phi H_x + 2u^2 + 2\kappa u - 2Hu + \kappa^2 + H^2 = 0$$

and $H(x,t)$ satisfies the system

$$H_{xx} + \kappa + H = 0, \qquad H_{tt} + \kappa^2 + \kappa H = 0.$$

Then

$$u(x,t) = \tfrac{1}{2}(A^2+B^2)^{1/2}\sinh[\pm x + w(t)] - \kappa + \tfrac{1}{2}(A\sin x + B\cos x),$$

where $A(t)$ and $B(t)$ satisfy

$$\frac{\mathrm{d}^2A}{\mathrm{d}t^2} + \kappa A = 0, \qquad \frac{\mathrm{d}^2B}{\mathrm{d}t^2} + \kappa B = 0$$

and $w(t)$ is an arbitrary function.

Case (g) $\gamma = 0$, $\beta = -1 - \alpha$, $\kappa = 0$ and $\alpha = (c_1-1)^2/c_1$, where $c_1 \neq 0, 1$. In this case

$$\phi(x,t,u) = u + \frac{c_2t + c_3}{c_1 - 1}\exp(-c_1x),$$

and so

$$u(x,t) = \begin{cases} w_1(t)\mathrm{e}^x - \frac{1}{2}(c_2t+c_3)x\mathrm{e}^x, & \text{if} \quad c_1 = -1, \\ w_2(t)\mathrm{e}^x + \left(1 - c_1^2\right)^{-1}(c_2t + c_3)\mathrm{e}^{-c_1x}, & \text{if} \quad c_1 \neq -1, \end{cases}$$

where $w_1(t)$ and $w_2(t)$ are arbitrary functions.

Case (h) $\gamma = 0$, $\beta = -1 - \alpha$, $c_1^2 + \alpha^2c_1 + 2c_1 + 4\alpha c_1 + 1 = 0$, $\alpha \neq 0, -2$ and $c_1 \neq 0, 1$. In this case $\phi(x,t,u)$ satisfies

$$\alpha\phi^2 - \alpha H_x\phi - \alpha u^2 + u\left[\kappa + H(1+2\alpha) - H_{xx}\right] - \kappa H - (\alpha+1)H^2 - H_{tt} = 0,$$

where $H(x,t)$ satisfies the system

$$H_{tt} + c_1\kappa H = 0 \qquad (\alpha+2)H_x \pm (1-c_1)(H+\kappa) = 0.$$

Then

$$u(x,t) = \left[\frac{1+2\alpha+c_1}{2\alpha}w(t) + g(x,t)\right]\exp\left\{\frac{(c_1-1)x}{\alpha+2}\right\} - \kappa,$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + c_1 \kappa w = 0,$$

and $g(x,t)$ satisfies

$$\alpha g_x^2 + \frac{2\alpha(c_1-1)}{\alpha+2} g g_x - \alpha(c_1+1)g^2 + (\alpha c_1 + c_1 + 1) w(t) g_x$$
$$-c_1(1+\alpha+c_1) w(t) g + \frac{c_1(c_1-1)(\alpha+2)}{4\alpha} w^2(t) = 0.$$

4.3 Exact solutions

In this subsection some exact solutions are presented. The infinitesimal $\phi(x,t,u)$ is given, possibly up to satisfying some equations, and then the solution, found by solving the invariant surface condition (16).

4.3.1 $\gamma = 0$ and $\phi = \dfrac{u}{x} + H_1(t)x + 3H_2(t)x^3 + \dfrac{H_3(t)}{x} + H_4(t)x^{2-\alpha}$.

Solving the invariant surface condition gives

$$u(x,t) = \begin{cases} x w(t) + H_1(t)x^2 + H_2(t)x^4 - H_3(t) + \dfrac{H_4(t)x^{3-\alpha}}{2-\alpha}, & \text{if} \quad \alpha \neq 2, \\ x\widetilde{w}(t) + H_1(t)x^2 + H_2(t)x^4 - H_3(t) + H_4(t)x\log x, & \text{if} \quad \alpha = 2. \end{cases}$$

Various types of solution are found, as seen in Table 4.1. The $H_i(t)$ are obtained from by the determining equations, $w(t)$ by substituting back into (1).

4.3.2 $\phi = B(x,t)$.

Case (a) $\gamma = 0$. In this case $B(x,t) = 4H_1(t)x^3 + 3H_2(t)x^2 + 2H_3(t)x + H_4(t)$, where $H_1(t)$, $H_2(t)$, $H_3(t)$ and $H_4(t)$ satisfy

$$\frac{\mathrm{d}^2 H_1}{\mathrm{d}t^2} - 24(6\beta + 4\alpha + 1)H_1^2 = 0,$$

$$\frac{\mathrm{d}^2 H_2}{\mathrm{d}t^2} - 24(6\beta + 4\alpha + 1)H_1 H_2 = 0,$$

$$\frac{\mathrm{d}^2 H_3}{\mathrm{d}t^2} - 24(2\beta + 2\alpha + 1)H_1 H_3 = 18(2\beta + \alpha)H_2^2 + 12\kappa H_1 + 12\mu \frac{\mathrm{d}^2 H_1}{\mathrm{d}t^2},$$

$$\frac{\mathrm{d}^2 H_4}{\mathrm{d}t^2} - 24(\alpha + 1)H_1 H_4 = 12(2\beta + \alpha)H_2 H_3 + 6\kappa H_2 + 6\mu \frac{\mathrm{d}^2 H_2}{\mathrm{d}t^2}.$$

Then

$$u(x,t) = w(t) + H_1(t)x^4 + H_2(t)x^3 + H_3(t)x^2 + H_4(t)x,$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 24 H_1 w = 2\kappa H_3 + 6\alpha H_2 H_4 + 4\beta H_3^2 + 2\mu \frac{\mathrm{d}^2 H_3}{\mathrm{d}t^2}.$$

Table 4.1

Parameters	$H_i(t)$ and $w(t)$ satisfy
$\alpha = \frac{1}{2},\ \beta = -\frac{1}{8}$	$H_1 = 4\kappa, \qquad H_2 = H_3 = \dfrac{\mathrm{d}^2 H_4}{\mathrm{d}t^2} = 0$ $32\dfrac{\mathrm{d}^2 w}{\mathrm{d}t^2} = -5H_4^2$
$\alpha = \frac{1}{2},\ \kappa = 0$	$H_1 = \mu H_2 = H_3 = 0$ $\dfrac{\mathrm{d}^2 H_2}{\mathrm{d}t^2} - 72(1+2\beta)H_2^2 = 0$ $16\dfrac{\mathrm{d}^2 H_4}{\mathrm{d}t^2} - 3(480\beta + 303)H_2 H_4 = 0$ $8\dfrac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 288 H_2 w = (50\beta + 5)H_4^2$
$\beta = \dfrac{\alpha^2 - \alpha}{3 - \alpha}$ $\kappa = 0, \quad \alpha \neq 3$	$H_1 = \mu H_2 = H_3 = 0$ $(\alpha - 3)\dfrac{\mathrm{d}^2 H_2}{\mathrm{d}t^2} + 24(2\alpha+3)(\alpha+1)H_2^2 = 0$ $\dfrac{\mathrm{d}^2 H_4}{\mathrm{d}t^2} + 3(\alpha+2)(\alpha+1)(\alpha^2 - \alpha - 4)H_2 H_4 = 0$ $\dfrac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 24(\alpha+1)H_2 w = 0$ $\dfrac{\mathrm{d}^2 \widetilde{w}}{\mathrm{d}t^2} - 72 H_2 \widetilde{w} = 90 H_2 H_4 (\text{if } \alpha = 2)$
$\alpha = -2, \quad \beta = \frac{6}{5}$	$H_1 = \frac{5}{6}\kappa, \qquad \mu H_2 = \dfrac{\mathrm{d}^2 H_4}{\mathrm{d}t^2} = 0$ $5\dfrac{\mathrm{d}^2 H_2}{\mathrm{d}t^2} - 24 H_2^2 = 0$ $5\dfrac{\mathrm{d}^2 H_3}{\mathrm{d}t^2} - 120 H_2 H_3 = -25\kappa^2$ $\dfrac{\mathrm{d}^2 w}{\mathrm{d}t^2} + 24 H_2 w = -30 H_3 H_4$

Case (b). In this case $B(x,t) = H_1(t) + 2H_2(t)x$ where $H_1(t)$ and $H_2(t)$ satisfy

$$\frac{\mathrm{d}^2 H_2}{\mathrm{d}t^2} - 12\gamma H_2^2 = 0, \tag{36}$$

$$\frac{\mathrm{d}^2 H_1}{\mathrm{d}t^2} - 12\gamma H_2 H_1 = 0. \tag{37}$$

Then

$$u(x,t) = w(t) + H_1(t)x + H_2(t)x^2,$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 4\gamma H_2 w = 2\kappa H_2 + 4(6\gamma\mu + \beta)H_2^2 + 2\gamma H_1^2. \tag{38}$$

Case (c) $\beta = 1-\alpha$. In this case $B(x,t) = cH_1(t)\mathrm{e}^{cx} + cH_2(t)\mathrm{e}^{-cx} + H_4(t)$, with $c^2 = -2\gamma$ and where $H_1(t)$, $H_2(t)$, $H_3(t)$ and $H_4(t)$ satisfy

$$\frac{\mathrm{d}^2 H_4}{\mathrm{d}t^2} = 0, \tag{39}$$

$$(1+2\gamma\mu)\frac{\mathrm{d}^2 H_1}{\mathrm{d}t^2} - 2\gamma c(2-\alpha)H_4 H_1 + 2\gamma\kappa H_1 = 0, \tag{40}$$

$$(1+2\gamma\mu)\frac{\mathrm{d}^2 H_2}{\mathrm{d}t^2} + 2\gamma c(2-\alpha)H_4 H_2 + 2\gamma\kappa H_2 = 0. \tag{41}$$

Then

$$u(x,t) = w(t) + H_1(t)\mathrm{e}^{cx} - H_2(t)\mathrm{e}^{-cx} + H_4(t)x,$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = 2\gamma H_4^2 - 16\gamma^2(1-\alpha)H_1 H_2. \tag{42}$$

Case (d) $\alpha = 2$ and $\beta = -1$. In this case $B(x,t) = cH_1(t)\mathrm{e}^{cx} + cH_2(t)\mathrm{e}^{-cx} + 2H_3(t)x + H_4(t)$, with $c^2 = -2\gamma$ and where $H_1(t)$, $H_2(t)$, $H_3(t)$ and $H_4(t)$ satisfy

$$\frac{\mathrm{d}^2 H_3}{\mathrm{d}t^2} - 12\gamma H_3^2 = 0, \tag{43}$$

$$\frac{\mathrm{d}^2 H_4}{\mathrm{d}t^2} - 12\gamma H_3 H_4 = 0,$$

$$(1+2\gamma\mu)\frac{\mathrm{d}^2 H_1}{\mathrm{d}t^2} - 12\gamma H_3 H_1 + 2\kappa\gamma H_1 = 0,$$

$$(1+2\gamma\mu)\frac{\mathrm{d}^2 H_2}{\mathrm{d}t^2} - 12\gamma H_3 H_2 + 2\kappa\gamma H_2 = 0. \tag{44}$$

Then

$$u(x,t) = w(t) + H_1(t)\mathrm{e}^{cx} - H_2(t)\mathrm{e}^{-cx} + H_3(t)x^2 + H_4(t)x,$$

where $w(t)$ satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 4\gamma H_3 w = 2\kappa H_3 + 2\gamma H_4^2 + 4(6\gamma\mu - 1)H_3^2 + 16\gamma^2 H_1 H_2. \tag{45}$$

4.3.3 $\gamma = \beta + \alpha + 1 = 0$ and $\phi = R(u + H_1(t) + H_2(t)e^{Rx} + H_3(t)e^{m_+x} + H_4(t)e^{m_-x})$.

Here $m_\pm = -\frac{1}{2}R(2 + \alpha \pm n)$, $n = \sqrt{\alpha(\alpha + 4)}$, with $R \neq \pm 1$ a non-zero constant. Solving the invariant surface condition yields

$$u(x,t) = w(t)e^{Rx} - H_1(t) + RH_2(t)e^{Rx}$$
$$- \frac{2H_3(t)}{4 + \alpha + n} \exp\left\{-\tfrac{1}{2}Rx(2 + \alpha + n)\right\} - \frac{2H_4(t)}{4 + \alpha - n} \exp\left\{-\tfrac{1}{2}Rx(2 + \alpha - n)\right\}.$$

The solutions are represented in Table 4.2

The equations that the various $H_i(t)$ satisfy in this subsection are all solvable, and the order in which a list of equations should be solved is from the top down. The only nonlinear equations all have either polynomial solutions (sometimes only in special cases of the parameters) or are equivalent to the Weierstrass elliptic function equation (29). The homogeneous part of any linear equation is either of Euler-type, is equivalent to the Airy equation [4],

$$\frac{d^2H}{dt^2}(t) + tH(t) = 0$$

or is equivalent to the Lamé equation [43]

$$\frac{d^2H}{dt^2}(t) - \{k + n(n+1)\wp(t)\}H(t) = 0. \tag{46}$$

The particular integral of any non-homogeneous linear equation may always be found, up to quadratures, using the method of variation of parameters.

For instance consider the solution of *4.3.2* case (b) above. There are essentially two separate cases to consider, either (i) $\gamma = 0$ or (ii) $\gamma \neq 0$.

Case (i) $\gamma = 0$. The functions $H_1(t)$ and $H_2(t)$ are trivially found from (36) and (37) to be $H_1(t) = c_1t + c_2$ and $H_2(t) = c_3t + c_4$, then (38) becomes

$$\frac{d^2w}{dt^2} = 2\kappa(c_3t + c_4) + 4\beta(c_3t + c_4)^2$$

which may be integrated twice to yield the exact solution

$$u(x,t) = \begin{cases} \dfrac{\kappa}{3c_3^2}(c_3t + c_4)^3 + \dfrac{\beta}{3c_3^2}(c_3t + c_4)^4 & \text{if} \quad c_3 \neq 0, \\ \qquad + c_5t + c_6 + (c_1t + c_2)x + (c_3t + c_4)x^2, & \\ \left(\kappa c_4 + 2\beta c_4^2\right)t^2 + c_5t + c_6 + (c_1t + c_2)x + c_4x^2, & \text{if} \quad c_3 = 0. \end{cases}$$

Case (ii) $\gamma \neq 0$. Equation (36) may be transformed into the Weierstrass elliptic function equation (29), hence $H_2(t)$ has solution $H_2(t) = \wp(t + t_0; 0, g_3)/(2\gamma)$. Now $H_1(t)$ satisfies the Lamé equation

$$\frac{d^2H_1}{dt^2} - 6\wp(t + t_0; 0, g_3)H_1 = 0,$$

which has general solution

$$H_1(t) = c_1\wp(t + t_0; 0, g_3) + c_2\wp(t + t_0; 0, g_3)\int^{t+t_0} \frac{ds}{\wp^2(s; 0, g_3)},$$

where c_1 and c_2 are arbitrary constants. Now $w(t)$ satisfies the inhomogeneous Lamé equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 2\wp(t+t_0; 0, g_3) w = Q(t), \tag{47}$$

where $Q(t) = 2\kappa H_2(t) + 4(6\gamma\mu + \beta) H_2^2(t) + 2\gamma H_1^2(t)$, with $H_1(t)$ and $H_2(t)$ as above. The general solution of the homogeneous part of this Lamé equation is given by

$$w_{\mathrm{CF}}(t) = c_3 w_1(t+t_0) + c_4 w_2(t+t_0),$$

where c_3 and c_4 are arbitrary constants,

$$w_1(t) = \exp\{-t\zeta(a)\} \frac{\sigma(t+a)}{\sigma(t)}, \qquad w_2(t) = \exp\{t\zeta(a)\} \frac{\sigma(t-a)}{\sigma(t)}$$

in which $\zeta(z)$ and $\sigma(z)$ are the Weierstrass zeta and sigma functions defined by the differential equations

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = -\wp(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z} \log \sigma(z) = \zeta(z)$$

together with the conditions

$$\lim_{z\to 0} \left(\zeta(z) - \frac{1}{z} \right) = 0, \qquad \lim_{z\to 0} \left(\frac{\sigma(z)}{z} \right) = 1$$

respectively (cf. [70]), and a is any solution of the transcendental equation

$$\wp(a) = 0$$

i.e., a is a zero of the Weierstrass elliptic function (cf. [43], p.379). Hence the general solution of (47) is given by

$$\begin{aligned} w(t) &= c_3 w_1(t+t_0) + c_4 w_2(t+t_0) \\ &\quad + \frac{1}{W(a)} \int^{t+t_0} \left[w_1(s) w_2(t+t_0) - w_1(t+t_0) w_2(s) \right] Q(s)\, \mathrm{d}s, \end{aligned} \tag{48}$$

where $W(a)$ is the non-zero Wronskian

$$W(a) = w_1 w_2' - w_1' w_2 = -\sigma^2(a) \wp'(a) \tag{49}$$

and $Q(t)$ is defined above. We remark that in order to verify that (48,49) is a solution of (47) one uses the following addition theorems for Weierstrass elliptic, zeta and sigma functions

$$\zeta(s \pm t) = \zeta(s) \pm \zeta(t) + \frac{1}{2} \left[\frac{\wp'(s) \mp \wp'(t)}{\wp(s) - \wp(t)} \right],$$

$$\sigma(s+t)\sigma(s-t) = -\sigma^2(s)\sigma^2(t)[\wp(s) - \wp(t)]$$

(cf. [70], p.451).

Table 4.2

Parameters	$H_i(t)$ and $w(t)$ satisfy
$\alpha=-4$ $\beta=3$	$H_3=H_4=\dfrac{\mathrm{d}^2H_1}{\mathrm{d}t^2}=0$ $(1-\mu R^2)\dfrac{\mathrm{d}^2H_2}{\mathrm{d}t^2}+R^2(R^2H_1-\kappa)H_2=0$ $(1-\mu R^2)\dfrac{\mathrm{d}^2w}{\mathrm{d}t^2}+R^2(R^2H_1-\kappa)w=2\kappa R^2H_2-4R^4H_1H_2+2\mu R^2\dfrac{\mathrm{d}^2H_2}{\mathrm{d}t^2}$
α arbitrary $j=\frac{7}{2}\pm\frac{1}{2}$ $i=\frac{7}{2}\mp\frac{1}{2}$	$H_2=H_j=\dfrac{\mathrm{d}^2H_1}{\mathrm{d}t^2}=0$ $(4-\mu(2+\alpha\pm n)^2R^2)\dfrac{\mathrm{d}^2H_i}{\mathrm{d}t^2}+R^2H_i$ $\times[R^2H_1((\alpha^2+4\alpha+2)(2+\alpha\pm n)^2-4)-\kappa(2+\alpha\pm n)^2]=0$ $(1-\mu R^2)\dfrac{\mathrm{d}^2w}{\mathrm{d}t^2}+R^2(R^2H_1-\kappa)w=0$
$\alpha=-3$ $\beta=2$	$H_2=\dfrac{\mathrm{d}^2H_1}{\mathrm{d}t^2}=0$ $(2+\mu R^2(1+\mathrm{i}\sqrt{3}))\dfrac{\mathrm{d}^2H_3}{\mathrm{d}t^2}-H_1H_3R^4(1-\mathrm{i}\sqrt{3})+\kappa H_3R^2(1+\mathrm{i}\sqrt{3})=0$ $(2+\mu R^2(1-\mathrm{i}\sqrt{3}))\dfrac{\mathrm{d}^2H_4}{\mathrm{d}t^2}-H_1H_4R^4(1+\mathrm{i}\sqrt{3})+\kappa H_4R^2(1-\mathrm{i}\sqrt{3})=0$ $(1-\mu R^2)\dfrac{\mathrm{d}^2w}{\mathrm{d}t^2}+R^2(R^2H_1-\kappa)w=6H_3H_4R^4$
$\alpha=-1$ $\beta=0$	$H_2=\dfrac{\mathrm{d}^2H_1}{\mathrm{d}t^2}=0$ $(2+\mu R^2(1-\mathrm{i}\sqrt{3}))\dfrac{\mathrm{d}^2H_3}{\mathrm{d}t^2}-H_1H_3R^4(1+\mathrm{i}\sqrt{3})+\kappa H_3R^2(1-\mathrm{i}\sqrt{3})=0$ $(2+\mu R^2(1+\mathrm{i}\sqrt{3}))\dfrac{\mathrm{d}^2H_4}{\mathrm{d}t^2}-H_1H_4R^4(1-\mathrm{i}\sqrt{3})+\kappa H_4R^2(1+\mathrm{i}\sqrt{3})=0$ $(1-\mu R^2)\dfrac{\mathrm{d}^2w}{\mathrm{d}t^2}+R^2(R^2H_1-\kappa)w=0$

5 Discussion

This paper has seen a classification of symmetry reductions of the nonlinear fourth order partial differential equation (1) using the classical Lie method and the nonclassical method due to Bluman and Cole. The presence of arbitrary parameters in (1) has led to a large variety of reductions using both symmetry methods for various combinations of these parameters. The use of the MAPLE package `diffgrob2` was crucial in this classification procedure. In the classical case it identified the special values of the parameters for which additional symmetries might occur. In the generic nonclassical case the flexibility of `diffgrob2` allowed the fully nonlinear determining equations to be solved completely, whilst in the so-called $\tau = 0$ case it allowed the salvage of many reductions from a somewhat intractable calculation.

An interesting aspect of the results in this paper is that the class of reductions given by the nonclassical method, which are not obtainable using the classical Lie method, were much more plentiful and richer than the analogous results for the generalized Camassa-Holm equation (2) given in [24].

An interesting problem this paper throws open is whether (1) is integrable, or perhaps more realistically for which values of the parameters is (1) integrable. Effectively, in finding the symmetry reductions of (1), we have provided a first step in using the Painlevé ODE test for integrability due to Ablowitz, Ramani and Segur [2, 3]. However the presence of so many reductions makes this a lengthy task and so the PDE test due to Weiss, Tabor and Carnevale [69] is a more inviting prospect. It is likely though that extensions of this test, namely "weak Painlevé analysis" [58, 59] and "perturbative Painlevé analysis" [28] will be necessary (for instance see [40]). We shall not pursue this further here.

The FFCH equation (11) may be thought of as an integrable generalization of the Korteweg-de Vries equation (4). Analogous integrable generalizations of the modified Korteweg-de Vries equation

$$q_t = q_x + q_{xxx} + 3\gamma q^2 q_x,$$

the nonlinear Schrödinger equation

$$\mathrm{i}q_t + q_{xx} + |q|^2 q = 0$$

and Sine-Gordon equation

$$q_{xt} = \sin q$$

are given by

$$u_t + \nu u_{xxt} = u_x + q_{xxx} + \gamma[(u + \nu u_{xx})(u^2 + \nu u_x^2)]_x, \tag{50}$$

$$\mathrm{i}u_t + \mathrm{i}u_x + \mu u_{xt} + u_{xx} + \kappa u|u|^2 - \mathrm{i}\kappa\mu|u|^2 u_x = 0, \tag{51}$$

$$u_{xt} = \sin(u + \mu u_{xx}) = 0, \tag{52}$$

respectively, where μ and κ are arbitrary constants [32, 33, 56].

Recently Clarkson, Gordoa and Pickering [17] derived $2+1$-dimensional generalization of the FFCH equation (11) given by

$$\tfrac{1}{2}u_y u_{xxxx} + u_{xy}u_{xxx} - \alpha\left(\tfrac{1}{2}u_y u_{xx} + u_x u_{xy}\right) + u_{xxxt} - \alpha u_{xt} = 0, \tag{53}$$

where α is an arbitrary constant. The FFCH equation (11) and is obtained from (53) under the reduction $\partial_y = \partial_x$, with $v = u_x$. The 2 + 1-dimensional FFCH equation (11) has the non-isospectral Lax pair

$$\begin{aligned} &4\psi_{xx} = [\alpha - \lambda (u_{xxx} - \alpha u_x)]\, \psi, \\ &\psi_t = \lambda^{-1}\psi_y - \tfrac{1}{2}u_y\psi_x + \tfrac{1}{4}u_{xy}\psi, \end{aligned}$$

with λ satisfying $\lambda_y = \lambda\lambda_t$. Clarkson, Gordoa and Pickering [17] also derived a 2-component generalisation of the FFCH equation (11) in 2 + 1-dimensions given by

$$\begin{aligned} &u_{xxxt} - \alpha u_{xt} = -\tfrac{1}{2}u_y u_{xxxx} - u_{xy}u_{xxx} + \alpha\left(\tfrac{1}{2}u_y u_{xx} + u_x u_{xy}\right) - \kappa u_{xxxy} + v_y, \\ &v_t = -v u_{xy} - \tfrac{1}{2}v_x u_y, \end{aligned} \tag{54}$$

which has the Lax pair

$$\begin{aligned} &4(1+\kappa\lambda)\psi_{xx} = \left[\alpha - \lambda (u_{xxx} - \alpha u_x) - \lambda^2 v\right] \psi, \\ &\psi_t = \lambda^{-1}\psi_y - \tfrac{1}{2}u_y\psi_x + \tfrac{1}{4}u_{xy}\psi, \end{aligned}$$

where the spectral parameter λ satisfies $\lambda_y = \lambda\lambda_t$.

We believe that a study of symmetry reductions of (50,51,52,53,54) would be interesting, though we shall not pursue this further here.

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Appendix A

In this appendix we list the determining equations that are generated in § 3.2 in the generic case when $\xi^2 + u \neq 0$.

$$\begin{aligned}
&\xi_u = 0 \\
&\alpha\xi^2\phi_u - 2\alpha\xi\xi_t + 4\phi_{uu}u^2 - \alpha\phi + 4\xi^4\phi_{uu} + \alpha\phi_u u + 8\xi^2\phi_{uu}u - 2\alpha\xi^2\xi_x = 0 \\
&\beta\xi^2\phi_u + 3\phi_{uu}u^2 + 3\xi^4\phi_{uu} + 6\xi^2\phi_{uu}u - \beta\phi + \beta\phi_u u - 2\beta\xi^2\xi_x - 2\beta\xi\xi_t = 0 \\
&12\xi^2\phi_{uuu}u + 3\alpha\phi_{uu}u + 6\phi_{uuu}u^2 + 2\beta\phi_{uu}u + 2\beta\xi^2\phi_{uu} + 3\alpha\xi^2\phi_{uu} + 6\xi^4\phi_{uuu} = 0 \\
&7\xi^2\xi_{xx}u - 10\xi\xi_x^2 u - 2\xi^2\xi_t\phi_u - 4\xi\xi_{xt}u - \alpha\xi^2\phi_x - 2\xi\phi\phi_u - 4\xi_t\xi_x u - \alpha\phi_x u + 2\xi_t\phi_u u \\
&\quad +2\xi^3\phi\phi_{uu} + 5\xi\xi_x\phi + 2\xi\phi_{tu}u - 6\xi^2\phi_{xu}u + 4\xi^2\xi_t\xi_x - 2\xi\xi_t^2 + 6\xi_{xx}u^2 + 2\xi^3\phi_{tu} + \xi_{tt}u \\
&\quad +\xi^4\xi_{xx} + \xi^2\xi_{tt} + 4\xi\xi_x\phi_u u - 4\phi_{xu}u^2 + 2\xi\phi\phi_{uu}u - 2\xi^4\phi_{xu} - 4\xi^3\xi_{xt} - \xi_t\phi = 0 \\
&\alpha\xi^2\phi_{uuu} + \alpha\phi_{uuu}u + 2\xi^2\phi_{uuuu}u + \xi^4\phi_{uuuu} + \phi_{uuuu}u^2 = 0 \\
&3\alpha\xi_{xx}u - 4\beta\xi^2\phi_{xu} - 12\phi_{xuu}u^2 - 3\alpha\phi_{xu}u + 6\xi\phi_{tuu}u + 3\alpha\xi^2\xi_{xx} - 18\xi^2\phi_{xuu}u \\
&\quad +9\xi_t\phi_{uu}u - 4\beta\phi_{xu}u - 6\xi\phi\phi_{uu} + 2\beta\xi_{xx}u + 6\xi^3\phi\phi_{uuu} - 15\xi^3\xi_x\phi_{uu} - 3\xi^2\xi_t\phi_{uu} \\
&\quad -3\alpha\xi^2\phi_{xu} + 12\xi^3\phi_u\phi_{uu} - 3\xi\xi_x\phi_{uu}u + 6\xi\phi\phi_{uuu}u + 12\xi\phi_u\phi_{uu}u + 2\beta\xi^2\xi_{xx} \\
&\quad -6\xi^4\phi_{xuu} + 6\xi^3\phi_{tuu} = 0
\end{aligned}$$

$$2\xi^3\phi_{xu}\phi_{xx} - \xi^3\xi_{xx}\phi_{xx} + 2\xi\xi_t\phi\phi_{xxu} - 2\xi\xi_t\phi_t + \xi_{xx}\phi_{xt}u + \xi^3\phi_{xxu}\phi_x + 2\xi_{xxt}\phi_x u$$
$$+\phi^2\phi_{uu}u - 2\xi_t\phi_x u + 2\xi^2\phi\phi_{tu} - \xi_{xx}\phi\phi_x - \phi_t\phi_{xxu}u - 2\xi_x\phi_{xxt}u - 2\xi\xi_t\xi_{xx}\phi_x$$
$$+2\phi\phi_{tu}u - \kappa\phi_{xx}u - 2\phi_{xt}\phi_{xu}u + 4\xi_{xt}\phi_{xx}u - 2\phi\phi_{xxtu}u + 4\xi\xi_t\phi_{xu}\phi_x + 2\xi\xi_t\phi_{xxt}$$
$$-2\xi^2\phi_{xt}\phi_{xu} + 4\xi_x\phi_t u - 2\xi^2\phi\phi_{xxtu} - 4\xi^2\phi_{xtu}\phi_x + 4\xi^2\xi_{xt}\phi_{xx} - \xi^2\phi^2\phi_{xxuu}$$
$$+\phi\phi_u\phi_{xx} - \xi^2\kappa\phi_{xx} + \xi^2\xi_{xx}\phi_{xt} - \xi^2\phi_t\phi_{xxu} + \xi\phi\phi_x - 2\phi\phi_{xu}^2u - 2\xi^2\phi_{uu}\phi_x^2$$
$$-2\xi_x\phi\phi_{xx} + 2\xi_t\phi_{xxx}u + 2\xi^2\xi_x\phi_{xu}\phi_x + \xi^2\phi^2\phi_{uu} - \phi^2\phi_{xxuu}u + 2\phi\phi_{xu}\phi_x$$
$$-\xi^2\phi_{xxxx}u + 2\xi^2\xi_{xxt}\phi_x - 2\xi^3\xi_x\phi_x - 2\xi^2\phi\phi_{xu}^2 - 2\phi_{tu}\phi_{xx}u + 4\xi_x^2\phi_{xx}u - 2\gamma\phi_{xx}u^2$$
$$+2\xi^2\xi_x\phi_t - 2\xi^2\phi_{tu}\phi_{xx} - \phi\phi_u\phi_{xxu}u - \xi\phi\phi_{xxx} - 2\xi\xi_t\phi\phi_u - 2\phi_{uu}\phi_x^2u - 4\phi_{xtu}\phi_x u$$
$$-2\phi\phi_{uu}\phi_{xx}u - 4\phi\phi_{xuu}\phi_x u - \xi^2\phi_{xxtt} + \xi^2\phi_{tt} - \phi_{xxxx}u^2 + 2\xi\xi_x\phi_{xxx}u + \xi_{xx}\phi\phi_{xu}u$$
$$+\xi^2\xi_{xx}\phi_u\phi_x + \xi_x\xi_{xx}\phi_x u - \xi\xi_{xx}\phi_{xx}u + \xi\phi_{xxu}\phi_x u + \xi_{xx}\phi_u\phi_x u - 2\xi_x\phi\phi_{xxu}u$$
$$-2\xi^2\phi\phi_{uu}\phi_{xx} + 4\xi_x\phi\phi_u u + 2\xi\phi_{xu}\phi_{xx}u - 2\xi_x\phi_u\phi_{xx}u - 2\xi^2\gamma\phi_{xx}u - 2\xi_x\phi_{xu}\phi_x u$$
$$-\xi^2\xi_x\xi_{xx}\phi_x + 2\xi^2\xi_x\phi\phi_u + 2\xi\xi_t\phi_u\phi_{xx} - 4\xi\xi_t\xi_x\phi_{xx} - 4\xi^2\phi\phi_{xuu}\phi_x - \phi_{xxtt}u$$
$$+\phi_{tt}u - \phi^2\phi_u + \phi\phi_{xxt} - \phi\phi_t + \phi^2\phi_{xxu} - 2\xi^2\phi_u\phi_{xu}\phi_x - \xi^2\phi\phi_u\phi_{xxu} - 4\xi\xi_x\phi_x u$$
$$+\xi^2\xi_{xx}\phi\phi_{xu} - 2\phi_u\phi_{xu}\phi_x u = 0$$

$$2\xi^3\xi_{xx}\phi_u - 4\xi^2\xi_{xxx}u - 8\xi\xi_t\xi_x^2 - 3\xi^3\xi_x\xi_{xx} + 4\xi\xi_t\xi_{xt} + \phi^2\phi_{uuu}u - \kappa\phi + 4\xi_{xt}\xi_x u$$
$$+2\phi_{tu}\phi_u u - 2\xi\xi_t u - 4\xi\phi_{xtu}u - 4\xi^3\phi\phi_{xuu} + 2\beta\xi^2\phi_{xx} - 4\xi^2\xi_x u + 4\xi_x\gamma u^2 + 2\xi^2\gamma\phi$$
$$-4\xi_{xt}\phi_u u + 2\phi\phi_{tuu}u - 4\xi^2\xi_x\phi_{tu} - 4\xi^2\xi_{xt}\phi_u - 5\xi^3\phi_{uu}\phi_x - 8\xi_x^2\phi_u u + 2\xi_x\phi_u^2u$$
$$-2\xi\xi_t\kappa - 4\xi^3\phi_u\phi_{xu} + 7\xi_t\xi_{xx}u + \xi^2\phi_t\phi_{uu} - 4\xi\xi_{xx}\phi + \phi_t\phi_{uu}u + \xi^2\phi^2\phi_{uuu}$$
$$+2\xi\xi_{xxt}u - \xi^2\xi_t\xi_{xx} + 2\xi^2\xi_t\phi_{xu} - 8\xi_t\phi_{xu}u + 6\xi^3\xi_x\phi_{xu} - 2\xi_x\phi_{tu}u - 2\xi\xi_t\phi\phi_{uu}$$
$$+2\xi^2\phi_{tu}\phi_u - 2\xi\xi_t\phi_u^2 + 7\xi^2\phi_{xxu}u + 8\xi\xi_t\xi_x\phi_u - 4\xi\phi_u\phi_{xu}u - 2\xi_x\phi\phi_{uu}u - 4\xi\xi_x\phi_{xu}u$$
$$+2\beta\phi_{xx}u + 2\xi^2\phi\phi_{tuu} + 2\xi\xi_{xx}\phi_u u + \xi^2\phi_{ttu} + 8\xi_x^3u - 4\xi^3\phi_{xtu} + 2\xi^3\xi_{xxt} + 5\xi\phi\phi_{xu}$$
$$+2\xi_{xt}\phi - 4\xi\xi_t\gamma u + \xi^4\phi_{xxu} + 5\xi\xi_x\xi_{xx}u - 5\xi\phi_{uu}\phi_x u - 4\xi\phi\phi_{xuu}u - 2\xi\xi_t\phi_{tu} + \phi_{ttu}u$$
$$-\phi^2\phi_{uu} + 3\xi^2\phi\phi_u\phi_{uu} + 3\phi\phi_u\phi_{uu}u - \phi\phi_u^2 - \phi\phi_{tu} + 2\xi_x\kappa u - 4\xi^2\xi_x\phi\phi_{uu} + 4\xi_x\phi\phi_u$$
$$+6\phi_{xxu}u^2 - 4\xi_{xxx}u^2 - 2\xi^2\xi_{xtt} - 2\xi^4\xi_x - 2\xi_{xtt}u - 4\xi_x^2\phi + \xi^2\phi + 8\xi^2\xi_{xt}\xi_x = 0$$

$$2\xi\phi_{tuuu}u - 5\xi^3\xi_x\phi_{uuu} - 3\alpha\phi_{xuu}u + 2\xi^3\phi\phi_{uuuu} + 6\xi\phi_{uu}^2u + 3\xi_t\phi_{uuu}u + 6\xi^3\phi_u\phi_{uuu}$$
$$+6\xi^3\phi_{uu}^2 + 2\xi^3\phi_{tuuu} - 2\xi\phi\phi_{uuu} - \xi^2\xi_t\phi_{uuu} - 3\alpha\xi^2\phi_{xuu} + 6\xi\phi_u\phi_{uuu}u$$
$$+2\xi\phi\phi_{uuuu}u - 6\xi^2\phi_{xuuu}u - \xi\xi_x\phi_{uuu}u - 4\phi_{xuuu}u^2 - 2\xi^4\phi_{xuuu} = 0$$

$$4\xi^2\phi_{tu}\phi_{uu} + \xi^2\phi_t\phi_{uuu} - \alpha\xi^2\xi_{xxx} + 4\xi^2\phi_{tuu}\phi_u - 6\xi_{xt}\phi_{uu}u - 2\gamma\phi + 4\phi_{tu}\phi_{uu}u$$
$$+2\gamma\phi_u u + \phi^2\phi_{uuuu}u + 2\phi\phi_{tuuu}u - \alpha\xi_{xxx}u + 5\xi\phi\phi_{xuu} + 2\xi^2\xi_t\phi_{xuu} + 4\xi_x\phi\phi_{uu}$$
$$-4\xi^2\xi_x\phi_{tuu} + 2\xi^2\gamma\phi_u + 4\xi^2\xi_x^2\phi_{uu} - 5\xi^3\phi_{uuu}\phi_x + 2\gamma\phi_{uu}u^2 + \xi^2\kappa\phi_{uu} + 6\xi^3\xi_x\phi_{xuu}$$
$$-8\xi^3\phi_u\phi_{xuu} - 4\xi^3\phi\phi_{xuuu} - 2\xi\xi_t\phi_{tuu} + 4\phi_{tuu}\phi_u u + \phi_t\phi_{uuu}u - 14\xi^3\phi_{xu}\phi_{uu}$$
$$+2\xi^2\phi\phi_{tuuu} - 8\xi_t\phi_{xuu}u - 6\xi^2\xi_{xt}\phi_{uu} + 6\xi^3\xi_{xx}\phi_{uu} + 3\alpha\phi_{xxu}u - 2\xi_x\phi_{tuu}u$$
$$-\xi^2\phi_{uu}u + 4\phi\phi_{uu}^2u + \kappa\phi_{uu}u + 4\xi_x\gamma u + 4\phi_u^2\phi_{uu}u - 8\xi\phi_u\phi_{xuu}u - 4\xi\phi_{xtuu}u$$
$$+\xi^2\phi^2\phi_{uuuu} + 7\xi^2\phi_{xxuu}u - 14\xi\phi_{xu}\phi_{uu}u - 2\xi_x\phi_u\phi_{uu}u + 2\xi^2\gamma\phi_{uu}u + 6\xi\xi_{xx}\phi_{uu}u$$
$$+4\xi^2\phi_u^2\phi_{uu} - 4\xi\xi_t\gamma - \phi^2\phi_{uuu} - \phi\phi_{tuu} + 6\phi_{xxuu}u^2 - 4\xi^3\phi_{xtuu} - 2\xi\xi_t\phi\phi_{uuu}$$
$$-4\xi\phi\phi_{xuuu}u - 4\xi_x^2\phi_{uu}u - 5\xi\phi_{uuu}\phi_x u - \xi^4\phi_{uu} - 8\xi^2\xi_x\phi_u\phi_{uu} + 5\phi\phi_u\phi_{uuu}u$$
$$+5\xi^2\phi\phi_u\phi_{uuu} + 8\xi\xi_t\xi_x\phi_{uu} - 2\xi_x\phi\phi_{uuu}u + 3\alpha\xi^2\phi_{xxu} + \xi^4\phi_{xxuu} - 3\phi\phi_u\phi_{uu}$$
$$-4\xi\xi_x\phi_{xuu}u - 4\xi^2\xi_x\phi\phi_{uuu} + \xi^2\phi_{ttuu} + \phi_{ttuu}u + 4\xi^2\phi\phi_{uu}^2 - 6\xi\xi_t\phi_u\phi_{uu} = 0$$

$$
\begin{aligned}
&2\phi_{xt}\phi_{uu}u+4\phi_{tuu}\phi_x u+2\xi_t\xi_x u+4\xi\phi\phi_{xxu}+8\xi\xi_x\phi_u u-2\xi_{xx}\gamma u^2+4\phi\phi_{xtuu}u\\
&\quad-6\xi\phi_{xu}^2u+6\phi_{tu}\phi_{xu}u-6\xi^3\phi_{xuu}\phi_x+2\phi_t\phi_{xuu}u+\xi^2\xi_t\phi_{xxu}+4\gamma\phi_x u\\
&\quad+4\phi_{xxxu}u^2+2\phi_{xttu}u+6\xi^2\phi\phi_u\phi_{xuu}-4\xi^2\xi_x\phi\phi_{xuu}+4\xi^2\phi_{xtu}\phi_u+4\xi^2\phi\phi_{xtuu}\\
&\quad-2\phi^2\phi_{xuu}+2\xi_{xx}\phi\phi_u+2\xi^2\phi_u^2\phi_{xu}-4\xi^3\phi_{uu}\phi_{xx}+2\phi_u^2\phi_{xu}u+5\xi_x^2\xi_{xx}u-2\xi^2\xi_t\phi_u\\
&\quad+2\xi\phi_{tu}u+\xi_{tt}u+2\xi^3\phi\phi_{uu}-2\xi^3\phi_u\phi_{xxu}-4\phi\phi_u\phi_{xu}-10\xi^2\xi_{xt}\phi_{xu}-4\xi\xi_t\phi_{xtu}\\
&\quad+4\xi^2\phi_{tuu}\phi_x+2\xi^2\xi_t\xi_x+4\gamma\phi_{xu}u^2+5\xi^2\xi_{xt}\xi_{xx}+\alpha\phi_{xxx}u+\xi\xi_x\phi+2\xi^2\phi^2\phi_{xuuu}\\
&\quad-4\xi\xi_t\phi\phi_{xuu}+6\xi^2\phi_{tu}\phi_{xu}+5\xi\xi_{xx}\phi_{xu}u+2\kappa\phi_{xu}u-10\xi_{xt}\phi_{xu}u+4\xi^2\gamma\phi_x-\xi_{xx}\kappa u\\
&\quad-3\xi_x\xi_{xx}\phi+2\xi^2\xi_{xxt}\xi_x+2\xi^2\kappa\phi_{xu}+\xi^3\xi_x\phi_{xxu}+2\xi^2\phi_{xt}\phi_{uu}+2\xi^2\xi_x^2\phi_{xu}-\xi\xi_{xxx}\phi\\
&\quad-2\xi^2\xi_{xxt}\phi_u-2\phi\phi_{uu}\phi_x-\xi^2\xi_{xx}\phi_u^2+2\xi_t\phi_u u+2\xi_t\xi_{xxx}u+5\xi^3\xi_{xx}\phi_{xu}+2\xi\xi_t\xi_{xxt}\\
&\quad-\xi^2\xi_x^2\xi_{xx}+4\phi\phi_{uuu}\phi_x u+6\phi_u\phi_{uu}\phi_x u-2\phi\phi_{xtu}+8\xi^2\phi\phi_{xu}\phi_{uu}-2\xi_x\xi_{xx}\phi_u u\\
&\quad-3\xi^2\xi_{xx}\phi_{tu}-2\xi\phi_{xxtu}u-\xi\xi_{xx}^2u-\xi_{xxtt}u-3\xi^2\xi_{xx}\phi\phi_{uu}+2\xi^3\phi_{tu}-\xi^3\xi_{xx}^2\\
&\quad-2\xi^3\phi\phi_{xxuu}-\xi^2\xi_{xxxx}u+2\xi^2\phi_t\phi_{xuu}-\xi_{xx}\phi_u^2u+6\xi_x\phi\phi_{xu}-2\xi_{xxt}\phi_u u\\
&\quad-4\xi^2\xi_x\phi_{xtu}-2\xi\phi\phi_u-\xi^2\xi_{xx}\kappa+5\xi_{xt}\xi_{xx}u+\alpha\xi^2\phi_{xxx}-8\xi\xi_t\phi_u\phi_{xu}-4\xi\xi_x^2u\\
&\quad-6\xi^3\phi_{xu}^2+6\phi\phi_u\phi_{xuu}u+8\phi\phi_{xu}\phi_{uu}u-4\xi^2\xi_x\phi_u\phi_{xu}+\xi_{xxt}\phi-2\xi^3\phi_{xxtu}\\
&\quad-10\xi_x^2\phi_{xu}u+4\phi_{xtu}\phi_u u-2\xi\xi_t^2-4\xi\xi_t\phi_{uu}\phi_x-3\xi_{xx}\phi_{tu}u+2\phi^2\phi_{xuuu}u\\
&\quad+12\xi\xi_t\xi_x\phi_{xu}-\xi_{xxxx}u^2-2\xi_x\phi_{uu}\phi_x u-2\xi^3\xi_x^2+2\xi^2\phi_{xttu}+4\xi^3\xi_x\phi_u-6\xi\phi_{xuu}\phi_x u\\
&\quad+2\xi\phi\phi_{uu}u-2\xi\phi_u\phi_{xxu}u-2\xi\phi\phi_{xxuu}u+4\xi^2\gamma\phi_{xu}u+2\xi^2\xi_x\xi_{xx}\phi_u-7\xi_t\phi_{xxu}u\\
&\quad-\xi^2\xi_{xxtt}+4\xi^2\phi_{xxxu}u-7\xi\xi_x\phi_{xxu}u+4\xi^2\phi\phi_{uuu}\phi_x-6\xi\xi_t\xi_x\xi_{xx}+\xi^2\xi_{tt}\\
&\quad-4\xi\phi_{uu}\phi_{xx}u-6\xi^2\xi_x\phi_{uu}\phi_x+6\xi^2\phi_u\phi_{uu}\phi_x-2\xi^2\xi_{xx}\gamma u-3\xi_{xx}\phi\phi_{uu}u+2\xi\xi_x\xi_{xxx}u\\
&\quad-\xi_t\phi+4\xi_x\phi_u\phi_{xu}u+4\xi\xi_t\xi_{xx}\phi_u=0
\end{aligned}
$$

References

[1] Ablowitz M.J. and Haberman R., *J. Math. Phys.*, 1975, V.16, 2301–2305.

[2] Ablowitz M.J., Ramani A. and Segur H., *Phys. Rev. Lett.*, 1978, V.23, 333–338.

[3] Ablowitz M.J., Ramani A. and Segur H., *J. Math. Phys.*, 1980, V.21, 715–721.

[4] Abramowitz M. and Stegun I.A., Handbook of Mathematical Functions, Dover, New York, 1965.

[5] Ames W.F., *Appl. Num. Math.*, 1992, V.10, 235–259.

[6] Arrigo D., Broadbridge P. and Hill J.M., *J. Math. Phys.*, 1993, V.34, 4692–4703.

[7] Bluman G.W and Cole J.D., *J. Math. Mech.*, 1969, V.18, 1025–1042.

[8] Boyd J.P., *Appl. Math. Comp.*, 1997, V.81, 173–187.

[9] Boussinesq J., *Comptes Rendus*, 1871, V.72, 755–759.

[10] Boussinesq J., *J. Pure Appl.*, 1872, V.7, 55–108.

[11] Camassa R. and Holm D.D., *Phys. Rev. Lett.*, 1993, V.71, 1661–1664.

[12] Camassa R., Holm D.D. and Hyman J.M., *Adv. Appl. Mech.*, 1994, V.31, 1–33.

[13] Caudrey P.J., *Phys. Lett. A*, 1980, V.79, 264–268.

[14] Caudrey P.J., *Physica D*, 1982, V.6, 51–66.

[15] Champagne B., Hereman W. and Winternitz P., *Comp. Phys. Comm.*, 1991, V.66, 319–340.

[16] Clarkson P.A., *Chaos, Solitons & Fractals*, 1995, V.5, 2261–2301.

[17] Clarkson P.A., Gordoa P.R. and Pickering A., *Inverse Problems*, 1997, V.13, 1463–1476.

[18] Clarkson P.A. and Kruskal M.D., *J. Math. Phys.*, 1989, V.30, 2201–2213.

[19] Clarkson P.A., Ludlow D.K. and Priestley T.J., *Meth. Appl. Anal.*, 1997, V.4, 173–195.

[20] Clarkson P.A. and Mansfield E.L., *Physica D*, 1994, V.70, 250–288.

[21] Clarkson P.A. and Mansfield E.L., *Nonlinearity*, 1994, V.7, 975–1000.

[22] Clarkson P.A. and Mansfield E.L., *SIAM J. Appl. Math.*, 1994, V.54, 1693–1719.

[23] Clarkson P.A. and Mansfield E.L., *Acta Appl. Math.*, 1995, V.39, 245–276.

[24] Clarkson P.A., Mansfield E.L. and Priestley T.J., *Math. Comput. Modell.*, 1997, V.25, 195–212.

[25] Constantin A., *J. Math. Anal. Appl.*, 1997, V.210, 215–230.

[26] Constantin A., *J. Diff. Eqns.*, 1997, V.41, 218–235.

[27] Constantin A., *J. Func. Anal.*, 1998, V.155, 352–363.

[28] Conte R., Fordy A.P. and Pickering A., *Physica D*, 1993, V.69, 33–58.

[29] Cooper F. and Shephard H., *Phys. Lett. A*, 1994, V.194, 246–250.

[30] Deift P., Tomei C. and Trubowitz E., *Commun. Pure Appl. Math.*, 1982, V.35, 567–628.

[31] Fornberg B. and Whitham G.B., *Phil. Trans. R. Soc. Lond. A*, 1978, V.289, 373–404.

[32] Fokas A.S., *Physica D*, 1994, V.87, 145–150.

[33] Fokas A.S., Olver P.J. and Rosenau P., A plethora of integrable bi-Hamiltonian equations, in Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman, Editors A.S. Fokas and I.M. Gel'fand, Progress in Nonlinear Differential Equations, Vol.26, Birkhäuser, Boston, p. 93–101, 1996.

[34] Fuchssteiner B., *Prog. Theo. Phys.*, 1981, V.65, 861–876.

[35] Fuchssteiner B., *Physica D*, 1996, V.95, 229–243.

[36] Fuchssteiner B. and Fokas A.S., *Physica D*, 1981, V.4, 47.

[37] Galaktionov V.A., *Diff. Int. Eqns.*, 1990, V.3, 863–874.

[38] Galaktionov V.A., Dorodnytzin V.A., Elenin G.G., Kurdjumov S.P. and Samarskii A.A., *J. Sov. Math.*, 1988, V.41, 1222–1292.

[39] Gardner C.S., Greene J.M., Kruskal M.D. and Muira R.M., *Phys. Rev. Lett.*, 1967, V.19, 1095–1097.

[40] Gilson C. and Pickering A., *J. Phys. A: Math. Gen.*, 1995, V.28, 2871–2888.

[41] Hereman W., *Euromath. Bull.*, 1994, V.1, no.2, 45–79.

[42] Hone A.N.W., Exact solutions of the associated Camassa-Holm equation, preprint, University of Rome III, 1998.

[43] Ince E.L., Ordinary Differential Equations, Dover, New York, 1956.

[44] Kovalev A.S., Bose–gas with nontrivial interparticle interaction and exotic magnetic solitons, in Proceedings of the Conference on Nonlinear Coherent Structures in Physics and Biology, Editors D.B. Duncan and J.C. Eilbeck, published on WWW: `http://www.ma.hw.ac.uk/ solitons/procs/`, 1995.

[45] Levi D. and Winternitz P., *J. Phys. A: Math. Gen.*, 1989, V.22, 2915–2924.

[46] Li Yi.A. and Olver P.J., *Discrete Cont. Dyn. Syst.*, 1997, V.3, 419–432.

[47] Li Yi.A. and Olver P.J., *Discrete Cont. Dyn. Syst.*, 1998, V.4, 159–191.

[48] Li Yi.A., Olver P.J. and Rosenau P., Non-analytic solutions of nonlinear wave models, preprint, University of Minnesota, Minneapolis, 1998.

[49] Mansfield E.L., `diffgrob2:` A symbolic algebra package for analysing systems of PDE using Maple, `ftp ftp.ukc.ac.uk`, login: anonymous, password: your email address, directory: `pub/maths/liz`, 1993.

[50] Mansfield E.L. and Clarkson P.A., *J. Symb. Comp.*, 1997, V.23, 517–533.

[51] Mansfield E.L. and Clarkson P.A., *Math. Comp. Simul.*, 1997, V.43, 39–55.

[52] Mansfield E.L. and Fackerell E.D., Differential Gröbner Bases, preprint **92/108**, Macquarie University, Sydney, Australia, 1992.

[53] Nucci M.C. and Clarkson P.A., *Phys. Lett. A*, 1992, V.164, 49–56.

[54] Olver P.J., Applications of Lie Groups to Differential Equations, 2nd edition, Graduate Text Maths., V.107, New York, Springer-Verlag, 1993.

[55] Olver P.J., *Proc. R. Soc. Lond. A*, 1994, V.444, 509–523.

[56] Olver P.J. and Rosenau P., *Phys. Rev. E*, 1996, V.53, 1900–1906.

[57] Pucci E., *J. Phys. A: Math. Gen.*, 1992, V.25, 2631–40.

[58] Ramani A., Dorizzi B. and Grammaticos B., *Phys. Rev. Lett.*, 1982, V.49, 1539–1541.

[59] Ranada A.F., Ramani A., Dorizzi B. and Grammaticos B., *J. Math. Phys.*, 1985, V.26, 708–710.

[60] Reid G.J., *J. Phys. A: Math. Gen.*, 1990, V.23, L853–L859.

[61] Reid G.J., *Eur. J. Appl. Math.*, 1991, V.2, 293–318.

[62] Rosenau P., *Phys. Rev. Lett.*, 1994, V.73, 1737–1741.

[63] Rosenau P., *Phys. Lett. A*, 1996, V.211, 265–275.

[64] Rosenau P., *Phys. Lett. A*, 1997, V.230, 305–318.

[65] Rosenau P. and Hyman J.M., *Phys. Rev. Lett.*, 1993, V.70, 564–567.

[66] Schiff J., *Physica D*, 1998, V.121, 24-43.

[67] Shokin Yu.I., The Method of Differential Approximation, Springer-Verlag, New York, 1983.

[68] Vorob'ev E.M., *Acta Appl. Math.*, 1991, V.24, 1–24.

[69] Weiss J., Tabor M. and Carnevale G., *J. Math. Phys.*, 1983, V.24, 522–526.

[70] Whittaker E.E. and Watson G.M., Modern Analysis, 4th edition, Cambridge University Press, Cambridge, 1927.

[71] Zakharov V.E., *Sov. Phys. JETP*, 1974, V.38, 108–110.