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Resonance Broadening Theory of Farley-Buneman Turbulence in the Auroral E-Region

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Abstract

The conventional theory of resonance broadening for a two-species plasma in a magnetic field is revised, and applied to an ionospheric turbulence case. The assumptions made in the conventional theory of resonance broadening have, in the past, led to replacing the frequency ω by $\omega + ik_\perp^2 D^*$ in the resonant part of the linear dielectric function to obtain the nonlinear dielectric function. Where D^* is an anomalous diffusion coefficient due solely to wave scattering of the particle orbits. We show that in general these assumptions are not valid, and consequently the straightforward substitution of frequencies is not legitimate. We remedy these problems and derive expressions for the time-dependent components of the diffusion tensor. The improved resonance broadening theory is developed in the context of an ionospheric problem, namely that of the Farley-Buneman turbulence in the auroral E-region. A kinetic description of the electrons is used. A general expression for the nonlinear dielectric function is derived in the special case where no parallel electric field is present, and the differences with the conventional dispersion relation are discussed.

1 Introduction

The problem of wave-particle interaction has always played a critical role when one tries to understand the saturation mechanisms for plasma instabilities. Linear theory, which is a single wave theory, which does not take into account the wave-particle interaction fails to conserve energy and momentum, and consequently does not and should not predict saturation. This is unphysical. But it is important to remember that the linear theory ceases to be valid after a trapping time; the time neccessary to a charged particle to bounce near the bottom of a potential well of a finite-amplitude wave. Consequently, the predictions of linear theory beyond the trapping time are not valid. A first remedy to this fundamental problem is provided by the so called Quasilinear theory, a weak turbulence theory which takes into consideration the wave-particle interaction and requires a slow time dependence of the background particle distribution function. The ensemble averaged distribution function is then a solution to a Fokker-Planck equation. In the absence of sources and sinks, quasilinear theory is then described through a diffusion equation which predicts saturation of the instabilities when the background distribution function becomes constant, or "plateaus", along the diffusion paths which are the characteristics of the partial differential equation. However, in the presence of sources and sinks, such as a background electric field or collisions, there is no saturation, since the sources and sinks tend to destroy the plateau, and consequently the unstable waves keep on growing until they become large enough for the other nonlinear processes to enter the picture. This is the case of the problem we have elected to address, namely that of the modified two stream Farley-Buneman instability, where the source of free energy is the electrojet and the sink is due to the collisions of the electrons with the background neutrals. In other words quasilinear theory does not provide an ultimate saturation mechanism much needed to describe steady state turbulence and predict reasonable saturation amplitudes. Nevertheless the quasilinear theory has stood the test of time when it comes to predicting the onset of instabilities to a certain degree of accuracy. A first attempt to improve on the quasilinear theory was described by Dupree [5] and Weinstock [20] for an unmagnetized plasma and by Dupree [6] and Dum and Dupree [7] for a magnetized plasma. Dupree [5] derived a perturbation theory based on the knowledge of the electric field which allowed him to find the exact particle orbits which were then used in the perturbation solutions of the Vlasov equation. The principal result of this improved perturbation theory is a broadening of the wave-particle resonance function, which in the conventional quasilinear theory is $\delta(\omega - \mathbf{k}\cdot\mathbf{v})$. This result has been the subject of many debates on strong turbulence theories in the past two decades. The validity of some of the assumptions made by Dupree [5] was questioned by a number of authors (Cook and Sanderson [3], Rolland [15]). A number of authors (Salat [18], Ishihara and Hirose [10], Ishihara *et al.* [11, 12] have since taken the task to study the problem of resonance broadening very thoroughly, and were able to show successfully the shortcomings of the conventional resonance broadening approach. They all addressed the resonance broadening problem in the case of unmagnetized plasmas, as well as the case related to drift waves in a shear magnetic field because of the mathematical tractability and its reduction to a form similar to the unmagnetized case. Kleva and Drake [21] addressed the problem of stochastic $\mathbf{E}\times\mathbf{B}$ particle transport using a dynamical system approach. More recently Kleva [14] investigated the problem of energy transport in a magnetically confined plasma by $\mathbf{E}\times\mathbf{B}$ flow generated by a spectrum of electrostatic waves stil using a dynamical system's approach. He found like Ishihara *et al.* [12] that the diffusion coefficient scales like $E^{4/3}$ instead of E^2 as predicted by the conventional quasilinear theory. The problem addressed by Dupree [6] and Dum and Dupree [7], that of the broadening of the wave-particle resonance in the presence of background magnetic field, has not been addressed since probably because of its complexity.

In this paper we have investigated the problem of resonance broadening for a turbulent magnetized plasma and compared our results to the classical calculation of Dum and Dupree [7]. We have investigated the implications of the improved results on the Farley-Buneman instability that occurs in ionospheric plasmas, and shown that one cannot obtain the nonlinear dielectric function by just substituting the frequency ω by $\omega + ik_\perp^2 D^*$ in the resonant part of the dielectric function. In Section 2 we describe the mathematical model used to investigate the Farley-Buneman instability, and derive the different components of the diffusion tensor. In Section 3 we derived the generalized dispersion relation, and finally in Section 4 we properly reinterpret the thresholds conditions for the Farley-Buneman instability in the absence of a parallel electric field.

2 The Mathematical Model

2.1 The Quasi-Linear Approximation

2.1.1 The Ion Description

The ions are assumed to be highly collisional in the region of interest, namely the auroral E-region, and unmagnetized. The ion convection is also assumed to be negligible, i.e., the nonlinear ion terms are neglected.

With these assumptions a linear fluid model is adopted for the ions, and is best described by the linearized momentum and continuity equations. Following Sudan [19] we write

$$
\begin{aligned}
m_i \frac{\partial \delta \mathbf{v}_i}{\partial t} &= -e\nabla\phi - \frac{T_i}{n_0}\nabla\delta n - \nu_{in} m_i \delta\mathbf{v}_i, \\
\frac{\partial \delta n}{\partial t} &+ n_0 \nabla \cdot \delta\mathbf{v}_i = 0
\end{aligned}
\tag{2.1}
$$

operating on the continuity equation (2.1) with $\left(\frac{\partial}{\partial t} + \nu_{in}\right)$ leads to

$$
\left(\frac{\partial}{\partial t} + \nu_{in}\right)\frac{\partial \delta n}{\partial t} = -n_0 \nabla \cdot \left(\frac{\partial}{\partial t} + \nu_{in}\right)\delta\mathbf{v}_i = n_0 \nabla \cdot \left\{\frac{e}{m_i}\nabla\phi + \frac{T_i}{n_0 m_i}\nabla\delta n\right\}. \tag{2.2}
$$

Equation (2.2) allows us to express the density fluctuation in terms of the electric field. Taking the Fourier transform in space and time of equation (2.2) leads to

$$
\left(\omega^2 - k^2\frac{T_i}{m_i} + i\nu_{in}\omega\right)\delta n_{\mathbf{k}\omega} = \frac{n_0 e}{m_i}k^2\phi_{\mathbf{k}\omega}. \tag{2.3}
$$

This equation will eventually be used in Poisson's equation to determine the dispersion relation.

2.1.2 The Electron Description

The electrons are described by the Vlasov equation with a relaxation model for collision operator. The electron distribution function satisfies the following equation

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla - \frac{e}{m_e}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}}\right) f_e(\mathbf{x},\mathbf{v},t) = \\
&\qquad -\nu_{en}\left(f_e(\mathbf{x},\mathbf{v},t) - \frac{n(\mathbf{x},t)}{n_0}f_0\right),
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \mathbf{E}_0 + \delta\mathbf{E}(\mathbf{x},t), \\
n(\mathbf{x},t) &= \int d\mathbf{v} f_e(\mathbf{x},\mathbf{v},t).
\end{aligned}
\tag{2.5}
$$

The electric field has been separated into two parts; $\mathbf{E}_0$ represents the electrojet background electric field, and $\delta\mathbf{E}$ the fluctuating field.

This leads to the following distribution function, which is in turn separated into a weakly space and tine dependent average distibution function $\langle f_e(\mathbf{x}, \mathbf{v}, t)\rangle$, and a fluctuating part $\delta f_e(\mathbf{x}, \mathbf{v}, t)$, that is

$$f_e(\mathbf{x}, \mathbf{v}, t) = \langle f_e(\mathbf{x}, \mathbf{v}, t)\rangle + \delta f_e(\mathbf{x}, \mathbf{v}, t). \tag{2.6}$$

We should point out that both the electric field and the distribution function decompositions are exact. One can easily write equation (2.4) in a compact form

$$\left(\frac{\partial}{\partial t} + i\mathcal{L}(\mathbf{x}, \mathbf{v}, t)\right) f_e(\mathbf{x}, \mathbf{v}, t) = \frac{e}{m_e}\delta\mathbf{E}(\mathbf{x}, t) \cdot \nabla_{\mathbf{v}} f_e(\mathbf{x}, \mathbf{v}, t), \tag{2.7}$$

where the operator $\mathcal{L}$ is given by:

$$i\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = \mathbf{v} \cdot \nabla - \frac{e}{m_e}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}} + \nu_{en} - \nu_{en}\frac{f_0}{n_0}\int d\mathbf{v}.$$

Note that the collision operator is a linear operator, and consequently its incorporation into the collisionless Vlasov equation is not as complex as if we would have used a Fokker-Plank or any more sofisticated collision operator.

Equation (2.7) can be solved using different techniques. In the quasi-linear approximation one considers only the wave particle interaction and neglects any other effects, such as wave coupling and radiation effects which are nonlinear effects of higher order in a perturbation analysis based on the amplitude of the wave electric field.

To solve the quasi-linear problem we start by ensemble averaging the collisional Valsov equation to obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nu_{en} + \mathbf{v} \cdot \nabla - \frac{e}{m_e}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}}\right)\langle f_e\rangle \\ &\quad = \nu_{en} f_0 + \frac{e}{m_e}\nabla_{\mathbf{v}} \cdot \langle\delta\mathbf{E}\delta f_e\rangle.\end{aligned} \tag{2.8}$$

The next step consists of substituting the expression for the distribution function into the collisional Vlasov equation to obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nu_{en} + \mathbf{v} \cdot \nabla - \frac{e}{m_e}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}}\right)(\langle f_e\rangle + \delta f_e) - \frac{e}{m_e}\delta\mathbf{E} \cdot \nabla_{\mathbf{v}}\langle f_e\rangle \\ &\quad = \nu_{en} f_0 + \nu_{en}\frac{f_0}{n_0}\delta n + \frac{e}{m_e}\nabla_{\mathbf{v}} \cdot (\delta\mathbf{E}\delta f_e),\end{aligned} \tag{2.9}$$

where δn is defined as

$$\delta n = \int d\mathbf{v}\delta f_e.$$

Taking the difference between equation (2.9) and equation (2.8) leads to

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nu_{en} + \mathbf{v} \cdot \nabla - \frac{e}{m_e}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}}\right)\delta f_e = \\ &\quad -\frac{e}{m_e}\delta\mathbf{E} \cdot \nabla_{\mathbf{v}}\langle f_e\rangle - \nu_{en}\frac{f_0}{n_0}\delta n = \frac{e}{m_e}\nabla_{\mathbf{v}} \cdot \{\delta\mathbf{E}\delta f_e - \langle\delta\mathbf{E}\delta f_e\rangle\}.\end{aligned} \tag{2.10}$$

The quasi-linear approximation consists of neglecting the right hand side of equation (2.10) which describes the nonlinear mode coupling terms. When this term is neglected, the equation to lowest order in the electric field becomes

$$\left(\frac{\partial}{\partial t} + \nu_{en} + \mathbf{v}\cdot\nabla - \frac{e}{m_e}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}}\right)\delta f_e = \nu_{en}\frac{f_0}{n_0}\delta n + \frac{e}{m_e}\delta\mathbf{E}\cdot\nabla_{\mathbf{v}}\langle f_e\rangle.$$

This equation can be solved formally using Fourier transforms and defining the following operator

$$\mathbf{G}_{\mathbf{k}\omega}(\mathbf{v}) = \left\{-i\omega + \nu_{en} + i\mathbf{k}\cdot\mathbf{v} - \frac{e}{m_e}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}}\right\}^{-1}.$$

The expression for the fluctuating distribution function can then be written in the following form

$$\delta f_{e\mathbf{k}\omega}(\mathbf{v}) = \frac{e}{m_e}\delta\mathbf{E}_{\mathbf{k}\omega}\cdot\mathbf{G}_{\mathbf{k}\omega}\nabla_{\mathbf{v}}\langle f_e\rangle + \nu_{en}\frac{\delta n_{\mathbf{k}\omega}}{n_0}\mathbf{G}_{\mathbf{k}\omega}f_0. \tag{2.11}$$

Integrating over velocity space one obtains an expression for the density fluctuation which when substituted into expression (2.11) for the fluctuating distribution function leads to

$$\delta f_{e\mathbf{k}\omega} = \frac{e}{m_e}\delta\mathbf{E}_{\mathbf{k}\omega}\cdot\mathbf{G}_{\mathbf{k}\omega}\nabla_{\mathbf{v}}\langle f_e\rangle + \nu_{en}\frac{\frac{e}{m_e}\delta\mathbf{E}_{\mathbf{k}\omega}\cdot\int d\mathbf{v}\mathbf{G}_{\mathbf{k}\omega}\nabla_{\mathbf{v}}\langle f_e\rangle}{n_0 - \nu_{en}\int d\mathbf{v}\mathbf{G}_{\mathbf{k}\omega}f_0}\mathbf{G}_{\mathbf{k}\omega}f_0 \tag{2.12}$$

when substituting expression (2.12) into the equation governing the evolution of the average distribution equation one obtains

$$\frac{\partial\langle f_e\rangle}{\partial t} = \frac{\partial}{\partial\mathbf{v}}\cdot\underline{\underline{\mathbf{D}}}\cdot\frac{\partial\langle f_e\rangle}{\partial\mathbf{v}} - \frac{\partial}{\partial\mathbf{v}}\cdot\underline{\mathbf{F}}\langle f_e\rangle.$$

In other words one obtains a Fokker-Planck equation with the diffusion and drag coefficients defined as follows

$$\underline{\underline{\mathbf{D}}} = \frac{e^2}{m_e^2}\left\langle\delta\mathbf{E}\delta\mathbf{E}\mathbf{G}\right\rangle,$$

$$\underline{\mathbf{F}}\langle f_e\rangle = -\frac{e}{m_e}\mathbf{E}_0\langle f_e\rangle - \frac{e^2}{m_e^2}\nu_{en}\left\langle\delta\mathbf{E}\delta\mathbf{E}\mathbf{G}\cdot\frac{\int d\mathbf{v}\mathbf{G}\frac{\partial\langle f_e\rangle}{\partial\mathbf{v}}}{n_0 - \nu_{en}\int d\mathbf{v}\mathbf{G}f_0}f_0\right\rangle.$$

It is clear that the drag term in the Fokker-Planck equation is solely due to the background electric field of the electrojet and to the electron-neutral collisions.

In the absence of sources or collisions the conventional quasi-linear theory predicts a saturation of the fluctuating fields when the distribution function becomes constant along the diffusion paths. That is, the quasi-linear theory predicts a zero growth rate. This however, is valid only when there are no sources or collisions; when such effects are present, like in our case, the distribution function never plateaus along the diffusion paths since the source and the collisions tend to destroy the plateau, which in turn leads to a non zero growth rate; the oscillation amplitude continues to grow until nonlinear processes enter the picture. In other words, there is no ultimate saturation of the fluctuations through the quasilinear process.

This leads us to conclude that the conventional quasi-linear theory is not the ultimate stabilization mechanism, and improvements on the theory are needed. One possible theory that has been suggested, as a first attempt to remedy the problem from which the quasi-linear theory suffers, is the resonance broadening theory which was first introduced by Dupree [5], and applied by a number of authors, see for example Sudan [19], Robinson [16], and Robinson and Honary [17], to the problem of irregularities in the E region.

In the next section we shall develop the resonance broadening in some details. We will also discuss, to a certain extent, the validity limits of the conventional resonance broadening. We remedy some of the problems, and show the shortcomings of this wave particle interaction, as well as the need to provide for a complete theory of plasma turbulence. The latter will be addressed in a companion paper describing a fully selfconsistent kinetic theory for the Farley-Buneman instability.

2.2 The Resonance Broadening Approximation

In this section we shall address the problem of wave particle interaction in a magnetized plasma through a nonlinear formalism that includes the nonlinear effects of the waves on the particle orbits, but not vice versa. The electric field and the corresponding distribution function are decomposed according to equations (2.5) and (2.6).

The fundamental assumption and the goal of the resonance broadening theory is to evaluate the modification of the quasilinear resonance between the particles and waves. This theory neglects the coherent contributions which arise from the coupling between the waves and the background oscillations, as well as interactions between the background oscillations.

We define the following operator $\mathcal{L}$ is then defined as follows

$$i\mathcal{L}(\mathbf{x},\mathbf{v},t) = \mathbf{v}\cdot\nabla - \frac{e}{m_e}\left(\mathbf{E}_0 + \delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} + \mathcal{O}^{coll}(\mathbf{x},\mathbf{v},t),$$

where the collision operator $\mathcal{O}^{coll}$ is given by

$$\mathcal{O}^{coll}(\mathbf{x},\mathbf{v},t) = \nu_{en} - \nu_{en}\frac{f_0(\mathbf{v})}{n_0}\int d\mathbf{v}'$$

which allows us to rewrite equation (2.4) in a compact form

$$\left(\frac{\partial}{\partial t} + i\mathcal{L}(\mathbf{x},\mathbf{v},t)\right) f_e(\mathbf{x},\mathbf{v},t) = 0.$$

The equation describing the evolution of the ensemble averaged electron distribution function is given by

$$\left(\frac{\partial}{\partial t} + i\mathcal{L}_0(\mathbf{x},\mathbf{v},t)\right)\langle f_e(\mathbf{x},\mathbf{v},t)\rangle = \frac{e}{m_e}\nabla_{\mathbf{v}}\cdot\langle\delta\mathbf{E}(\mathbf{x},t)\delta f_e(\mathbf{x},\mathbf{v},t)\rangle,$$

where $\mathcal{L}_0$ is a linear operator defined by

$$i\mathcal{L}_0(\mathbf{x},\mathbf{v},t) = \mathbf{v}\cdot\nabla + \left(-\frac{e}{m_e}\mathbf{E}_0 + \mathbf{\Omega}_e\times\mathbf{v}\right)\cdot\nabla_{\mathbf{v}} + \nu_{en} - \nu_{en}\frac{f_0(\mathbf{v})}{n_0}\int d\mathbf{v}'.$$

The equation governing the evolution of the fluctuating part of the electron distribution function can be written as follows

$$\left(\frac{\partial}{\partial t} + i\mathcal{L}(\mathbf{x},\mathbf{v},t)\right)\delta f_e(\mathbf{x},\mathbf{v},t) = \frac{e}{m_e}\delta\mathbf{E}(\mathbf{x},t)\cdot\nabla_{\mathbf{v}}\langle f_e(\mathbf{x},\mathbf{v},t)\rangle, \tag{2.13}$$

where as one can see the nonlinear term $\delta\mathbf{E}\delta f_e$. This term will take into account the effects of the waves on the electrons in the resonance broadening approximation, but will not address or investigate the effects of the electrons on the waves, neither does it take into consideration the wave-wave interaction. The "Resonance Broadening" approximation assumes that the quadratic nonlinearity leads to a nonlinear correction to the particle orbits, which should be taken into account when the fields become large enough.

Equation (2.13) can be solved using a Green's function analysis, i.e., we define the Green's function $\mathcal{G}$ which satisfies

$$\left(\frac{\partial}{\partial t} + i\mathcal{L}(\mathbf{x},\mathbf{v},t)\right)\mathcal{G}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t') = \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{v}-\mathbf{v}')\delta(t-t'). \tag{2.14}$$

Following Ishihara *et al.* [12] the operator $i\mathcal{L}$ can be rewritten as follows

$$\begin{aligned} i\mathcal{L} &= i\mathcal{L}_0 + i\mathcal{L}_1, \\ i\mathcal{L}_0 &= \mathbf{v}\cdot\nabla + \left(-\frac{e}{m_e}\mathbf{E}_0 + \boldsymbol{\Omega}_e\times\mathbf{v}\right)\cdot\nabla_{\mathbf{v}} + \mathcal{O}^{coll}, \\ i\mathcal{L}_1 &= -\frac{e}{m_e}\delta\mathbf{E}(\mathbf{x},t)\cdot\nabla_{\mathbf{v}}. \end{aligned} \tag{2.15}$$

The electron distribution function is then obtained

$$f_e(\mathbf{x},\mathbf{v},t) = \int d\mathbf{x}'d\mathbf{v}'\mathcal{G}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t_0)f_e(\mathbf{x}',\mathbf{v}',t_0).$$

Before we get into the details of the derivation, we should point out that throughout the calculation we will omit the collisions. We will discuss the impact of collisions on the dispersion relation when we apply the results to the case of Farley-Byneman turbulence using the relaxation model for the collision operator already shown above.

We now separate the Green's function into two parts; an unperturbed part $\mathcal{G}^{(0)}$ and a perturbed one $\mathcal{G}^{(1)}$, i.e.,

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(1)} \tag{2.16}$$

satisfying the following equations

$$\left(\frac{\partial}{\partial t} + i\mathcal{L}_0(\mathbf{x},\mathbf{v},t)\right)\mathcal{G}^{(0)}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t') = \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{v}-\mathbf{v}')\delta(t-t'). \tag{2.17}$$

The solution to equation (2.17) can be written in the following formal form

$$\mathcal{G}^{(0)}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t') = \Theta(t-t')\delta(\mathbf{x}(t)-\mathbf{x}'(t'))\delta(\mathbf{v}(t)-\mathbf{v}'(t')), \tag{2.18}$$

where $\Theta(t-t')$ is the Heavyside step function. Substracting equation (2.17) from equation (2.14) leads to the following expression for the perturbed part of the Green's function

$$\begin{aligned} &\mathcal{G}^{(1)}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t') \\ &\quad = -i\int_{t'}^{t} dt''\int d\mathbf{x}''\int \mathbf{v}''\mathcal{G}^{(0)}(\mathbf{x},\mathbf{v},t;\mathbf{x}'',\mathbf{v}'',t'')\mathcal{L}_1(\mathbf{x}'',\mathbf{v}'',t'')\mathcal{G}(\mathbf{x}'',\mathbf{v}'',t'';\mathbf{x}',\mathbf{v}',t'). \end{aligned}$$

Define the average Green's function $G(\mathbf{x},\mathbf{v},t;t')$ as follows

$$G(\mathbf{x},\mathbf{v},t;t') = \int d\mathbf{x}' \int d\mathbf{v}' \, \langle \mathcal{G}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t')\rangle,$$

where $\langle\cdots\rangle$ represent an ensemble average. Taking the ensemble average of equation (2.14) and integrating over $\mathbf{x}'$ and $\mathbf{v}'$ leads to the following equation

$$\frac{\partial G(\mathbf{x},\mathbf{v},t;t')}{\partial t} + \int d\mathbf{x}' \int d\mathbf{v}' \, \langle i\mathcal{L}(\mathbf{x},\mathbf{v},t)\mathcal{G}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t')\rangle = \delta(t-t'). \tag{2.19}$$

Substituting equations (2.15), (2.16), (2.18) and (2.17) into equation (2.19) we obtain

$$\begin{aligned}\frac{\partial G(\mathbf{x},\mathbf{v},t;t')}{\partial t} &+ \int d\mathbf{x}' \int d\mathbf{v}' \int_{t'}^{t} dt'' \int d\mathbf{x}'' \int d\mathbf{v}'' \\ &\times\langle \mathcal{L}_1(\mathbf{x},\mathbf{v},t)\mathcal{G}^{(0)}(\mathbf{x},\mathbf{v},t;\mathbf{x}'',\mathbf{v}'',t'')\mathcal{L}_1(\mathbf{x}'',\mathbf{v}'',t'')\mathcal{G}(\mathbf{x}'',\mathbf{v}'',t'';\mathbf{x}',\mathbf{v}',t')\rangle.\end{aligned} \tag{2.20}$$

Following the work of Ishihara *et al.* [12] and references therein we can simplify equation (2.20) after making the following approximations in order to evaluate the second term on the left hand side. We assume that the process under consideration is Gaussian, which leads to

$$\begin{aligned}&\int d\mathbf{x}' \int d\mathbf{v}' \, \langle \mathcal{L}_1(\mathbf{x},\mathbf{v},t)\mathcal{G}^{(0)}(\mathbf{x},\mathbf{v},t;\mathbf{x}'',\mathbf{v}'',t'')\mathcal{L}_1(\mathbf{x}'',\mathbf{v}'',t'')\mathcal{G}(\mathbf{x}'',\mathbf{v}'',t'';\mathbf{x}',\mathbf{v}',t')\rangle \\ &\quad = \langle \mathcal{L}_1(\mathbf{x},\mathbf{v},t)\mathcal{G}^{(0)}(\mathbf{x},\mathbf{v},t;\mathbf{x}'',\mathbf{v}'',t'')\mathcal{L}_1(\mathbf{x}'',\mathbf{v}'',t'')\rangle G(\mathbf{x}'',\mathbf{v}'',t'';t').\end{aligned} \tag{2.21}$$

The direct interaction approximation makes a further approximation and replaces $\mathcal{G}^{(0)}$ in the right hand side of equation (2.21) by $\mathcal{G}$. Finally, the last approximation consists of replacing $G(\mathbf{x}'',\mathbf{v}'',t'';t')$ by $G(\mathbf{x},\mathbf{v},t;t')$, and $\mathcal{L}_1(\mathbf{x}'',\mathbf{v}'',t'')$ by $\mathcal{L}_1(\mathbf{x},\mathbf{v},t'')$ to obtain

$$\begin{aligned}\frac{\partial G(\mathbf{x},\mathbf{v},t;t')}{\partial t} &+ \int_{t'}^{t} dt'' \int d\mathbf{x}'' \int d\mathbf{v}'' \\ &\times\langle \mathcal{L}_1(\mathbf{x},\mathbf{v},t)\mathcal{G}(\mathbf{x},\mathbf{v},t;\mathbf{x}'',\mathbf{v}'',t'')\mathcal{L}_1(\mathbf{x},\mathbf{v},t'')\rangle G(\mathbf{x},\mathbf{v},t;t') = 0.\end{aligned} \tag{2.22}$$

Noting that $\mathcal{L}_1(\mathbf{x},\mathbf{v},t)$ is given by equation (2.15) we can rewrite equation (2.22) as a diffusion equation. However before we do this we will assume that the ensemble averaged distribution function changes very slowly through secular changes of the integrals (constants) of the unperturbed motion, i.e., we assume that

$$\langle f_e(\mathbf{x},\mathbf{v},t)\rangle = \langle f_e(\tilde{\mathbf{x}},v_\perp,v_\parallel,t)\rangle,$$

where $\tilde{\mathbf{x}}$ represent the guiding center coordinates of the electrons and $v_\perp$, $v_\parallel$ the perpendicular and parallel velocity components, respectively. This assumption allows us to express the velocity gradient in terms of gradients in the guiding center coordinates. The xpression for $i\mathcal{L}_1$ is as follows

$$\delta\mathbf{E}_\mathbf{k}(t)\cdot\nabla_\mathbf{v} = -i\Phi_\mathbf{k}(t)\mathbf{k}\cdot\nabla_\mathbf{v}$$

which requires

$$\mathbf{k}\cdot\nabla_\mathbf{v} = \mathbf{k}_\perp\cdot\mathbf{v}_\perp\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel} + \frac{\mathbf{k}\times\hat{\mathbf{z}}}{\Omega_e}\cdot\tilde{\nabla}, \tag{2.23}$$

where $\tilde{\nabla}$ represents the gradient with respect to the guiding center coordinates $\tilde{\mathbf{x}}$. Note that the last term on the right hand side of equation (2.23) comes from the $\delta\mathbf{E} \times \mathbf{B}$ contribution. The diffusion equation can then be written in a compact form as follows

$$\frac{\partial G(\mathbf{X}, t; t')}{\partial t} = \sum_{\alpha,\beta} \frac{\partial}{\partial X_\alpha} D_{\alpha\beta} \frac{\partial}{\partial X_\beta},$$

where $\mathbf{X}$ represents the guiding center coordinates as well as the perpendicular and parallel velocity components, i.e.,

$$\mathbf{X} \equiv \left\{\tilde{\mathbf{x}}, v_\perp, v_\parallel\right\}$$

and where by definition the diffusion coefficients are given by

$$D_{\alpha\beta} = \frac{1}{2}\frac{d}{dt}\langle \Delta X_\alpha \Delta X_\beta \rangle$$

more explicitly the diffusion coefficients are given by (when setting $t' = t_0 = 0$)

$$\begin{aligned}
\tilde{\mathbf{D}}_{\perp\perp} &= \frac{1}{2}\frac{d}{dt}\langle \delta\tilde{\mathbf{x}}_\perp(t)\delta\tilde{\mathbf{x}}_\perp(t)\rangle \\
&= \int_0^t dt'' \int d\mathbf{x}'' \int d\mathbf{v}'' \; \langle \mathbf{v}_E(\mathbf{x}, t)\mathcal{G}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}'', \mathbf{v}'', t'')\mathbf{v}_E(\mathbf{x}, t'')\rangle,
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}_{\perp v_\parallel} &= \frac{1}{2}\frac{d}{dt}\langle \delta\tilde{\mathbf{x}}_\perp(t)\delta v_\parallel(t)\rangle \\
&= -\frac{e}{m_e}\int_0^t dt'' \int d\mathbf{x}'' \int d\mathbf{v}'' \; \langle \mathbf{v}_E(\mathbf{x}, t)\mathcal{G}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}'', \mathbf{v}'', t'')\delta\mathbf{E}_\parallel(\mathbf{x}, t'')\rangle,
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}_{v_\parallel v_\parallel} &= \frac{1}{2}\frac{d}{dt}\langle \delta v_\parallel(t)\delta v_\parallel(t)\rangle \\
&= \frac{e^2}{m_e^2}\int_0^t dt'' \int d\mathbf{x}'' \int d\mathbf{v}'' \; \langle \delta\mathbf{E}_\parallel(\mathbf{x}, t)\mathcal{G}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}'', \mathbf{v}'', t'')\delta\mathbf{E}_\parallel(\mathbf{x}, t'')\rangle.
\end{aligned}$$

The other diffusion coefficients, $\mathbf{D}_{v_\perp v_\perp}$, $\mathbf{D}_{v_\perp v_\parallel}$, $\mathbf{D}_{v_\perp \perp}$ can be expressed in a similar way.

One can write a formal solution to the Green's function equation as follows

$$\mathcal{G}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t') = \Theta(t - t')\delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{v} - \mathbf{v}')U(t, t'),$$

where the time propagator $U(t, t')$ is given by

$$U(t, t') = \exp\left(-i\int_{t'}^t \mathcal{L}(\mathbf{x}, \mathbf{v}, t'')\; dt''.\right) \tag{2.24}$$

This allows us to simplify the expressions for the diffusion coefficients

$$\tilde{\mathbf{D}}_{\perp\perp} = \int_0^t dt'' \; \langle \mathbf{v}_E(\mathbf{x}, t)U(t, t'')\mathbf{v}_E(\mathbf{x}, t'')\rangle,$$

$$\mathbf{D}_{\perp v_\parallel} = -\frac{e}{m_e}\int_0^t dt'' \; \langle \mathbf{v}_E(\mathbf{x}, t)U(t, t'')\delta\mathbf{E}_\parallel(\mathbf{x}, t'')\rangle,$$

$$\mathbf{D}_{v_\parallel v_\parallel} = \frac{e^2}{m_e^2} \int_0^t dt'' \, \langle \delta \mathbf{E}_\parallel(\mathbf{x}, t) U(t, t'') \delta \mathbf{E}_\parallel(\mathbf{x}, t'') \rangle.$$

The solution to the equation (2.13) can be formally written in terms of the time propagator defined through equation (2.24), that is

$$\delta f_e(\mathbf{x}, \mathbf{v}, t) = U(t, 0) \delta f_e(\mathbf{x}, \mathbf{v}, 0) - i \int_0^t dt' \, U(t, t') \mathcal{L}_1 \langle f_e(\tilde{\mathbf{x}}, v_\perp, v_\parallel, t') \rangle.$$

This leads to the following Fourier component of the fluctuating part of the distribution function

$$\delta f_{e\mathbf{k}\omega}(\mathbf{v}) = -i \mathcal{L}_{1\mathbf{k}\omega} \langle f_e(\tilde{\mathbf{x}}, v_\perp, v_\parallel, t) \rangle \int_{t_0}^t dt' e^{i\omega(t-t')} \langle \, e^{-i\mathbf{k}\cdot\mathbf{x}} U(t, t') e^{i\mathbf{k}\cdot\mathbf{x}} \rangle, \tag{2.25}$$

where we have pulled out the velocity derivative of the averaged distribution assuming that the time dependence of the average distribution function is much slower than the time dependence of the orbits, and neglected the initial condition.

Notice that the solution presented through equation (2.25) has imbeded in it two time scales. A fast time scale (associated with ω, i.e., fast oscillation time scale), and a slow time scale that should be related to the growth time scale. The slow time scale is the classical quasi-linear time scale required for saturation. The absence of a slow time scales leads to the absence of saturation. The Fourier transform is clearly a transform over the fast time scale.

At this point we need to use a fundamental property of the time propagator, which is

$$\langle U(t, t') g(\mathbf{x}, \mathbf{v}, t) \rangle = g \left(\mathbf{x}'(t'), \mathbf{v}'(t') \right)$$

and

$$U(t, t') = \exp \left(-i \int_{t'}^t dt'' \, \mathcal{L}(t'') \right)$$

$$= \exp \left(i \int_{t_0}^{t'} dt'' \, \mathcal{L}(t'') \right) \exp \left(-i \int_{t_0}^t dt'' \, \mathcal{L}(t'') \right) = U^{-1}(t', t_0) U(t, t_0).$$

Substituting this result into equation (2.25) leads to the following result

$$\langle e^{-i\mathbf{k}\cdot\mathbf{x}} U(t, t') e^{i\mathbf{k}\cdot\mathbf{x}} \rangle = \langle e^{-i\mathbf{k}\cdot\mathbf{x}} U^{-1}(t', t_0) U(t, t_0) e^{i\mathbf{k}\cdot\mathbf{x}} \rangle = \langle e^{-i\mathbf{k}\cdot\mathbf{x}(t')} e^{i\mathbf{k}\cdot\mathbf{x}(t)} \rangle,$$

where $\mathbf{x}(t)$ and $\mathbf{x}(t')$ represent the exact orbits of the electrons. These orbits can be decomposed into the unperturbed orbits $\mathbf{x}_0(t)$ plus the perturbation due to the nonlinear effects of the random "bath" of waves $\delta\mathbf{x}(t)$, i.e.,

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \delta\mathbf{x}(t),$$

where the perturbation can be expressed as follows

$$\delta\mathbf{x}(t) = \delta\tilde{\mathbf{x}}(t) + \delta\mathbf{x}_\parallel + \frac{\delta\mathbf{v}_\perp}{\Omega_e}.$$

This finally leads to the following expression for the perturbed distribution function

$$\delta f_{e\mathbf{k}\omega}(\mathbf{v}) = -i\frac{e}{m_e}\Phi_{\mathbf{k}\omega}\left(\mathbf{k}_\perp\cdot\mathbf{v}_\perp\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel} + \frac{\mathbf{k}\times\hat{\mathbf{z}}}{\Omega_e}\cdot\tilde{\nabla}\right)\langle f_e\rangle \times \int_{t_0}^{t} dt'\; e^{(i\omega(t-t')-i\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t')))}\langle\exp\left(i\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{x}(t,t')\right)\rangle, \tag{2.26}$$

where

$$\boldsymbol{\Delta}\mathbf{x}(t,t') = \delta\mathbf{x}(t) - \delta\mathbf{x}(t').$$

At this point one can use the cumulant expansion (see for example Weinstock [20]) to write

$$\langle\exp\left(i\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{x}(t,t')\right)\rangle = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n!}\langle\left[i\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{x}(t,t')\right]^n\rangle_c\right),$$

where $\langle\cdots\rangle_c$ is the cummulant, and note that this expansion can be reduced to a single term in the case of a random variable of a Gaussian process, that is

$$\langle\exp\left(i\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{x}(t,t')\right)\rangle = \exp\left(-\frac{1}{2}\langle\left[\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{x}(t,t')\right]^2\rangle\right). \tag{2.27}$$

The exponent can now be expanded in the following form

$$\begin{aligned}
\langle\left[\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{x}(t,t')\right]^2\rangle &= k_\parallel^2\left(\langle\delta x_\parallel^2(t)\rangle + \langle\delta x_\parallel^2(t')\rangle - 2\langle\delta x_\parallel(t)\delta x_\parallel(t')\rangle\right)\\
&+\mathbf{k}_\perp\cdot\left(\langle\delta\tilde{\mathbf{x}}_\perp^2(t)\rangle + \langle\delta\tilde{\mathbf{x}}_\perp^2(t')\rangle - 2\langle\delta\tilde{\mathbf{x}}_\perp(t)\delta\tilde{\mathbf{x}}_\perp(t')\rangle\right)\cdot\mathbf{k}_\perp\\
&+2k_\parallel\left(\langle\delta x_\parallel(t)\delta\tilde{\mathbf{x}}_\perp(t)\rangle + \langle\delta x_\parallel(t')\delta\tilde{\mathbf{x}}_\perp(t')\rangle - \langle\delta x_\parallel(t)\delta\tilde{\mathbf{x}}_\perp(t')\rangle\right.\\
&\left.-\langle\delta x_\parallel(t')\delta\tilde{\mathbf{x}}_\perp(t)\rangle\right)\cdot\mathbf{k}_\perp\\
&+\mathbf{k}_\perp\cdot\left(\frac{\langle\delta\mathbf{v}_\perp^2(t)\rangle}{\Omega_e^2} + \frac{\langle\delta\mathbf{v}_\perp^2(t')\rangle}{\Omega_e^2} - 2\frac{\langle\delta\mathbf{v}_\perp(t)\delta\mathbf{v}_\perp(t')\rangle}{\Omega_e^2}\right)\cdot\mathbf{k}_\perp\\
&+2\mathbf{k}_\perp\cdot\left(\frac{\langle\delta\tilde{\mathbf{x}}_\perp(t)\delta\mathbf{v}_\perp(t)\rangle}{\Omega_e} + \frac{\langle\delta\tilde{\mathbf{x}}_\perp(t')\delta\mathbf{v}_\perp(t')\rangle}{\Omega_e} - \frac{\langle\delta\tilde{\mathbf{x}}_\perp(t)\delta\mathbf{v}_\perp(t')\rangle}{\Omega_e}\right.\\
&\left.-\frac{\langle\delta\mathbf{v}_\perp(t)\delta\tilde{\mathbf{x}}_\perp(t')\rangle}{\Omega_e}\right)\cdot\mathbf{k}_\perp\\
&+2\mathbf{k}_\perp\cdot\left(\frac{\langle\delta\mathbf{v}_\perp(t)\delta x_\parallel(t)\rangle}{\Omega_e} + \frac{\langle\delta\mathbf{v}_\perp(t')\delta x_\parallel(t')\rangle}{\Omega_e} - \frac{\langle\delta\mathbf{v}_\perp(t)\delta x_\parallel(t')\rangle}{\Omega_e}\right.\\
&\left.-\frac{\langle\delta\mathbf{v}_\perp(t')\delta x_\parallel(t)\rangle}{\Omega_e}\right)k_\parallel,
\end{aligned} \tag{2.28}$$

where the first two terms and the fourth term on the right hand side represent the parallel and perpendicular correlation functions, while the other terms represent the cross correlation terms.

The diffusion coefficients are expressed in terms of the correlation function of the corresponding random forces due to the background oscillations. They can also be defined more accurately as the rate of time change of velocity variance around the mean value. The components of the velocity diffusion tensor can be written in the form:

$$D_{v_\alpha v_\beta} = \frac{1}{2}\frac{d}{dt}\langle \delta v_\alpha(t)\delta v_\beta(t)\rangle,$$

where the subscripts α and β represent the parallel and perpendicular components. The mean value of the parallel velocity is zero, while the mean value of the perpendicular velocity is given by the $\mathbf{E}_0 \times \mathbf{B}$ drift due to the electric field of the electrojet.

The random velocity components on the other hand are given by

$$\delta v_\parallel(t) = -\frac{e}{m_e}\int_0^t dt'\ \delta E_\parallel(\mathbf{x}(t'), t'),$$

$$\delta \mathbf{v}_\perp(t) = -\frac{c}{B}\delta\mathbf{E}(\mathbf{x}(t), t) \times \hat{\mathbf{z}},$$

where we have set the initial time to be $t_0 = 0$. We then have an expression for the guiding center spatial displacements

$$\delta x_\parallel(t) = -\frac{e}{m_e}\int_0^t dt_1 \int_0^{t_1} dt_2\ \delta E_\parallel(\mathbf{x}(t_2), t_2),$$

$$\delta\tilde{\mathbf{x}}_\perp(t) = -\frac{c}{B}\int_0^t dt_1\ (\delta\mathbf{E}(\mathbf{x}(t_1), t_1) \times \hat{\mathbf{z}}).$$

Parallel Velocity Diffusion. Following the work of Ishihara *et al.* [12], the parallel diffusion coefficient $D_{v_\parallel v_\parallel}$ can be obtained by evaluating the parallel velocity correlation function

$$\begin{aligned}\langle \delta v_\parallel^2(t)\rangle &= \frac{e^2}{m_e^2}\int_0^t dt' \int_0^t dt''\ \langle \delta E_\parallel(\mathbf{x}(t'), t')\delta E_\parallel(\mathbf{x}(t''), t'')\rangle \\ &= \frac{e^2}{m_e^2}\int_0^t ds \int_{s-t}^{s} ds'\ \langle \delta E_\parallel(\mathbf{x}(s), s)\delta E_\parallel(\mathbf{x}(s-s'), s-s')\rangle.\end{aligned}$$

Using the definition of the exact orbits leads to

$$\langle \delta v_\parallel^2(t)\rangle = \frac{e^2}{m_e^2}\int_0^t ds \int_{s-t}^{s} ds' \sum_{\mathbf{k}} |\delta E_{\parallel\mathbf{k}}|^2 e^{i(\mathbf{k}\cdot(\mathbf{x}_0(s)-\mathbf{x}_0(s-s'))-\omega_{\mathbf{k}}s')}\langle e^{i\mathbf{k}\cdot\mathbf{\Delta x}(s,s-s')}\rangle.$$

It is clear from this equation that one needs the correlation of the spatial displacements to evaluate the parallel as well as the perpendicular and cross diffusion coefficients. In general, we need three correlations functions: $\langle \delta x_\parallel(\tau)\delta x_\parallel(\tau')\rangle$, $\langle \delta x_\parallel(\tau)\delta \mathbf{x}_\perp(\tau')\rangle$, and $\langle \delta \mathbf{x}_\perp(\tau)\delta \mathbf{x}_\perp(\tau')\rangle$. The first correlation function is given by

$$\langle \delta x_\parallel(\tau)\delta x_\parallel(\tau')\rangle = \frac{e^2}{m_e^2}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \int_0^{\tau'} dt_1' \int_0^{t_1'} dt_2'\ \langle \delta E_\parallel(\mathbf{x}(t_2), t_2)\delta E_\parallel(\mathbf{x}(t_2'), t_2')\rangle$$

following the arguments of Ishihara *et al.* [12], one obtains the following result for the parallel correlation function

$$\langle \delta x_{\parallel}(\tau)\delta x_{\parallel}(\tau')\rangle = \begin{cases} \frac{1}{3} D_{\parallel\parallel}\tau'^2(3\tau-\tau') & \text{for} \quad \tau \geq \tau', \\ \frac{1}{3} D_{\parallel\parallel}\tau^2(3\tau'-\tau) & \text{for} \quad \tau' \geq \tau. \end{cases}$$

Perpendicular Guiding Center Spatial Diffusion. We now proceed to evaluate the perpendicular correlation function

$$\langle \delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle = \frac{c^2}{B^2}\int_0^{\tau} dt_1 \int_0^{\tau'} dt_2 \, \langle (\delta\mathbf{E}(\mathbf{x}(t_1),t_1)\times\hat{\mathbf{z}})\delta\mathbf{E}(\mathbf{x}(t_2,t_2)\times\hat{\mathbf{z}})\rangle.$$

It is important to note that this correlation function was approximated by Dum and Dupree [7] by

$$\langle \delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle_{\text{Dum-Dupree}} \propto \tilde{\mathbf{D}}_{\perp\perp}(\tau-\tau')$$

using a Markovian approximation, and assuming the diffusion coefficient to be independent of time. However, such an approximation has its physical limitations as shown by Salat [18] and Ishihara *et al.* [12].

We now proceed to evaluate the perpendicular correlation function by noting that by definition the diffusion coefficient describing the diffusion of the electron guiding centers is given

$$\begin{aligned} \tilde{\mathbf{D}}_{\perp\perp}(t) &= \frac{1}{2}\frac{d}{dt}\langle \delta\tilde{\mathbf{x}}_{\perp}(t)\delta\tilde{\mathbf{x}}_{\perp}(t)\rangle \\ &= \frac{1}{2}\frac{c^2}{B^2}\frac{d}{dt}\int_0^{t} dt_1 \int_0^{t} dt_2 \, \langle (\delta\mathbf{E}(\mathbf{x}(t_1),t_1)\times\hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(t_2),t_2)\times\hat{\mathbf{z}})\rangle \\ &= \frac{c^2}{B^2}\int_0^{t} dt_1 \, \langle (\delta\mathbf{E}(\mathbf{x}(t_1),t_1)\times\hat{\mathbf{z}})\delta\mathbf{E}(\mathbf{x}(t),t)\times\hat{\mathbf{z}})\rangle. \end{aligned} \tag{2.29}$$

This in turn leads, in the case $\tau \geq \tau'$, to

$$\begin{aligned} \langle \delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle &= \frac{c^2}{B^2}\int_0^{\tau'} dt_1 \int_0^{\tau'} dt_2 \, \langle (\delta\mathbf{E}(\mathbf{x}(t_1),t_1)\times\hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(t_2),t_2)\times\hat{\mathbf{z}})\rangle \\ &+\frac{c^2}{B^2}\int_{\tau'}^{\tau} dt_1 \int_0^{\tau'} dt_2 \, \langle (\delta\mathbf{E}(\mathbf{x}(t_1),t_1)\times\hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(t_2),t_2)\times\hat{\mathbf{z}})\rangle. \end{aligned}$$

The first term can be approximated using the definition of the diffusion coefficient

$$\begin{aligned} \langle \delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle &= 2\tilde{\mathbf{D}}_{\perp\perp}(\tau')\tau' \\ &+\frac{c^2}{B^2}\int_{\tau'}^{\tau} dt_1 \int_0^{\tau'} dt_2 \, \langle (\delta\mathbf{E}(\mathbf{x}(t_1),t_1)\times\hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(t_2),t_2)\times\hat{\mathbf{z}})\rangle. \end{aligned}$$

Using the definition of the diffusion coefficient given by equation (2.29) and changing variables allows us to rewrite the integral in the following form

$$\begin{aligned} \langle \delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle &= 2\tilde{\mathbf{D}}_{\perp\perp}(\tau')\tau' + \frac{c^2}{B^2}\int_0^{\tau-\tau'} d\xi \int_0^{\tau'} d\eta \\ &\times\langle (\delta\mathbf{E}(\mathbf{x}(\xi+\tau'),\xi+\tau')\times\hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(\eta),\eta)\times\hat{\mathbf{z}})\rangle. \end{aligned} \tag{2.30}$$

At this stage we would like to consider the case where $\tau \approx \tau'$, i.e., we shall consider the case where $|\tau - \tau'|/\tau'$ is a small parameter. This assumption is critical to the rest of the calculations in this paper. It is also important to note that in general τ and τ' are independent variables, which makes our assumption a critical one indeed. We then can write

$$\delta\mathbf{E}\left(\mathbf{x}(\xi+\tau'), \xi+\tau'\right) = \delta\mathbf{E}(\mathbf{x}(\tau'), \tau') + \xi\frac{d}{d\tau'}\delta\mathbf{E}(\mathbf{x}(\tau'), \tau'),$$

where the convective derivative is defined by

$$\frac{d}{d\tau'} = \left(\frac{\partial}{\partial\tau'} + \mathbf{v}(\tau')\cdot\nabla\right).$$

The first term when substituted into equation (2.30) leads to $\tilde{\mathbf{D}}_{\perp\perp}(\tau')(\tau-\tau')$ and therefore the expression becomes

$$\begin{aligned}\langle\delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle &= \tilde{\mathbf{D}}_{\perp\perp}(\tau')(\tau+\tau') \\ &+\frac{c^2}{B^2}\int_0^{\tau-\tau'}\xi\, d\xi\int_0^{\tau'} d\eta\,\langle\left(\frac{d}{d\tau'}\delta\mathbf{E}(\mathbf{x}(\tau'),\tau')\times\hat{\mathbf{z}}\right)(\delta\mathbf{E}(\mathbf{x}(\eta),\eta)\times\hat{\mathbf{z}})\rangle.\end{aligned} \tag{2.31}$$

We now use the following property of the diffusion coefficient

$$\begin{aligned}\frac{d}{dt}\tilde{\mathbf{D}}_{\perp\perp}(t) &= \frac{c^2}{B^2}\langle(\delta\mathbf{E}(\mathbf{x}(t),t)\times\hat{\mathbf{z}})\,(\delta\mathbf{E}(\mathbf{x}(t),t)\times\hat{\mathbf{z}})\rangle \\ &+\frac{c^2}{B^2}\int_0^t dt_1\,\langle\left(\frac{d}{dt}\delta\mathbf{E}(\mathbf{x}(t),t)\times\hat{\mathbf{z}}\right)(\delta\mathbf{E}(\mathbf{x}(t_1),t_1)\times\hat{\mathbf{z}})\rangle.\end{aligned} \tag{2.32}$$

Consequently we can express the two-time correlation function by substituting expression (2.32) into expression (2.31)

$$\begin{aligned}\langle\delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle &= \tilde{\mathbf{D}}_{\perp\perp}(\tau')(\tau+\tau') + \frac{(\tau-\tau')^2}{2}\left(\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau')\right. \\ &\left.-\frac{c^2}{B^2}\langle(\delta\mathbf{E}(\mathbf{x}(\tau'),\tau')\times\hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(\tau'),\tau')\times\hat{\mathbf{z}})\rangle\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\langle\delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau)\rangle + \langle\delta\tilde{\mathbf{x}}_{\perp}(\tau')\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle - 2\langle\delta\tilde{\mathbf{x}}_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle \\ &= 2\tilde{\mathbf{D}}_{\perp\perp}(\tau)\tau + 2\tilde{\mathbf{D}}_{\perp\perp}(\tau')\tau' - 2\tilde{\mathbf{D}}_{\perp\perp}(\tau')(\tau+\tau') \\ &-(\tau-\tau')^2\left(\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau') - \frac{c^2}{B^2}\langle(\delta\mathbf{E}(\mathbf{x}(\tau'),\tau')\times\hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(\tau'),\tau')\times\hat{\mathbf{z}})\rangle\right).\end{aligned} \tag{2.33}$$

The first four terms on the right hand side of expression (2.33) can be rewritten, after expanding $\tilde{\mathbf{D}}_{\perp\perp}(\tau)$ around τ', in the following form,

$$2\left(\tilde{\mathbf{D}}_{\perp\perp}(\tau) - \tilde{\mathbf{D}}_{\perp\perp}(\tau')\right)\tau - (\tau-\tau')^2\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau') = (\tau^2 - \tau'2)\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau'), \tag{2.34}$$

where we have used

$$\tilde{\mathbf{D}}_{\perp\perp}(\tau) = \tilde{\mathbf{D}}_{\perp\perp}(\tau') + (\tau - \tau')\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau').$$

Finally equation (2.33) can be written as

$$\begin{aligned}\langle\delta\tilde{\mathbf{x}}_\perp(\tau)\delta\tilde{\mathbf{x}}_\perp(\tau)\rangle + \langle\delta\tilde{\mathbf{x}}_\perp(\tau')\delta\tilde{\mathbf{x}}_\perp(\tau')\rangle - 2\langle\delta\tilde{\mathbf{x}}_\perp(\tau)\delta\tilde{\mathbf{x}}_\perp(\tau')\rangle = (\tau^2 - \tau'^2)\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau') \\ +(\tau - \tau')^2\frac{c^2}{B^2}\langle(\delta\mathbf{E}(\mathbf{x}(\tau'), \tau') \times \hat{\mathbf{z}})(\delta\mathbf{E}(\mathbf{x}(\tau'), \tau') \times \hat{\mathbf{z}})\rangle.\end{aligned} \tag{2.35}$$

The second term can easily be identified with $\langle\delta\mathbf{v}_\perp(\tau')\delta\mathbf{v}_\perp(\tau')\rangle$, and can therefore be identified with a velocity space diffusion coefficient, which decribes random changes of the gyroradius and phase angle, and can therefore be written as

$$\langle\delta\mathbf{v}_\perp(\tau')\delta\mathbf{v}_\perp(\tau')\rangle = 2D_{v_\perp v_\perp}(\tau')\tau'. \tag{2.36}$$

Substituting expression (2.36) into equation (2.35) leads to

$$\begin{aligned}&\langle\delta\tilde{\mathbf{x}}_\perp(\tau)\delta\tilde{\mathbf{x}}_\perp(\tau)\rangle + \langle\delta\tilde{\mathbf{x}}_\perp(\tau')\delta\tilde{\mathbf{x}}_\perp(\tau')\rangle - 2\langle\delta\tilde{\mathbf{x}}_\perp(\tau)\delta\tilde{\mathbf{x}}_\perp(\tau')\rangle \\ &\quad = (\tau^2 - \tau'^2)\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau') + 2\Omega_e^2(\tau - \tau')^2\tau'\left(\frac{D_{v_\perp v_\perp}(\tau')}{\Omega_e^2}\right).\end{aligned} \tag{2.37}$$

One can finally use the approximation of Dum and Dupree [7], that is the velocity diffusion makes approximately an equal contribution to the diffusion of the guiding centers, i.e.,

$$\left(\frac{D_{v_\perp v_\perp}(\tau')}{\Omega_e^2}\right) \approx \tilde{\mathbf{D}}_{\perp\perp}(\tau') \tag{2.38}$$

which finally leads to the expression for the perpendicular correlation functions (2.37)

$$\begin{aligned}&\langle\delta\tilde{\mathbf{x}}_\perp(\tau)\delta\tilde{\mathbf{x}}_\perp(\tau)\rangle + \langle\delta\tilde{\mathbf{x}}_\perp(\tau')\delta\tilde{\mathbf{x}}_\perp(\tau')\rangle - 2\langle\delta\tilde{\mathbf{x}}_\perp(\tau)\delta\tilde{\mathbf{x}}_\perp(\tau')\rangle \\ &\quad = (\tau^2 - \tau'^2)\frac{d}{d\tau'}\tilde{\mathbf{D}}_{\perp\perp}(\tau') + 2\Omega_e^2(\tau - \tau')^2\tau'\tilde{\mathbf{D}}_{\perp\perp}(\tau').\end{aligned} \tag{2.39}$$

In the case $\tau' \geq \tau$ we just exchange the role of τ and τ' in the expression (2.39).

Perpendicular Velocity Diffusion. For the velocity diffusion, with $\tau \geq \tau'$ we need to evaluate the following expression

$$\langle\delta\mathbf{v}_\perp(\tau)\delta\mathbf{v}_\perp(\tau')\rangle \simeq \langle\delta\mathbf{v}_\perp(\tau')\delta\mathbf{v}_\perp(\tau')\rangle + \frac{(\tau - \tau')}{2}\frac{\partial}{\partial\tau'}\langle\delta\mathbf{v}_\perp(\tau')\delta\mathbf{v}_\perp(\tau')\rangle. \tag{2.40}$$

Using the definition of the velocity diffusion we can express the right hand side of equation (2.40) as follows

$$\langle\delta\mathbf{v}_\perp(\tau)\delta\mathbf{v}_\perp(\tau')\rangle \simeq 2\mathbf{D}_{v_\perp v_\perp}(\tau')\tau' + (\tau' - \tau)\frac{\partial(\mathbf{D}_{v_\perp v_\perp}(\tau')\tau')}{\partial\tau'}$$

which finally leads to the contribution of perpendicular velocity diffusion

$$\begin{aligned}&\langle\delta\mathbf{v}_\perp(\tau)\delta\mathbf{v}_\perp(\tau)\rangle + \langle\delta\mathbf{v}_\perp(\tau')\delta\mathbf{v}_\perp(\tau')\rangle - 2\langle\delta\mathbf{v}_\perp(\tau)\delta\mathbf{v}_\perp(\tau')\rangle \\ &\quad \simeq 2\mathbf{D}_{v_\perp v_\perp}(\tau)\tau - 2\mathbf{D}_{v_\perp v_\perp}(\tau')\tau' - 2(\tau - \tau')\mathbf{D}_{v_\perp v_\perp}(\tau') - 2\tau'(\tau - \tau')\frac{\partial\mathbf{D}_{v_\perp v_\perp}(\tau')}{\partial\tau'}.\end{aligned}$$

Expanding the first term on the right hand side leads to

$$\mathbf{D}_{v_\perp v_\perp}(\tau) \simeq \mathbf{D}_{v_\perp v_\perp}(\tau') + (\tau - \tau')\frac{\partial \mathbf{D}_{v_\perp v_\perp}(\tau')}{\partial \tau'}$$

which leads to

$$\frac{\langle \delta \mathbf{v}_\perp(\tau)\delta \mathbf{v}_\perp(\tau)\rangle}{\Omega_e^2} + \frac{\langle \delta \mathbf{v}_\perp(\tau')\delta \mathbf{v}_\perp(\tau')\rangle}{\Omega_e^2} - 2\frac{\langle \delta \mathbf{v}_\perp(\tau)\delta \mathbf{v}_\perp(\tau')\rangle}{\Omega_e^2} \simeq 2(\tau-\tau')^2\frac{\partial \tilde{\mathbf{D}}_{\perp\perp}(\tau')}{\partial \tau'},$$

where we have used equation (2.38).

Parallel Velocity-Perpendicular Spatial Cross Diffusion. We have now evaluated the parallel and perpendicular correlation functions, we therefore only need to calculate the cross correlation, which is given by

$$\langle \delta x_\parallel(\tau)\delta \tilde{\mathbf{x}}_\perp(\tau')\rangle = \frac{e}{m_e}\frac{c}{B}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \int_0^{\tau'} dt_1' \ \langle \delta E_\parallel(\mathbf{x}(t_2),t_2)\delta \mathbf{E}(\mathbf{x}(t_1'),t_1') \times \hat{\mathbf{z}}\rangle. \tag{2.41}$$

We proceed in a similar fashion to the previous cases, and start by defining the cross diffusion coefficient

$$\begin{aligned}\underline{D}_{v_\parallel\perp}(t) &= \frac{1}{2}\frac{d}{dt}\langle \delta v_\parallel(t)\delta \tilde{\mathbf{x}}_\perp(t)\rangle \\ &= \frac{1}{2}\frac{e}{m_e}\frac{c}{B}\int_0^t dt_1 \ \langle \delta E_\parallel(\mathbf{x}(t_1),t_1)\delta \mathbf{E}(\mathbf{x}(t),t)\times \hat{\mathbf{z}}\rangle \\ &+ \frac{1}{2}\frac{e}{m_e}\frac{c}{B}\int_0^t dt_1 \ \langle \delta E_\parallel(\mathbf{x}(t),t)\delta \mathbf{E}(\mathbf{x}(t),t)\times \hat{\mathbf{z}}\rangle.\end{aligned}$$

In the case $\tau \geq \tau'$ we can decompose expression (2.41) and write it in the following form

$$\langle \delta v_\parallel(t)\delta \mathbf{x}_\perp(t)\rangle = \frac{1}{2}\frac{e}{m_e}\frac{c}{B}\int_0^t dt_1 \int_0^t dt_2 \ \langle \delta E_\parallel(\mathbf{x}(t_1),t_1)\delta \mathbf{E}(\mathbf{x}(t_2),t_2)\times \hat{\mathbf{z}}\rangle$$

which allows us to approximate the integrant of equation (2.41) as follows

$$\begin{aligned}&\frac{e}{m_e}\frac{c}{B}\int_0^{t_1} dt_2 \int_0^{\tau'} dt_1' \ \langle \delta E_\parallel(\mathbf{x}(t_2),t_2)\delta \mathbf{E}(\mathbf{x}(t_1'),t_1')\times \hat{\mathbf{z}}\rangle \\ &\quad \approx \frac{e}{m_e}\frac{c}{B}\int_0^{\min(t_1,\tau')} dt_1 \int_0^{\min(t_1,\tau')} dt_1' \ \langle \delta E_\parallel(\mathbf{x}(t_2),t_2)\delta \mathbf{E}(\mathbf{x}(t_1'),t_1')\times \hat{\mathbf{z}}\rangle.\end{aligned}$$

The right hand side of this equation can easily be identified as

$$\langle \delta v_\parallel(t)\delta \mathbf{x}_\perp(t)\rangle_{t=\min(t_1,\tau')} \approx 2\underline{D}_{v_\parallel\perp}\min(t_1,\tau').$$

Expression (2.41) can now be approximated as

$$\langle \delta x_\parallel(\tau)\delta \tilde{\mathbf{x}}_\perp(\tau')\rangle = \int_0^{\tau'} dt_1 \ 2\underline{D}_{v_\parallel\perp}t_1 + \int_{\tau'}^{\tau} dt_1 \ 2\underline{D}_{v_\parallel\perp}\tau' = \underline{D}_{v_\parallel\perp}(2\tau-\tau')\tau'.$$

On the other hand

$$\langle \delta x_\parallel(\tau')\delta \tilde{\mathbf{x}}_\perp(\tau)\rangle = \frac{e}{m_e}\frac{c}{B}\int_0^{\tau'} dt_1 \int_0^{t_1} dt_2 \int_0^{\tau} dt_1' \ \langle \delta E_\parallel(\mathbf{x}(t_2),t_2)\left(\delta \mathbf{E}(\mathbf{x}(t_1'),t_1')\times \hat{\mathbf{z}}\right)\rangle.$$

This expression can be evaluated for $\tau \geq \tau'$ following the same procedure as before to obtain

$$\langle \delta x_{\|}(\tau')\delta\tilde{\mathbf{x}}_{\perp}(\tau)\rangle = \int_0^{\tau'} dt_1\, 2\underline{D}_{v_{\|}\perp}\min(t_1,\tau) = \underline{D}_{v_{\|}\perp}{\tau'}^2.$$

This finally allows to sum all the cross correlation functions that appear in the expression (2.28)

$$\begin{aligned}&\langle \delta x_{\|}(t)\delta\tilde{\mathbf{x}}_{\perp}(t)\rangle + \langle \delta x_{\|}(t')\delta\tilde{\mathbf{x}}_{\perp}(t')\rangle\\ &\quad = \langle \delta x_{\|}(t)\delta\tilde{\mathbf{x}}_{\perp}(t')\rangle + \langle \delta x_{\|}(t')\delta\tilde{\mathbf{x}}_{\perp}(t)\rangle + \underline{D}_{v_{\|}\perp}(t')(t-t')^2 + (t-t')t^2\frac{\partial \underline{D}_{v_{\|}\perp}}{\partial t'}.\end{aligned}$$

Cross Perpendicular Velocity-Space Diffusion. One can easily show that the cross perpendicular diffusion (velocity-space) can be written as follows

$$\begin{aligned}&\langle \delta v_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau)\rangle + \langle \delta v_{\perp}(\tau')\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle\\ &\quad = \langle \delta v_{\perp}(\tau)\delta\tilde{\mathbf{x}}_{\perp}(\tau')\rangle + \langle \delta v_{\perp}(\tau')\delta\tilde{\mathbf{x}}_{\perp}(\tau)\rangle + 2(\tau-\tau')^2\frac{\partial \mathbf{D}_{v_{\perp}\perp}(\tau')}{\partial \tau'}.\end{aligned}$$

Cross Velocity diffusion

$$\begin{aligned}\langle \delta x_{\|}(\tau)\delta\mathbf{v}_{\perp}(\tau')\rangle &= \underline{D}_{v_{\|}v_{\perp}}(\tau')\tau'(2\tau-\tau'),\\ \langle \delta x_{\|}(\tau')\delta\mathbf{v}_{\perp}(\tau)\rangle &= \underline{D}_{v_{\|}v_{\perp}}(\tau'){\tau'}^2.\end{aligned}$$

which finally leads to the following expression

$$\begin{aligned}&\langle \delta x_{\|}(t)\delta\mathbf{v}_{\perp}(t)\rangle + \langle \delta x_{\|}(t')\delta\mathbf{v}_{\perp}(t')\rangle\\ &\quad = \langle \delta x_{\|}(t)\delta\mathbf{v}_{\perp}(t')\rangle + \langle \delta x_{\|}(t')\delta\mathbf{v}_{\perp}(t)\rangle + \underline{D}_{v_{\|}v_{\perp}}(t')(t-t')^2 + (t-t')t^2\frac{\partial \underline{D}_{v_{\|}v_{\perp}}}{\partial t'}.\end{aligned}$$

We now are able to express equation (2.34) in terms of components of the diffusion tensor, that is for $t \geq t'$

$$\begin{aligned}\langle[\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{x}(t,t')]^2\rangle &= k_{\|}^2\left(\frac{2}{3}D_{v_{\|}v_{\|}}(t^3+{t'}^2(2t'-3t))\right) + \mathbf{k}_{\perp}\cdot\left(\left[(t^2-{t'}^2)\frac{d}{dt'}\tilde{\mathbf{D}}_{\perp\perp}(t')\right.\right.\\ &\quad\left.\left. +\ 2(t-t')^2\frac{\partial}{\partial t'}\left\{\tilde{\mathbf{D}}_{\perp\perp}(t') + \frac{\mathbf{D}_{v_{\perp}\perp}(t')}{\Omega_e}\right\}\right] + 2\Omega_e^2(t-t')^2t'\tilde{\mathbf{D}}_{\perp\perp}(t')\right)\cdot\mathbf{k}_{\perp}\\ &\quad +2k_{\|}\left(\left[\underline{D}_{v_{\|}\perp}(t') + \frac{\underline{D}_{v_{\|}v_{\perp}}(t')}{\Omega_e}\right](t-t')^2\right.\\ &\quad\left. +t^2(t-t')\frac{\partial}{\partial t'}\left[\underline{D}_{v_{\|}\perp}(t') + \frac{\underline{D}_{v_{\|}v_{\perp}}(t')}{\Omega_e}\right]\right)\cdot\mathbf{k}_{\perp}.\end{aligned} \tag{2.42}$$

At this stage we need to solve self-consistently for the diffusion coefficients introduced above. First, we start with the parallel diffusion

$$
\begin{aligned}
D_{v_\parallel v_\parallel} &= \frac{1}{2}\frac{d}{dt}\langle \delta v_\parallel(t)\delta v_\parallel(t)\rangle \\
&= \frac{1}{2}\frac{d}{dt}\frac{e^2}{m_e^2}\int_0^t ds \int_{s-t}^{s} ds' \sum_{\mathbf{k}} |\delta E_{\parallel\mathbf{k}}|^2 e^{i(\mathbf{k}\cdot(\mathbf{x}_0(s)-\mathbf{x}_0(s-s'))-\omega_{\mathbf{k}}s')}\langle e^{i\mathbf{k}\cdot\mathbf{\Delta x}(s,s-s')}\rangle \\
&= \frac{1}{2}\frac{d}{dt}\frac{e^2}{m_e^2}\int_0^t ds \int_{s-t}^{0} ds' \sum_{\mathbf{k}} |\delta E_{\parallel\mathbf{k}}|^2 e^{i(\mathbf{k}\cdot(\mathbf{x}_0(s)-\mathbf{x}_0(s-s'))-\omega_{\mathbf{k}}s')}\langle e^{i\mathbf{k}\cdot\mathbf{\Delta x}(s,s-s')}\rangle \\
&+ \frac{1}{2}\frac{d}{dt}\frac{e^2}{m_e^2}\int_0^t ds \int_{0}^{s} ds' \sum_{\mathbf{k}} |\delta E_{\parallel\mathbf{k}}|^2 e^{i(\mathbf{k}\cdot(\mathbf{x}_0(s)-\mathbf{x}_0(s-s'))-\omega_{\mathbf{k}}s')}\langle e^{i\mathbf{k}\cdot\mathbf{\Delta x}(s,s-s')}\rangle .
\end{aligned}
\tag{2.43}
$$

Note that different expressions ought to be used for $\mathbf{\Delta x}(s, s-s')$ for each of the last two integrals of expression (2.43). Then using equation (2.27) and defining the following diffusion coefficient we obtain

$$
\tilde{\mathcal{D}}_{\perp\perp} = \tilde{\mathbf{D}}_{\perp\perp} + \frac{\mathbf{D}_{v_\perp\perp}}{\Omega_e},
$$

$$
\underline{\tilde{\mathcal{D}}}_{\parallel\perp} = \underline{D}_{v_\parallel\perp} + \frac{\underline{D}_{v_\parallel v_\perp}}{\Omega_e}.
$$

Case when $s' > 0$

$$
\begin{aligned}
\exp\left(-\frac{1}{2}\langle[\mathbf{k}\cdot\mathbf{\Delta x}(s,s-s')]^2\rangle\right) &= \exp\Bigg\{-\frac{k_\parallel^2}{3}D_{v_\parallel v_\parallel}s'^2(3s-2s') \\
&-k_\parallel\left[s'^2\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(s-s') + s^2 s'\frac{\partial\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(s-s')}{\partial s}\right]\cdot\mathbf{k}_\perp\Bigg\} \\
&\times\exp\Bigg\{-\mathbf{k}_\perp\cdot\Bigg[\Omega_e^2 s'^2(s-s')\tilde{\mathbf{D}}_{\perp\perp}(s-s') + s'\left(s-\frac{s'}{2}\right)\frac{d}{ds}\tilde{\mathbf{D}}_{\perp\perp}(s-s') \\
&+s'^2\frac{\partial\tilde{\mathcal{D}}_{\perp\perp}(s-s')}{\partial s}\Bigg]\cdot\mathbf{k}_\perp\Bigg\}.
\end{aligned}
\tag{2.44}
$$

Case when $s' < 0$

$$
\begin{aligned}
\exp\left(-\frac{1}{2}\langle[\mathbf{k}\cdot\mathbf{\Delta x}(s,s-s')]^2\rangle\right) &= \exp\Bigg\{-\frac{k_\parallel^2}{3}D_{v_\parallel v_\parallel}s'^2(3s-s') \\
&-k_\parallel\left[s'^2\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(s) - s'(s-s')^2\frac{\partial\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(s)}{\partial s}\right]\cdot\mathbf{k}_\perp\Bigg\} \\
&\times\exp\Bigg\{-\mathbf{k}_\perp\cdot\Bigg[\Omega_e^2 s'^2 s\tilde{\mathbf{D}}_{\perp\perp}(s) - s'\left(s-\frac{s'}{2}\right)\frac{d}{ds}\tilde{\mathbf{D}}_{\perp\perp}(s) \\
&+s'^2\frac{\partial\tilde{\mathcal{D}}_{\perp\perp}(s)}{\partial s}\Bigg]\cdot\mathbf{k}_\perp\Bigg\}.
\end{aligned}
\tag{2.45}
$$

Using the last two expressions (2.44) and (2.45), and changing variables in the second integral, we obtain the following expression for the parallel diffusion coefficient

$$D_{v_\parallel v_\parallel} = \frac{e^2}{m_e^2}\sum_{\mathbf{k}} |\delta E_{\parallel\mathbf{k}}|^2 \int_0^t ds\ \cos\left(\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)) - \omega_{\mathbf{k}} s\right)$$
$$\times \exp\left\{-\frac{k_\parallel^2}{3} D_{v_\parallel v_\parallel} s^2(3t-2s) - k_\parallel\left[s^2\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t-s) + t^2 s\frac{\partial\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t-s)}{\partial t}\right]\cdot\mathbf{k}_\perp\right\}$$
$$\times \exp\left\{-\mathbf{k}_\perp\cdot\left[\Omega_e^2 s^2(t-s)\tilde{\mathbf{D}}_{\perp\perp}(t-s) + s(t-\frac{s}{2})\frac{d}{dt}\tilde{\mathbf{D}}_{\perp\perp}(t-s)\right.\right.$$
$$\left.\left.+ s^2\frac{\partial\tilde{\mathcal{D}}_{\perp\perp}(t-s)}{\partial t}\right]\cdot\mathbf{k}_\perp\right\}$$

A similar calculation for the perpendicular components of the diffusion tensor as well as the cross components, leads to

$$\tilde{\mathbf{D}}_{\perp\perp}(t) = \frac{c^2}{B^2}\sum_{\mathbf{k}}(\delta\mathbf{E}_{\mathbf{k}}\times\hat{\mathbf{z}})(\delta\mathbf{E}^*_{\mathbf{k}}\times\hat{\mathbf{z}}) \int_0^t ds\ \cos\left(\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)) - \omega_{\mathbf{k}} s\right)$$
$$\times \exp\left\{-\frac{k_\parallel^2}{3} D_{v_\parallel v_\parallel} s^2(3t-2s) - k_\parallel\left[s^2\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t-s) + t^2 s\frac{\partial\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t-s)}{\partial t}\right]\cdot\mathbf{k}_\perp\right\}$$
$$\times \exp\left\{-\mathbf{k}_\perp\cdot\left[\Omega_e^2 s^2(t-s)\tilde{\mathbf{D}}_{\perp\perp}(t-s) + s(t-\frac{s}{2})\frac{d}{dt}\tilde{\mathbf{D}}_{\perp\perp}(t-s)\right.\right.$$
$$\left.\left.+ s^2\frac{\partial\tilde{\mathcal{D}}_{\perp\perp}(t-s)}{\partial t}\right]\cdot\mathbf{k}_\perp\right\}$$

and similar expressions for the cross diffusion coefficients.

It is clear from these results that the resonance broadening effects due to scattering of electrons by the Modified Two Stream Farley-Buneman waves can not be accounted for by just replacing ω by $\omega + ik_\perp^2 D^*$ in the resonant part of the dispersion relation. It is obvious from the expressions above that there is a complex time dependence of the diffusion coefficients. Moreover, most of the published work (Dum and Dupree [7], Sudan [19], Robinson [16], Robinson and Honary [17]) ignores the cross correlation and consequently the cross diffusion.

3 The Nonlinear Dielectric Function

In order to obtain the dielectric function one has to use the results of the previous section. The expression for the fluctuating part of the electron distribution function (2.26) can now be written in the following form after using equation (2.42)

$$
\begin{aligned}
\delta f_{e\mathbf{k}\omega}(\mathbf{v}) &= -i\mathcal{L}_{1\mathbf{k}\omega}\langle f_e\rangle \\
&\times \int_{t_0}^{t} dt'\, e^{(i\omega(t-t')-i\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t')))} \exp\left\{-\frac{k_\parallel^2}{3}D_{v_\parallel v_\parallel}(t^3+t'^2(2t'-3t))\right\} \\
&\times \exp\left\{-\mathbf{k}_\perp\cdot\left(\frac{(t^2-t'^2)}{2}\frac{d}{dt'}\tilde{\mathbf{D}}_{\perp\perp}(t')+(t-t')^2\frac{\partial\tilde{\mathcal{D}}_{\perp\perp}(t')}{\partial t'}\right.\right. \\
&+ \left.\Omega_e^2(t-t')^2t'\tilde{\mathbf{D}}_{\perp\perp}(t')\right)\cdot\mathbf{k}_\perp - k_\parallel\left(\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t')(t-t')^2\right. \\
&+ \left.\left. t^2(t-t')\frac{\partial\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t')}{\partial t'}\right)\cdot\mathbf{k}_\perp\right\}
\end{aligned}
\tag{3.1}
$$

using this expression (3.1) along with equation (2.3) in Poisson's equation, and changing the integration variable from t' to $s = t - t'$ leads to the nonlinear dielectric function

$$
\begin{aligned}
\epsilon(\mathbf{k},\omega) &= 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2\frac{T_i}{m_i} + i\nu_{in}\omega} - i\frac{\omega_{pe}^2}{k^2}\int d\mathbf{v}\left(\mathbf{k}\cdot\mathbf{v}_\perp\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel}\right. \\
&+ \left.\frac{\mathbf{k}\times\hat{\mathbf{z}}}{\Omega_e}\cdot\tilde{\nabla}_\perp\right)\langle f_e\rangle\int_0^t ds\, e^{(i\omega s - i\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)))}\exp\left\{-\frac{k_\parallel^2}{3}D_{v_\parallel v_\parallel}s^2(3t-2s)\right\} \\
&\times \exp\left\{-k_\parallel\left(s^2\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t-s) + t^2 s\frac{\partial\underline{\tilde{\mathcal{D}}}_{\parallel\perp}(t-s)}{\partial t}\right)\cdot\mathbf{k}_\perp\right\} \\
&\times \exp\left\{-\mathbf{k}_\perp\cdot\left(s(t-\frac{s}{2})\frac{d}{dt}\tilde{\mathbf{D}}_{\perp\perp}(t-s) + \Omega_e^2 s^2(t-s)\tilde{\mathbf{D}}_{\perp\perp}(t-s)\right.\right. \\
&+ \left.\left. s^2\frac{\partial\tilde{\mathcal{D}}_{\perp\perp}(t-s)}{\partial t}\right)\cdot\mathbf{k}_\perp\right\}.
\end{aligned}
\tag{3.2}
$$

This dispersion relation differs considerably from that used by Robinson and Honary [17] and Robinson [16] in many aspects, and therefore the consequences on the physics of irregularities in the auroral as well as equatorial E regions are significantly different. We will show in the next section how the accurate resonance broadening calculation affects the threshold for Farley-Buneman instablity. We will also show how it affects the important problem of aspect angles.

In order to extract the information hidden in the dispersion relation we have to evaluate the time integral in the expression (3.2), a nontrivial calculation.

A final note on the time dependence. As mentioned earlier in the paper, the time dependence that appears in the right hand side of the dispersion relation is a slow time dependence necessary for energy and momentum balance. In other words, the time dependence is necessary for wave saturation. This is a classical problem. Linear theory ignores the slow time dependence and predicts a time independent growth rate which in turn suggests that waves will grow indefinetly. Quasi-linear theory remedies this critical problem by introducing a slow time dependence in the background distribution function. We have retained both the fast and the slow time dependence and Fourier transformed over the fast time scale. The slow time scale is associate with the diffusion time scale in the classical quasi-linear theory. This same time scale reappears in the resonance broadening analysis.

4 The Farley-Buneman Case

In order to simplify the results and obtain a direct comparison with the classical Farley-Buneman results we will assume that $\delta E_{\|} = 0$, and the we have isotropic turbulence. This allows us to eliminate the parallel and cross diffusion effects, and use a simplified dispersion relation

$$\epsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2\frac{T_i}{m_i} + i\nu_{in}\omega} - i\frac{\omega_{pe}^2}{k^2}\int d\mathbf{v}\left(\mathbf{k}\cdot\mathbf{v}_\perp\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + \frac{\mathbf{k}\times\hat{\mathbf{z}}}{\Omega_e}\cdot\tilde{\nabla}_\perp\right)\langle f_e\rangle$$
$$\times\int_0^t ds\, e^{(i\omega s - i\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)))}\exp\left\{-k_\perp^2\left(s(t-\frac{s}{2})\frac{d}{dt}D^*(t-s)\right.\right.$$
$$\left.\left.+\,\Omega_e^2 s^2(t-s)D^*(t-s) + s^2\frac{\partial\tilde{\mathcal{D}}_{\perp\perp}(t-s)}{\partial t}\right)\right\},$$

where we have assumed an isotropic spectrum for simplicity, and replaced $\mathbf{D}_{\perp\perp}$ by D^*. Then assuming a slow time dependence of the diffusion coefficient D^* we can further simplify the expression for the dielectric function to obtain

$$\epsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2\frac{T_i}{m_i} + i\nu_{in}\omega} - i\frac{\omega_{pe}^2}{k^2}\int d\mathbf{v}\left(\mathbf{k}\cdot\mathbf{v}_\perp\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + \frac{\mathbf{k}\times\hat{\mathbf{z}}}{\Omega_e}\cdot\nabla_\perp\right)\langle f_e\rangle$$
$$\times\int_0^t ds\, e^{(i\omega s - i\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)))}\exp\left\{-k_\perp^2\Omega_e^2 s^2(t-s)D^*(t-s)\right\}.$$

At this point one can explicitly express the unperturbet orbits of the electrons in terms of Bessel functions, that is

$$e^{(i\omega s - i\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)))} = \sum_{n,m=-\infty}^{+\infty} J_n\left(k_\perp\rho_\perp\right)J_m\left(k_\perp\rho_\perp\right)$$
$$\times\exp\left\{i\left(\omega - \mathbf{k}\cdot\mathbf{v}_E^{(0)} - n\Omega_e\right)s + i(n-m)\phi - i(n-m)\Omega_e t\right\}$$

the integral over the velocity allows us to reduce the double sum to a single sum by integrating over ϕ to obtain

$$\epsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2\frac{T_i}{m_i} + i\nu_{in}\omega} - i\frac{\omega_{pe}^2}{k^2}\int v_\perp\, dv_\perp\sum_{n=-\infty}^{+\infty} J_n^2(k_\perp\rho_\perp)$$
$$\times R(\omega - \mathbf{k}\cdot\mathbf{v}_E^{(0)} - n\Omega_e)\left(\frac{n\Omega_e}{v_\perp}\frac{\partial}{\partial v_\perp} + \frac{\mathbf{k}\times\hat{\mathbf{z}}}{\Omega_e}\cdot\tilde{\nabla}_\perp\right)\langle f_e\rangle,$$

where the resonance function R is given by

$$R(\omega - \mathbf{k}\cdot\mathbf{v}_E^{(0)} - n\Omega_e) = \int_0^t ds\ \exp\left\{i\left(\omega - \mathbf{k}\cdot\mathbf{v}_E^{(0)} - n\Omega_e\right)s - k_\perp^2\Omega_e^2 s^2(t-s)D^*(t-s)\right\}. \tag{4.1}$$

It is clear from the expression (4.1) that the resonance function is similar to that of the unmagnetized case derived by Ishihara *et al.* [12] and Salat [18]. Therefore one can extract similar properties of the time integral and consequently obtain some useful information regarding the broadening of the wave-particle resonance. If we were to make the same change of variables

$$\tau_K = (k_\perp^2 \Omega_e^2 D^*)^{-\frac{1}{3}}, \qquad T = \frac{t}{\tau_K}, \qquad U = \omega - \mathbf{k}\cdot\mathbf{v}_E^{(0)} - n\Omega_e, \qquad \tilde{R} = \frac{R}{\tau_K}$$

then the resonance function can be written as

$$\tilde{R}(U,T) = \int_0^T dS\ \exp\left\{iUS - S^2(T-S)\right\}.$$

Note that only the real part of R appears in the expressions for the diffusion tensor components.

The final question to be addressed regarding this problem is that related to collisions between electrons and neutrals. It has been established through linear theory, Kadomtsev [13], Coppi and Rosenbluth [2] and Hendel *et al.* [9], that collisional effects enter the dispersion relation through the introduction of a collisional damping d_e^{coll} depending on the wave frequency, and which can be expressed as follows for electron neutral collisions

$$d_e^{coll} = \frac{k^2 v_e^2}{2(\nu_{en} - i\omega)} = k_\perp^2 D_e^{coll},$$

where D_e^{coll} is the collisional diffusion coefficient.

When combining the last two equations, we obtain the correct dispersion relation for Farley-Buneman waves. To be more specific we can integrate the Vlasov equation with the Bhatnagar collision operator, one can formally solve for the distribution function and therefore the charge dewnsity to obtain the dispersion relation. The steps of this procedure are described below.

We start with the collisional Vlasov equation, which we write in the following form

$$\frac{d\delta f_e}{dt} + \nu_{en}\delta f_e = -\frac{e}{m_e}\nabla\Phi\cdot\nabla_{\mathbf{v}} f_{e0} + \nu_{en}\frac{f_0}{n_0}\delta n(\mathbf{x},t),$$

where the total time derivative d/dt is the derivative along the perturbed particle orbits. The solution to this equation can be written in an integral form

$$\begin{aligned}\delta f_e &= f_{e0}(\mathbf{v})e^{-\nu_{en}t}\\ &\quad +\nu_{en}e^{-\nu_{en}t}\int_0^t dt'\left\{-\frac{e}{m_e\nu_{en}}\nabla\Phi(\mathbf{x}(t'),t')\cdot\nabla_{\mathbf{v}} f_{e0}(\mathbf{v}) + \frac{f_0(\mathbf{v})}{n_0}\delta n(\mathbf{x}(t'),t')\right\}e^{-\nu_{en}t'}\end{aligned}$$

writing

$$\delta f_e(\mathbf{x},\mathbf{v},t) = \delta f_{e\mathbf{k}\omega}\exp\{i\,[\mathbf{k}\cdot\mathbf{x}(t) - \omega t]\}, \qquad \Phi(\mathbf{x},t) = \Phi_{\mathbf{k}\omega}\exp\{i\,[\mathbf{k}\cdot\mathbf{x}(t) - \omega t]\}$$

leads to

$$\begin{aligned}\delta f_{e\mathbf{k}\omega} &= f_{e0}\exp\{-i\,[\mathbf{k}\cdot\mathbf{x}(t) - (\omega + i\nu_{en})t]\}\\ &\quad + \int_0^t dt'\left\{i\frac{e\Phi_{\mathbf{k}\omega}}{T_e} f_{e0}\mathbf{k}\cdot\mathbf{v} + \frac{f_0}{n_0}\delta n_{\mathbf{k}\omega}\right\} e^{i(\omega+i\nu_{en})(t-t')}\langle e^{-i\left[\mathbf{k}\cdot(\mathbf{x}(t)-\mathbf{x}(t'))\right]}\rangle,\end{aligned} \tag{4.2}$$

where $\mathbf{x}(t) = \mathbf{x}_0 + \delta\mathbf{x}(t)$, with $\mathbf{x}_0$ representing the unperturbed orbits, while $\delta\mathbf{x}$ is the perturbation due to the random electric fields. Note that we have assumed that f_{e0} is a drifting Maxwellian. Neglecting the first term in equation (4.2) for long times $t \to \infty$, and using the results of the previous section we obtain

$$\delta f_{e\mathbf{k}\omega} = \int_0^t ds \left\{ i\frac{e\Phi_{\mathbf{k}\omega}}{T_e} f_{e0}\mathbf{k}\cdot\mathbf{v} + \nu_{en}\frac{f_0}{n_0}\delta n_{\mathbf{k}\omega} \right\}$$
$$\times e^{i((\omega+i\nu_{en})(t-t')-\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)))} \exp\left(-\frac{1}{2}\langle[\mathbf{k}\cdot\mathbf{\Delta x}(t,t-s)]^2\rangle\right).$$

Using the results of the previous section where the Resonance function was calculated, we can deduce the expression for the fluctuating part of the distribution function, and then integrate over velocity to obtain an expression for the density

$$\frac{\delta\mathbf{n}_{\mathbf{k}\omega}}{n_0} = i\frac{e\Phi_{\mathbf{k}\omega}}{T_e}\int d\mathbf{v}\int_0^t ds\ \frac{f_{e0}}{n_0}\mathbf{k}\cdot\mathbf{v}e^{i((\omega+i\nu_{en})(t-t')-\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)))}$$
$$\times\exp\left(-\frac{1}{2}\langle[\mathbf{k}\cdot\mathbf{\Delta x}(t,t-s)]^2\rangle\right)$$
$$+\nu_{en}\frac{\delta\mathbf{n}_{\mathbf{k}\omega}}{n_0}\int d\mathbf{v}\int_0^t ds\ \frac{f_0}{n_0}e^{i((\omega+i\nu_{en})(t-t')-\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s)))}$$
$$\times\exp\left(-\frac{1}{2}\langle[\mathbf{k}\cdot\mathbf{\Delta x}(t,t-s)]^2\rangle\right).$$

Substituting this expression along with the expression for the ion charge density in Poisson's equation we obtain the dispersion relation

$$\epsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2\frac{T_i}{m_i} + i\nu_{in}\omega}$$
$$+\frac{i}{k^2\lambda_{De}^2}\frac{\int d\mathbf{v}\int_0^t ds\frac{f_{e0}}{n_0}\mathbf{k}\cdot\mathbf{v}e^{i\left((\omega+i\nu_{en})(t-t')-\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s))-\frac{1}{2}\langle[\mathbf{k}\cdot\mathbf{\Delta x}(t,t-s)]^2\rangle\right)}}{\left(1-\nu_{en}\int d\mathbf{v}\int_0^t ds\frac{f_0}{n_0}e^{i\left((\omega+i\nu_{en})(t-t')-\mathbf{k}\cdot(\mathbf{x}_0(t)-\mathbf{x}_0(t-s))-\frac{1}{2}\langle[\mathbf{k}\cdot\mathbf{\Delta x}(t,t-s)]^2\rangle\right)}\right)}.$$

5 Summary

We have shown that the components of the electron diffusion coefficient are time dependent and the conventional result that suggested replacing ω by $\omega + ik_\perp^2 D^*$ in the resonant part of dielectric function is not valid. This alters a number of results obtained through the application of the classical resonance broadening calculations of Dum and Dupree [7], such as Sudan's results [19] and Robinson's results [17] concerning the thresholds of the Farley-Buneman and Gradient drift instabilities in the ionosphere. The correct results will be presented in a subsequent paper to be submitted in the near future. We have aslo added the parallel diffusion as well as the cross diffusion coefficients. The former is identical to the one dimensional analog derived by Ishihara *et al.* [12] and Salat [18], the latter has never been calculated explicitly.

Finally, we have explicitely derived the dispersion relation for the Farley-Buneman waves using the improved resonance broadening formalism. Further details on the Farley-Buneman thresholds and transport will be published in the near future.

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