Journal of Nonlinear Mathematical Physics 1999, V.6, N 1, 35–50. Article

Variational Methods for Solving Nonlinear Boundary Problems of Statics of Hyper-Elastic Membranes

V.A. TROTSENKO

Institute of Mathematics of the National Academy of Sciences of Ukraine,
3 Tereshchenkivska Str., 252601 Kyiv-4, Ukraine

Received June 10, 1998; Accepted July 13, 1998

Abstract

A number of important results of studying large deformations of hyper-elastic shells are obtained using discrete methods of mathematical physics [1]–[6]. In the present paper, using the variational method for solving nonlinear boundary problems of statics of hyper-elastic membranes under the regular hydrostatic load, we investigate peculiarities of deformation of a circular membrane whose mechanical characteristics are described by the Bidermann-type elastic potential. We develop an algorithm for solving a singular perturbation of nonlinear problem for the case of membrane loaded by heavy liquid. This algorithm enables us to obtain approximate solutions both in the presence of boundary layer and without it. The class of admissible functions, on which the variational method is realized, is chosen with account of the structure of formal asymptotic expansion of solutions of the corresponding linearized equations that have singularities in a small parameter at higher derivatives and in the independent variable. We give examples of calculations that illustrate possibilities of the method suggested for solving the problem under consideration.

1 Definition of the strained deformed state of a circular membrane under the regular hydrostatic load

Let us consider a circular membrane of the radius R_0 of incompressible, isotropic and hyperelastic material having the small constant width h_0. Let a hydrostatic load Q be applied to the membrane with an inflexibly fixed contour. To describe the geometry of the deformed membrane, let us introduce the cylindrical coordinate system $Oz\eta r$, the axis Oz of which coincides with the symmetry axis of the membrane. It follows from the axial symmetry of the problem that principal deformation directions at any point will coincide with meridians, parallels and normals to the deformed surface and all the parameters of the strained deformed state will be functions of initial distances s of points of the membrane from its symmetry axis only. Denote the principal degrees of lengthenings in these directions by λ_1, λ_2 and λ_3, respectively. Then, according to the nonlinear theory of elastic membrane [1], interior stresses T_1 and T_2 of the deformed shell in the direction of its meridian and parallel can be determined from the relations

$$
\begin{gathered}
T_i = 2h_0\lambda_3\left(\lambda_i^2 - \lambda_3^2\right)\left(\frac{\partial W}{\partial I_1} + \lambda_{3-i}^2\frac{\partial W}{\partial I_2}\right) \qquad (i = 1, 2), \\
\lambda_1 = \sqrt{\left(\frac{dz}{ds}\right)^2 + \left(\frac{dr}{ds}\right)^2}, \qquad \lambda_2 = \frac{r(s)}{s}, \qquad \lambda_3 = \frac{1}{\lambda_1\lambda_2} = \frac{h(s)}{h_0}, \\
I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}, \quad W = W(I_1, I_2).
\end{gathered}
\tag{1.1}
$$

Here $z(s)$ and $r(s)$ are axial and radial components of the radius of the generatix vector of the deformed shell, W is a deformation energy function for the material of the shell, I_1 and I_2 are deformation invariants, and $h(s)$ is the width of the membrane in the deformed state.

Let us choose the deformation energy function $W(I_1, I_2)$ in the form of its four-term approximation proposed by V.L. Biderman [7], i.e.,

$$
W = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(I_1 - 3)^2 + C_4(I_1 - 3)^3, \tag{1.2}
$$

where C_i $(i = 1, \ldots, 4)$ are physical constants that are determined experimentally.

By using function (1.2) one can sufficiently precisely describe the law of deforming for a number of types of elastic materials up to their break.

If the origin of the coordinate system coincides with a pole of the deformed membrane, then the hydrostatic pressure acting on the shell, is of the form

$$
Q = C^* - Dz, \qquad C^* = P^* - P^0, \qquad D = \rho g. \tag{1.3}
$$

Here ρ is the density of liquid, g is the acceleration of gravity, P^* is a constant component of pressure in liquid, and P^0 is the pressure of gas onto the shell.

Let us introduce the following dimensionless parameters:

$$
\begin{gathered}
\{s^*, r^*, z^*\} = \{s, r, z\}/R_0, \qquad T_i^* = T_i/(2C_1h_0), \\
W^* = W/C_1, \qquad Q^* = QR_0/(2C_1h_0).
\end{gathered}
$$

In what follows, we shall use these parameters omitting asterisks for the sake of simplicity.

Movements corresponding to the static equilibrium state are singled out from all admissible movements by the fact that stationary values of the functional [8]

$$
I = \int_0^1 \left[W(I_1, I_2)s - Qr^2\frac{dz}{ds}\right] ds \tag{1.4}
$$

correspond to them.

Having placed the origin of the coordinate system $Oz\eta r$ at the center of the undeformed membrane, we present the hydrostatic load as

$$
Q = C - Dz,
$$

where difference between the constants C and C^* in formula (1.3) is due to the shift of the origin of the coordinate system. In this case, we should seek a stationary value of functional (1.4) within the set of functions satisfying the boundary conditions

$$\left.\frac{dz}{ds}\right|_{s=0} = 0, \qquad z(1) = 0, \qquad r(0) = 0, \qquad r(1) = 1. \tag{1.5}$$

Utilizing usual tools of the calculus of variations, one can show that the Euler equations for functional (1.4) are:

$$\frac{d}{ds}(rT_1) = T_2\frac{dr}{ds}, \qquad k_1T_1 + k_2T_2 = Q. \tag{1.6}$$

Here k_1 and k_2 are principle curvatures of the deformed median surface of the membrane calculated by the formulae

$$k_1 = \left(\frac{d^2r}{ds^2}\frac{dz}{ds} - \frac{dr}{ds}\frac{d^2z}{ds^2}\right)\lambda_1^{-3}, \qquad k_2 = -(r\lambda_1)^{-1}\frac{dz}{ds}.$$

Following Rietz to find an extremum of functional (1.4) we present functions $z(s)$ and $r(s)$ in the form of the following expansions:

$$z(s) = \sum_{k=1}^{m} x_k u_k(s), \qquad r(s) = s + \sum_{k=1}^{m} x_{k+m} v_k(s), \tag{1.7}$$

where in view of (1.5), the coordinate functions $u_k(s)$ and $v_k(s)$ have to obey the conditions

$$u_k'(0) = u_k(1) = v_k(0) = v_k(1) = 0. \tag{1.8}$$

Then constants x_k $(k = 1, 2, \ldots, 2m)$ forming $2m$-component vector $\vec{x}$ will be determined from the nonlinear algebraic system

$$\vec{g}(\vec{x}) = 0. \tag{1.9}$$

In this case, components of the vector-function $\vec{g}$ are of the form

$$\begin{aligned} g_i &= \int_0^1 \left[U(\lambda_1, \lambda_2)\frac{dz}{ds}\frac{du_i}{ds} - Q\lambda_2\frac{dr}{ds}u_i\right] s\,ds, \\ g_{i+m} &= \int_0^1 \left[U(\lambda_1, \lambda_2)\frac{dr}{ds}\frac{dv_k}{ds} + \left(U(\lambda_2, \lambda_1)\frac{\lambda_2}{s} + Q\lambda_2\frac{dz}{ds}\right) v_i\right] s\,ds. \end{aligned} \tag{1.10}$$

Here

$$U(\lambda_1, \lambda_2) = \left(1 - \lambda_1^{-4}\lambda_2^{-2}\right)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2\frac{\partial W}{\partial I_2}\right).$$

We solve algebraic system (1.9) by using the Newton method according to which correction of an approximate solution at each iteration step is performed by the scheme

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} - H^{-1}\left(\vec{x}^{(k)}\right) g\left(\vec{x}^{(k)}\right) \quad (k = 1, 2, \ldots), \tag{1.11}$$

where $H(\vec{x})$ is the symmetric Jacobi matrix of functions $g_1,\ g_2,\ldots,g_{2m}$ of variables x_1, $x_2,\ldots,x_{2m}$, whose elements are calculated as follows

$$h_{ij}=\int\limits_0^1\left\{\left[\frac{\partial U(\lambda_1,\lambda_2)}{\lambda_1\partial\lambda_1}\left(\frac{dz}{ds}\right)^2+U(\lambda_1,\lambda_2)\right]\frac{du_i}{ds}\frac{du_j}{ds}+D\lambda_2\frac{dr}{ds}u_iu_j\right\}s\,ds,$$

$$h_{i+m,j+m}=\int\limits_0^1\left\{\left[\frac{\partial U(\lambda_1,\lambda_2)}{\lambda_1\partial\lambda_1}\left(\frac{dz}{ds}\right)^2+U(\lambda_1,\lambda_2)\right]\frac{dv_i}{ds}\frac{dv_j}{ds}\right.$$

$$\left.+\frac{1}{s}\frac{dr}{ds}\frac{\partial U(\lambda_1,\lambda_2)}{\partial\lambda_2}\frac{d}{ds}(v_iv_j)+\left[\frac{\partial U(\lambda_2,\lambda_1)}{\lambda_2\partial\lambda_2}+U(\lambda_2,\lambda_1)+Qs\frac{dz}{ds}\right]\frac{v_iv_j}{s^2}\right\}s\,ds,$$

$$(j=\overline{1,m};\ i\geq j),$$

$$h_{i,m+j}=\int\limits_0^1\left\{\frac{\partial U(\lambda_1,\lambda_2)}{\lambda_1\partial\lambda_1}\frac{dz}{ds}\frac{dr}{ds}\frac{du_i}{ds}\frac{dv_j}{ds}\right.$$

$$\left.+\left[\frac{\partial U(\lambda_1,\lambda_2)}{s\partial\lambda_2}\frac{dz}{ds}+Q\lambda_2\right]\frac{du_i}{ds}v_j-D\lambda_2\frac{dz}{ds}u_iv_j\right\}s\,ds\qquad(i,j=1,\ldots,m).$$

So starting from the potential energy stationarity principle, we reduce the problem of determining finite deformations of the circular membrane to calculating integrals forming algebraic system and elements of the Jacobi matrix with subsequent solving problems of linear algebra (1.11) at each step of the Newton iteration process. The algorithm may be effective if one chooses properly systems of coordinate functions $\{u_k(s)\}$ and $\{v_k(s)\}$ that would allow one to get solutions with high precision, provided the dimension of the nonlinear algebraic system is not large.

To this end, let us expand solutions to be found in a Taylor series in a neighbourhood of the point $s=0$, using sequential differentiation of equilibrium equations and geometric relationships of the shell in order to get coefficients of the expansion. With regard to symmetry conditions at the pole of the deformed shell

$$\lambda_1=\lambda_2=\lambda,\qquad T_1=T_2=T,\qquad k_1=k_2=k\qquad(s=0),$$

we can show that solutions boundered under $s\to 0$ have the following structure:

$$\begin{aligned}z(s)&=a_1+a_2s^2+a_3s^4+\cdots,\\ r(s)&=b_1s+b_2s^3+b_3s^5+\cdots.\end{aligned}\tag{1.12}$$

Taking into account expansions (1.12), sequences of coordinate functions $\{u_k(s)\}$ and $\{v_k(s)\}$ satisfying boundary conditions (1.8) take the form

$$u_k(s)=(s^2-1)s^{2k-2},\qquad v_k(s)=(s^2-1)s^{2k-1}\qquad(k=1,\ldots,m).\tag{1.13}$$

Let us present some results of calculations according to the algorithm proposed. To illustrate convergence of the Rietz method, we write out values of functions $z(s)$ and

Figure 1

$r(s)$ and their first two derivatives, depending on the number m of approximations in expansions (1.7) at the point $s = 0.2$ for $C = 1.7$, $D = 0$ in Table 1. Ratios of elastic constants in Biderman's potential are chosen as follows:

$$\Gamma_1 = 0.02, \qquad \Gamma_2 = -0.015, \qquad \Gamma_3 = 0.00025.$$

In the last column of the table we give the relative error of the obtained approximation to the solution of the second equilibrium equation

$$\Delta = |k_1 T_1 + k_2 T_2 - Q|/|C|. \tag{1.14}$$

Table 1

m	z	r	$-z'$	r'	$-z''$	$-r''$	Δ
1	0.7016	0.2865	0.2923	1.3966	1.4617	0.5408	$2 \cdot 10^{-1}$
2	0.7824	0.3026	0.3975	1.4644	1.9279	0.7252	$6 \cdot 10^{-2}$
3	0.7913	0.3064	0.4255	1.4753	2.0248	0.8345	$1 \cdot 10^{-2}$
4	0.7926	0.3068	0.4350	1.4759	2.0421	0.8560	$1 \cdot 10^{-3}$
5	0.7926	0.3069	0.4362	1.4758	2.0415	0.8591	$6 \cdot 10^{-4}$
6	0.7926	0.3069	0.4362	1.4757	2.0406	0.8596	$2 \cdot 10^{-5}$

Solutions possess the similar convergence at other points of the interval of integration for these equations. Lengthening of the generatrix of the deformed shell affects weakly on the degree of convergence of the variational method. For a chosen system of coordinate functions, solutions have the uniform convergence as well as their first two derivatives for sufficiently wide range of parameters.

Graphic dependence of the sag of the shell at its pole $f = z(0)$, upon the applied uniform pressure ($D = 0$) is shown in Figure 1. The characteristic indicates the possibility of existence of several equilibrium forms for the same value of the load. It is clearly seen that the diagram "load-sag" for the material under consideration has two extreme points A and B. If the parameter of the load increases monotonely and reaches the value of the upper critical load, then there will be an abrupt passage of the system into a new equilibrium state, after which sags of the shell will again increase smoothly. The monotone decrease of the load C (unloading stage) leads to the backward distortion of the shell to the initial equilibrium form at lower critical load. Equilibrium states described by a falling part of the curve from the point A to the point B are unstable and they are not realized at all in the way of loading under consideration.

In conclusion, we consider to the main principles of choosing an initial approximation of the Newton iteration process that allows us to get solutions of nonlinear algebraic system (1.9). If solutions of these equations are far from singular points A and B, then one can construct an initial approximation, taking advantage of rapid convergence of solutions. As we can see from Table 1, the first Rietz approximation is a good solution to the initial problem. In this case, we have to solve a system of two nonlinear algebraic equations. The initial approximation is determined from physically noncontradictory values of x_1 and x_2 (to the positive load there corresponds the positive sag of the shell). Then we use the obtained solution to construct P-th approximation, setting the first and $(P+1)$-th components of the vector $\vec{x}$ equal to obtained values of x_1 and x_2, and putting other components equal to zero. The initial approximation constructed in this way improved by employing the Newton method with 4–5 iterations. In the case, when it is necessary to obtain a series of equilibrium states, when choosing initial values of the vector $\vec{x}$ one can use different extrapolational formulae to prolong solutions in the parameter C. However, when implementing the procedure of prolonging a solution with approximation to critical points A and B, the convergence of the Newton process sharply comes down, and then the divergence appears clearly. To go through these equilibrium states of the shell one should change prolongation parameters. The main point of this approach is that starting from some moment, the parameter of the load is supposed to be unknown, and as a given parameter we choose the value of the sag of the shell at the pole. Artificial introduction of a new parameter provides that the uniqueness condition for a solution in a neighbourhood of a fixed parameter is fulfilled, and, therefore, Jacobian is nondegenerate, which is a necessary condition for the Newton method to be applicable.

The algorithm proposed is also effective for other rotation shells in the nondeformed state, the generatrix of which intersects the symmetry axis at the right angle and principle curvatures at its pole are equal.

2 Adaptive Rietz method under conditions of singular perturbance of the initial problem

Let us consider the case of loading a membrane by a hydrostatic load for $C^* \ll 1$ and $D \gg 1$ (heavy liquid). For such parameters of the load, stresses in the median surface of the deformed shell will be small, i.e., $\{T_1, T_2\} \ll 1$, and the shell will take a nearly plane form with the exception of a narrow domain near the fixed contour of the membrane. The increase of the constant component of the load C^* leads to the increase of stresses in the membrane, that begin to play the prevalent role in forming its deformed surface. This is accompanied by increasing the width of a domain with rapidly changing functions of movements and stresses, and turning a plane part of the surface of the shell into a curvilinear surface.

After introducing a small parameter $\mu = 1/D$, equilibrium equations (1.6) in the coordinate system associated with the center of the nondeformed membrane can be rewritten as follows:

$$\begin{aligned} &L_1(s, r, z', r', z'', r'') = 0, \\ &\mu\,[L_2(s, r, z', r', z'', r'') - C^*] = l - z, \end{aligned} \tag{2.1}$$

where

$$L_1 = \frac{dT_1}{ds} + \frac{1}{r}\frac{dr}{ds}(T_1 - T_2), \qquad L_2 = k_1 T_1 + k_2 T_2, \qquad l = z(0).$$

In the case under consideration, the presence of a small parameter at higher derivatives transfers the initial regular problem into a class of nonlinear singular perturbed problems. In this case, solutions will have domains both with fast and slow variability. As a result, when approximating solutions by means of polynomial basis, the convergence of successive approximations in the Rietz method sharply comes down, and the increase of the number of coordinate functions in expansions (1.7) leads to a loss of calculation stability before the required precision is achieved. The situation is also aggravate by that the singularity degree of the problem depends on a solution itself that is a priori unknown. Thus, there arise the principal problem of extending a class of admissible functions by functions that allow getting within the framework of a unique algorithm approximate solutions with the equally high precision both in the presence of a narrow area of great changes in solutions and without it.

The problem indicated plays a key role in realizing the method of variations when solving singularly perturbed boundary problems. It is one of suffuciently difficult problems of mathematical physics and theory of function approximation. Apparently for this reason, variantional methods (unlike finite difference methods [9]) are not developed properly for solving even linear boundary problems with a small parameter the higher derivative.

One of the most important merits of the method of variations is that when constructing approximate solutions, one can use an information about problem, that can be obtained by means of preliminary analysis of required solutions. On the other hand, methods of a small parameter may be also used to determine a structure of a solution and characterize its degeneration with the parameter at the higher derivative tending to zero. With this objective in view, let us carry out an asymptotic analysis of the initial equations.

Searching for a direct expansion of solutions to system (2.1) in the form

$$z(s) = \sum_{k=0}^{\infty} \mu^k z^{(k)}(s), \qquad r(s) = \sum_{k=0}^{\infty} \mu^k r^{(k)}(s), \tag{2.2}$$

let us expand operators L_i $(i = 1, 2)$ in a Taylor series in the parameter μ

$$L_i = (L_i)_{\mu=0} + \frac{\mu}{1!}\left(\frac{dL_i}{d\mu}\right)_{\mu=0} + \cdots + \frac{\mu^{(n)}}{n!}\left(\frac{d^n L_i}{d\mu^n}\right)_{\mu=0} + \cdots. \tag{2.3}$$

Having substituted (2.2) and (2.3) into (2.1) and having equated coefficients of expansions of the same degrees of μ, we come to

$$\begin{aligned} &(L_1)_{\mu=0} = 0, \qquad z^{(0)}(s) = l, \\ &\left(\frac{dL_1}{d\mu}\right)_{\mu=0} = 0, \qquad (L_2)_{\mu=0} - C^* = -z^{(1)}(s), \\ &\left(\frac{d^n L_1}{d\mu^n}\right)_{\mu=0} = 0, \qquad \frac{1}{(n-1)!}\left(\frac{d^{(n-1)} L_2}{d\mu^{(n-1)}}\right)_{\mu=0} = -z^{(n)}(s) \qquad (n = 2, 3, \ldots). \end{aligned} \tag{2.4}$$

We can obtain a bounded solution of the nonlinear equation $(L_1)_{\mu=0} = 0$ with respect to the function $r^{(0)}(s)$ by expanding it in a Taylor series in a neighbourhood of the point $s = 0$ with regard to the symmetry at a pole of the deformed shell. In this case, we can show that

$$r^{(0)}(s) = a_1 s, \qquad z^{(0)}(s) = l, \tag{2.5}$$

where a_1 is an arbitrary constant.

Furthermore, with regard for expressions (2.5), linear equations of the first approximation allow us to get expressions for functions $r^{(1)}(s)$ and $z^{(1)}(s)$:

$$r^{(1)}(s) = a_2 s, \qquad z^{(1)}(s) = C^*.$$

Having performed several transformations that are simple but sufficiently awkward, we can show that

$$r^{(n)}(s) = a_n s, \qquad z^{(n)}(s) = 0 \qquad (n = 2, 3, \ldots),$$

where a_n are arbitrary constants.

Collecting the results obtained, we come to the general form of a solution bounded for $s = 0$ as a result of direct expansion (2.2):

$$r(s) = A(\mu)s, \qquad z(s) = C/D, \qquad C = C^* + lD, \tag{2.6}$$

where $A(\mu)$ is a function of μ; C is the constant component of the hydrostatic pressure, that differs from C^* at the expense of that the origin of the coordinate system passes into the center of the nondeformed membrane.

Proceeding from equations (2.1), it is not possible to establish the structure of solutions in the boundary area, since equations of the first approximation are nonlinear and they are not integrated in the explicit form. For this reason, in what follows, using the results of the asymptotic analysis of the linearized equilibrium equations with respect to some equilibrium state that is described by an analytic function of the form (1.12), we propose to establish the structure of their solutions and after that to endow solutions of the nonlinear problem with these properties.

Suppose that the membrane is in the deformed state as a result of applying the hydrostatic load $Q = C^* - Dz$. Let the shell have passed into a new state at the expense of changing the constant pressure component by a small value δC^*. Denote by u and w projections of a movement of the shell from this load onto directions of its generatrix and outer normal, respectively. We shall characterize the main strained deformed state by stresses T_1 and T_2, and principal degrees of lengthenings λ_1, λ_2 and λ_3. Having performed linearization of elasticity relations (1.1), we can establish the following relation between additional stresses δT_1 and δT_2 and deformations of the shell:

$$\delta T_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2, \qquad \delta T_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2, \tag{2.7}$$

where

$$\varepsilon_1 = \frac{1}{\lambda_1}\frac{du}{ds} + k_1 w, \qquad \varepsilon_2 = \frac{1}{r\lambda_1}\frac{dr}{ds}u + k_2 w;$$

$$c_{11} = f_1(\lambda_1,\lambda_2), \qquad c_{12} = f_2(\lambda_1,\lambda_2), \qquad c_{21} = f_2(\lambda_2,\lambda_1), \qquad c_{22} = f_1(\lambda_2,\lambda_1);$$

$$f_1(\lambda_1,\lambda_2) = \left(\lambda_1^2\lambda_3 + 3\lambda_3^3\right)\frac{\partial W}{\partial I_1} + \left(\lambda_1\lambda_2 + 3\lambda_2^2\lambda_3^3\right)\frac{\partial W}{\partial I_2}$$
$$+2\lambda_3\left(\lambda_1^2 - \lambda_3^2\right)\left(A_{11} + 2A_{12}\lambda_2^2 + A_{22}\lambda_2^4\right),$$

$$f_2(\lambda_1,\lambda_2) = \left(3\lambda_3^3 - \lambda_1^2\lambda_3\right)\frac{\partial W}{\partial I_1} + \left(\lambda_1\lambda_2 + \lambda_2^2\lambda_3^3\right)\frac{\partial W}{\partial I_2}$$
$$+2\lambda_3\left(\lambda_1^2 - \lambda_3^2\right)\left(\lambda_2^2 - \lambda_3^2\right)\left(A_{11} + A_{12}\left(\lambda_2^2 + \lambda_2^2\right) + A_{22}\lambda_1^2\lambda_2^2\right),$$

$$A_{ik} = \frac{\partial^2 W}{\partial I_i \partial I_k} \qquad (i,k = 1,2).$$

Linearized equilibrium equations in movements, describing the perturbed state of the shell are of the form

$$\begin{aligned} &L_{11}(u) + L_{12}(w) = 0,\\ &\varepsilon^2\left[L_{21}(u) + L_{22}(w)\right] + r\left(\frac{dr}{ds}w + \frac{dz}{ds}u\right) = \left(\varepsilon^2\delta C^* + \Delta l\right)r\lambda_1, \end{aligned} \tag{2.8}$$

where

$$L_{11}(u) = -\frac{d}{ds}\left(\alpha_1\frac{du}{ds}\right) + \alpha_2 u, \qquad L_{12}(w) = \alpha_3\frac{dw}{ds} + \alpha_4 w,$$

$$L_{22}(w) = -\frac{d}{ds}\left(\beta_1\frac{dw}{ds}\right) + \beta_2 w, \qquad L_{21}(u) = \beta_3\frac{du}{ds} + \beta_4 u,$$

$$\alpha_1 = \frac{c_{11}r}{\lambda_1}, \qquad \alpha_2 = \alpha_2^{(1)} + \frac{d\alpha_2^{(2)}}{ds}, \qquad \alpha_2^{(1)} = \frac{c_{22}}{r\lambda_1}\left(\frac{dr}{ds}\right)^2 - r\lambda_1 k_1 k_2 T_2,$$

$$\alpha_2^{(2)} = -\frac{c_{12}+T_1}{\lambda_1}\frac{dr}{ds}, \qquad \alpha_3 = -r(c_{11}k_1 + c_{21}k_2), \qquad \alpha_4 = \alpha_4^{(1)} + \frac{d\alpha_4^{(2)}}{ds},$$

$$\alpha_4^{(1)} = [(c_{21}+T_2)k_1 + c_{22}k_2]\frac{dr}{ds}, \qquad \alpha_4^{(2)} = -r[c_{11}k_1 + (c_{12}+T_1)k_2],$$

$$\beta_1 = \frac{rT_1}{\lambda_1}, \qquad \beta_2 = r\lambda_1\left[(c_{11}-T_1)k_1^2 + (c_{12}+c_{21})k_1k_2 + (c_{22}-T_2)k_2^2\right],$$

$$\beta_3 = -\alpha_3, \qquad \beta_4 = \beta_4^{(1)} + \frac{d\beta_4^{(2)}}{ds}, \qquad \beta_4^{(1)} = [c_{12}k_1 + (c_{22}-T_2)k_2]\frac{dr}{ds},$$

$$\beta_4^{(2)} = rT_1k_1, \qquad \varepsilon^2 = \mu, \qquad \Delta l = w(0).$$

One can show that the structure of a direct expansion of solutions to equations (2.8) in the parameter ε corresponds to structure (2.6) established for the system of initial nonlinear equations. The subject of further research is to construct integrals with high variability of a system of linear equations with variable coefficients (2.8). Since a direct expansion is an approximation to some solution of initial equations (2.8), we shall seek boundary layer part of an asymptotics as an approximation to a solution of the relative homogeneous system of equations. In so doing, we shall take into account that functions $r(s)$ and $z(s)$ characterizing the geometry of the shell in the main deformed state are analytic functions in a certain domain and according to (1.12) they are expanded in power series in even and odd powers of s, respectively. On the basis of this fact, we can rewrite the initial homogeneous system of equations as follows:

$$
\begin{aligned}
&s^2\frac{d^2u}{ds^2} + sp_{11}\frac{du}{ds} + sp_{12}\frac{dw}{ds} + q_{11}u + q_{12}w = 0, \\
&\varepsilon^2 s^2\frac{d^2w}{ds^2} + \varepsilon^2\left(sp_{21}\frac{du}{ds} + sp_{22}\frac{dw}{ds}\right) + q_{21}u + q_{22}w = 0,
\end{aligned}
\tag{2.9}
$$

where $p_{ij}(s)$ and $q_{ij}(s)$ are analytic functions for $s \in [0,1]$.

Proceeding from the form of system of equations (2.9), we conclude that this system has singularities both in the parameter and unknown variable. The presence of a small parameter at the higher derivative results in existence of a narrow area in a neighbourhood of the point $s = 1$, in which solutions of the system have large gradients. The point $s = 0$, in its turn, is a regular point for equations (2.9), that assumes the relative asymptotics of solutions when $s \to 0$. Since our final goal is to construct an algorithm for solving a problem with the uniform convergence in the parameter ε, in what follows it is necessary to take into account both of these facts.

We have not succeededin constructing solutions a singular perturbed equation, with a regular singular point, that belongs to the class of equations under consideration containing an exponent factor [10]. Simplified equations obtained from the initial ones while only the first terms of expansion of their coefficients remain unchanged, may serve as a guiding line for an asymptotic representation of solutions to system (2.9). In this case, we can establish that a regular solution of these equations is expressed by Bessel functions of order zero and their first derivatives. According to what has been said, we seek solutions of homogeneous system of equations (2.8) with a large variability index in the following form:

$$
\begin{aligned}
&u = u_1(s,\varepsilon)J_0(x) + u_2(s,\varepsilon)J_0'(x), \qquad w = w_1(s,\varepsilon)J_0(x) + w_2(s,\varepsilon)J_0'(x), \\
&x = \frac{\tau}{\varepsilon}, \qquad \tau = \int\limits_0^s P(t)dt.
\end{aligned}
\tag{2.10}
$$

Here $u_i(s,\varepsilon)$, $w_i(s,\varepsilon)$ $(i = 1,2)$ and $P(t)$ are functions to be determined.

The advisability of representing solutions in the form (2.10) will be warranted if functions $u_i(s,\varepsilon)$ and $w_i(s,\varepsilon)$ that we obtain are regular and can be presented as a direct expansion in the parameter ε.

Having substituted expressions (2.10) into homogeneous system of equations (2.8) and set equal to zero coefficients of $J_0(x)$ and $J_0'(x)$, we get four equations for the functions

$u_i(s,\varepsilon)$, $w_i(s,\varepsilon)$ and $P(s)$:

$$\begin{aligned}
&L_{11}(u_1) + L_{22}(w_1) + (\alpha_1 P u_2)'/\varepsilon + (u_1 P^2 \alpha_1)/\varepsilon^2 \\
&\qquad +\alpha_1 \left(u_2' P - \frac{u_2 P^2}{\tau} \right) /\varepsilon - (\alpha_3 w_2 P)/\varepsilon = 0, \\
&L_{11}(u_2) + L_{12}(w_2) + (\alpha_1 P u_2/\tau)' - (\alpha_1 P u_1)'/\varepsilon + (\alpha_3 P w_1)/\varepsilon - (\alpha_3 P w_2)/\tau \\
&\qquad +\alpha_1 u_2 P^2 (1/\varepsilon^2 - 1/\tau^2) + P\alpha_1 (u_1 P/\tau - u_1')/\varepsilon + u_2' P \alpha_1/\tau = 0, \\
&\varepsilon^2 \left[L_{21}(u_2) + L_{22}(w_2) + (\beta_1 w_2 P/\tau)' + P(\beta_1 w_2' - \beta_3 u_2)/\tau \right] + r w_2 r' + r u_2 z' \\
&\qquad +\varepsilon \left[\beta_3 u_1 P - (\beta_1 w_1 P)' + \beta_1 P (w_1 P/\tau - w_1') \right] + \beta_1 P^2 w_2 (1 - \varepsilon^2/\tau^2) = 0, \\
&\varepsilon^2 [L_{21}(u_1) + L_{22}(w_1)] + \varepsilon \left[(\beta_1 w_2 P)' + \beta_1 P (w_2' - w_2 P/\tau) - \beta_3 u_2 P \right] \\
&\qquad +w_1 P^2 \beta_1 + r w_1 r' + r u_1 z' = 0.
\end{aligned} \tag{2.11}$$

We shall seek a solution to system of equations (2.11) in the form

$$\begin{aligned}
u_1 &= \sum_{k=0}^{\infty} \varepsilon^{2k+2} u_1^{(k)}(s), \qquad u_2 = \sum_{k=0}^{\infty} \varepsilon^{2k+1} u_2^{(k)}(s), \\
w_1 &= \sum_{k=0}^{\infty} \varepsilon^{2k} w_1^{(k)}(s), \qquad w_2 = \sum_{k=0}^{\infty} \varepsilon^{2k+1} w_2^{(k)}(s).
\end{aligned} \tag{2.12}$$

To determine functions $u_i^{(k)}$, $w_i^{(k)}$ and $P(s)$ we substitute expansions (2.12) into equations (2.11) and set coefficients of the same degrees of ε equal to zero.

In the zeroth-order approximation, i.e., for functions $u_i^{(0)}$ and $w_i^{(0)}$, we have

$$\begin{aligned}
&P^2 \alpha_1 u_1^{(0)} - \alpha_3 P w_2^{(0)} = \psi, \qquad P\alpha_1 u_2^{(0)} + \alpha_3 w_1^{(0)} = 0, \qquad (P^2 \beta_1 + r r') w_1^{(0)} = 0, \\
&-\frac{d}{ds} \left(\beta_1 P w_1^{(0)} \right) + \beta_1 P \left(\frac{P w_1^{(0)}}{\tau} - \frac{d w_1^{(0)}}{ds} \right) + (P^2 \beta_1 + r r') w_2^{(0)} + r z' u_2^{(0)} = 0, \\
&\psi = -L_{12}(w_1^{(0)}) - \frac{d}{ds} \left(\alpha_1 P u_2^{(0)} \right) - \alpha_1 \left(P \frac{d u_2^{(0)}}{ds} - \frac{1}{\tau} u_2^{(0)} P^2 \right).
\end{aligned} \tag{2.13}$$

$$u_2^{(0)} = -\frac{\alpha_3 w_1^{(0)}}{\alpha_1 P}, \qquad P^2 = -\frac{r}{\beta_1} \frac{dr}{ds} = -\frac{\lambda_1^2 \cos\alpha}{T_1}. \tag{2.14}$$

Here α is the angle formed by the outer normal to the deformed shell and its symmetry axis.

It follows from (2.14) that $P(s)$ is an even and pure imaginary function for $-\dfrac{\pi}{2} \le \alpha \le \dfrac{\pi}{2}$. The fourth equation with regard for (2.13) takes the form

$$2\frac{d w_1^{(0)}}{ds} = \left(\frac{P}{\tau} - \frac{1}{\beta_1 P} \frac{d}{ds} (\beta_1 P) + \varphi(s) \right) w_1^{(0)}, \qquad \varphi(s) = \frac{\alpha_3}{\alpha_1} \frac{dz}{ds} \Big/ \frac{dr}{ds}.$$

Hence up to an arbitrary constant, we obtain a regular solution for function $w_1^{(0)}$:

$$w_1^{(0)} = \sqrt{\frac{\tau}{\beta_1 P} \exp \int \varphi(s) ds}. \tag{2.15}$$

Therefore, after the zeroth-order approximation, the function $w_2^{(0)}$ remains a free function that will be determined at the second step of the asymptotic integration of equations (2.11).

Without to finding explicit expressions for terms in expansions (2.12) of higher approximations, one may conclude from the form of system (2.11) with regard to expressions for its coefficients, that $u_2^{(k)}(s)$ and $w_1^{(k)}(s)$ are even functions, and $u_1^{(k)}(s)$ and $w_2^{(k)}(s)$ are odd ones. Specific expressions for these functions after the zeroth-order and first approximations show that they are regular for $s = 0$.

Putting together the results obtained, we conclude that a solution to system (2.8), bounded for $s = 0$ and having the large variability, can be presented in the form of the following formal expansions:

$$\begin{aligned} u(s,\varepsilon) &= I_0(y) \sum_{k=0}^{\infty} \varepsilon^{2k+2} u_1^{(k)}(s) + I_0'(y) \sum_{k=0}^{\infty} \varepsilon^{2k+1} u_2^{(k)}(s), \\ w(s,\varepsilon) &= I_0(y) \sum_{k=0}^{\infty} \varepsilon^{2k} w_1^{(k)}(s) + I_0'(y) \sum_{k=0}^{\infty} \varepsilon^{2k+1} w_2^{(k)}(s), \end{aligned} \tag{2.16}$$

where I_0 is a Bessel function of the order zero of pure imaginary argument; y is a function regular and odd in argument s.

The constructed asymptotic representations of solutions (2.6) and (2.16) quantitatively present the structure of a solution, that we use in a sequel to construct systems of coordinate functions when solving the initial problem by the method of variations.

Taking into account the asymptotics of expansions (2.16) for $s \to 0$, and also the asymptotics of fucntions $I_n(x)$ for large values of the argument, after passing from normal and tangent movements of the shell to its movements in axial and radial directions, the general form of solutions to the initial nonlinear problem can be presented as follows:

$$\begin{aligned} z(s) &= a_0 + \varphi_p(s)(a_1 + a_2 s^2 + a_3 s^4 + \cdots), \\ r(s) &= b_0 s + \varphi_p(s)(b_1 s + b_2 s^3 + b_3 s^5 + \cdots), \\ \varphi_p(s) &= I_0 \left(\sum_{k=0}^{n} p_k s^{2k-1} \right), \end{aligned} \tag{2.17}$$

where a_i, b_i and p_i are arbitrary constants.

Subjecting expressions (2.17) to boundary conditions (1.5), coordinate systems $\{u_k(s)\}$ and $\{v_k(s)\}$ take the form

$$\begin{aligned} &u_1(s,p_i) = \left(1 - \frac{\varphi_p(s)}{\varphi_p(1)}\right), \qquad u_2(s,p_i) = \frac{\varphi_p(s)}{\varphi_p(1)}(s^2-1), \\ &u_k(s,p_i) = s^2 u_{k-1}(s,p_i) \qquad (k = 3,4,\ldots), \qquad v_k(s,p_i) = s u_k(s,p_i). \end{aligned} \tag{2.18}$$

If constants p_i are known, then solving nonlinear algebraic system (1.9) by means of the iteration Newton method, we can construct an approximate solution for the problem. However, the question of specific values of parameters p_i for the nonlinear boundary problem under consideration remains open. Hence, from the conditions $\partial I/\partial p_i = 0$, $i = 1, 2, \ldots, n$ we can get n additional equations with respect to parameters p_i. These equations are of the following form:

$$\int_0^1 \left[U(\lambda_1, \lambda_2) \frac{dz}{ds} \frac{d}{ds} \left(\frac{dz}{dp_i} \right) - Q\lambda_2 \frac{dr}{ds} \frac{dz}{dp_i} + U(\lambda_1, \lambda_2) \frac{dr}{ds} \frac{d}{ds} \left(\frac{dr}{dp_i} \right) \right.$$
$$\left. + \left(U(\lambda_2, \lambda_1) \frac{\lambda_2}{s} + Q\lambda_2 \frac{dz}{ds} \right) \frac{dr}{dp_i} \right] s\, ds. \tag{2.19}$$

It is convenient to look for a solution of system (2.19) by the generalized secant method [11]. The whole algorithm is reduced to that at each step of changing parameters p_i (outer iteration process) one needs to solve nonlinear algebraic system (1.9).

Therefore, unlike the traditional Rietz method of solving nonlinear boundary problems, a peculiarity of the given approach is that at the expense of choosing parameters p_i that characterize the variability of required solutions, by using a computer, we construct a system of coordinate functions, that approximates solutions to the initial problem in a way optimal in some sense.

In conclusion, let us present results of calculations, giving an idea of possibilities and effectiveness of the algorithm proposed. To illustrate the convergence of successive approximations, in Table 2 we write out values of functions $z(s)$ and $r(s)$ and their first two derivatives depending on the number m of terms in expansions (1.7) at the point $s = 0.9$. In the last column we give the relative error of an approximate solution for problem (1.14). Parameters of the hydrostatic load in the coordinate system associated with the center of the nondeformed membrane are $C = 0.5$, $D = 10$. Ratios of constants in elastic potential (1.2) are chosen as follows:

$$\Gamma_1 = 0.1, \qquad \Gamma_2 = \Gamma_3 = 0.$$

All calculations are performed with regard for only one parameter P_1 in coordinate systems (2.18).

Table 2

m	$z \cdot 10$	$-z'$	$-z''$	r	r'	$-r''$	Δ
1	0.32896	0.18816	1.9342	0.90532	0.97550	0.38008	$4 \cdot 10^{-1}$
2	0.36718	0.18415	2.4584	0.90694	0.99159	0.46372	$1 \cdot 10^{-2}$
3	0.36533	0.18145	2.3754	0.90692	0.99250	0.42182	$1 \cdot 10^{-2}$
4	0.36451	0.17825	2.3497	0.90693	0.99278	0.41433	$3 \cdot 10^{-4}$
5	0.36450	0.17830	2.3498	0.90693	0.99276	0.41456	$2 \cdot 10^{-4}$
6	0.36448	0.17841	2.3461	0.90693	0.99275	0.41404	$3 \cdot 10^{-5}$

Solutions possess also the analogous convergence at other points from the interval of integration of the initial equations. Here the high precision of an approximate solution is

Figure 2

achieved at the expense of both increasing the dimension of algebraic system (1.9) and choosing parameters of the optimal coordinate system. It turns out that it is sufficient to restrict oneself to one parameter p_1 only. Calculations show that requirements to the precision of determining parameters p_i concern with input data of the hydrostatic load with high singularity in solutions of the problem only. As the width of the boundary layer increases, the rate of convergence of the algorithm is preserved, but influence of the precision of determining parameters for the optimal coordinate system on the final result decreases. There comes a moment when for some value of the hydrostatic load we do not need any more to organize the outer iteration process for seeking for values of parameters p_i. In this case, the rate of convergence of the given algorithm coincides with the rate of convergence of the algorithm based on the use of an exponent basis. It is worth noting that a polynomial basis for the example considered above allows one to obtain the relative error Δ for the first eight terms in expansions (1.7), that is equal to $\approx 10^{-2}$. The further increase of the dimension of algebraic system (1.9) in this case leads to the loss of the stability of calculations.

While performing the calculating experiment, it came out that functional (1.4) in the space of parameters p_i had only one minimum. This means that for the approximation under consideration there exists only one optimal coordinate system, which considerably simplify its search by means of a computer.

Therefore, the proposed approach allows one to obtain approximate solutions with a suffuciently high degree of precision for the small dimension of an algebraic system both in the presence of the singular perturbation of a problem and without it.

Meridional cuts of the deformed shell for different values of the hydrostatic load C are presented in Figure 2. The other input data coincide with those used for calculations presented in Table 2. Dashed lines correspond to paths of passage of points of the non-deformed membrane to points of its deformed surface. Curves of dependence of principal degrees of lengthenings λ_1 and stresses T_1 (continuous curves) and also λ_2 and stresses T_2 (dashed curves) upon the parameter s for different values of the parameter C are presented in Figures 3 and 4. From figures one can see that at the starting stage of deforming the membrane, the main part of its surface is nearly plane that is in the homogeneous biaxial

Figure 3

Figure 4

strained state. Only in a neighbourhood of the support contour of the shell its surface and intererior stresses undergo considerable changes. As the constant component of the hydrostatic load C increases, the plate part of the surface gradually decreases.

Since it is convenient to investigate behaviour of material of elastomers for large deformations by means of experiments on homogeneous biaxial extension of a thin sheet [1] (that often causes some difficulties when realizing such a type of deformations for experimental tests), the presence of the homogeneous strained state in the central part of the deformed membrane under the load of heavy liquid may be used in establishing a functional dependence of an elastic potential upon deformation invariants.

Acknowledgements

The author is grateful to the DFG for partial financial support.

References

[1] Green A. and Adkins J., Large Elastic Deformations and Nonlinear Mechanics of Solid Medium, Moscow, Mir, 1965, 456 p. (in Russian: Translation from English).

[2] Gospodarikov A.B., Terent'ev V.F. and Chernyh K.F., Numerical Investigation of Some Applied Problems of Elastic Statics for Axisymmetrical Membrane of Elastomers, *Mechanics of Deformed Media*, 1979, N 3, 6–10 (in Russian).

[3] Solodilov Yu.I., Large Deformations of a Spherical Shell Loaded by the Pressure of Liquid, *Engineering J. Mechanics of Solid*, 1966, N 4, 151–153 (in Russian).

[4] Trushina V.M., Large Deformations and Carring Capacity of a Shell of the Initial Cylindrical Form, *Izvestiya AN SSSR. Mechanics of Solid*, 1971, N 5, 193–199 (in Russian).

[5] Oden J., Finite Elements of Nonlinear Continua, Moscow, Mir, 1976, 464 p. (in Russian: Translation from English).

[6] Yang W. and Feng W., On Axisymmetrical Movements of Nonlinear Momentless Shells, *Applied Mechanics. Ser. E*, 1970, V.37, N 4, 108–115 (in Russian).

[7] Biderman V.L., Problems of Calculating Rubber Details, *Calculations on Strength*, 1958, Issue 3, 40–87 (in Russian).

[8] Trotsenko V.A., Axisymmetrical Problem on Equilibrium of a Circle Membrane under the Hydrostatic Load, in: Physics and Technical Applications of Boundary Problems, Kyiv, Naukova Dumka, 1978, 126–140 (in Russian).

[9] Doolan E., Miller J. and Schilders W., Uniform Numerical Methods for Solving Problems with Boundary Layers, Moscow, Mir, 1983, 200 p. (in Russian: Translation from English).

[10] Lomov S.A. and Yudina A.S., Structure of the Fundamental System of Solutions for a Singular Perturbed Equation with a Regular Singular Point, *Izvestiya AN SSSR. Math.*, 1982, V.46, N 5, 1124–1134 (in Russian).

[11] Ortega J. and Rheinboldt W., Iterative Methods for Solving Nonlinear Systems of Equations in Several Variables, Moscow, Mir, 1975, 558 p. (in Russian: Translation from English).