

# Quantum Differential Forms

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*Dedicated with gratitude to my teacher,  
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## Abstract

Formalism of differential forms is developed for a variety of Quantum and noncommutative situations.

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## § 1. Introduction

In its appearance, the algebraic apparatus of Quantum mechanics seems quite dissimilar from the familiar powerful machinery of Classical mechanics/calculus of functions of several variables. The crucial difference stems from the variables  $p$ 's and  $q$ 's no longer commuting between themselves, thus rendering useless all the comfortable tools of commutative mathematics. Or so it seems, though it's mostly true. But not entirely. At any rate, the practical problems of Quantum mathematics, for example those of Quantum integrable systems, require one to establish missing Quantum analogs of versatile Classical tools. This paper represents the second part of the project to develop such tools; the first part [11] has dealt with motion equations. Here I take up the problem of constructing Quantum differential forms, the exterior differential  $d$ , the Poincaré Lemma, and various useful maps and relations between these.

As in the preceding paper, the basic philosophy is to look at everything with noncommutative eyes and to utilize useful noncommutative constructions whenever feasible. The next two Sections can be considered as a deleted Appendix from the noncommutative textbook [12]; they set up the differential forms, Lie derivatives, and the Poincaré Lemma in general noncommutative polynomial rings. Section 4 generalizes all that to the  $\mathbf{Z}_2$ -graded case, and in the process establishes what I think is the true form of the classical E. Cartan formula for the exterior differential  $d$ .

One of the main tools used in §§ 2–4 is a construction of the homotopy operator. Such an operator no longer exists in Quantum mechanics, § 5; to establish there the Poincaré Lemma, I use instead elementary arguments of normal quantization.

§ 6 establishes a Quantum version of what is called *Clebsch representations* in [10], – but only for *finite-dimensional* Lie algebras, not differential ones. It's a bit unclear to me at the moment how to quantize the differential case, or indeed if it is at all possible. The device of Quantum Clebsch representations allows one to derive plausible rules for the generators and relations of a differential-forms complex attached to a finite-dimensional Lie algebra  $\mathcal{G}$  with its fixed representation on a vector space  $V$ ; this is the subject of § 7. In contrast to the familiar complex of differential forms associated to  $\mathcal{G}$  and  $V$ , we get now a variety of Quantum-inspired ghosts. For very special Lie algebras these ghosts can be avoided, as is done §§ 8, 9 for the affine Lie algebra  $aff(1)$  and the Lie algebra  $gl(V)$  respectively; for the Lie algebra  $so(V)$ , the number of ghosts can be reduced, § 10.

§§ 11, 12 consider the Quantum spaces of  $Q$ -type, where the commutation relations between the variables  $x_i$ 's are of the form

$$x_i x_j = Q_{ij} x_j x_i, \quad \forall i, j,$$

with some invertible constants  $Q_{ij}$ 's. These are the typical relations of Quantum vector spaces in the theory of Quantum Groups. In § 12 the variables  $x_i$ 's depend also on a discrete lattice index. This prepares the grounds for the Quantum Variational calculus, the subject of a future paper.

## § 2. Differential forms over noncommutative polynomial rings

Let  $R$  be a fixed associative ring with an unity and a  $\mathbf{Q}$ -algebra, – the algebra of coefficients. Denote by  $R\langle x \rangle = R\langle x_1, \dots, x_n \rangle$  the ring of polynomials in the *noncommuting* variables  $x_1, \dots, x_n$ ; the coefficients from  $R$  *do commute* with the  $x$ 's. The ring, and a

$R\langle x \rangle$ -bimodule, of differential forms on  $R\langle x \rangle$ , denoted  $\Omega^* = \Omega^* R\langle x \rangle$  is the noncommutative ring

$$\Omega^* R\langle x \rangle = R\langle x, y \rangle = R\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle, \quad (2.1)$$

$y_i$  denoting  $dx_i$ . The differential  $d : R\langle x \rangle \rightarrow \Omega^* R\langle x \rangle$  is an  $R$ -linear map and a derivation, satisfying the commutation rule

$$dx_i = y_i + x_i d, \quad i = 1, \dots, n. \quad (2.2)$$

The wedge product sign  $\wedge$  is suppressed from the notation as not pertinent or advantageous.

There are various grading degrees attached to an element

$$\omega = \left\{ \sum f_1 y_{i(1)} f_2 y_{i(2)} \dots f_{\ell+1} \mid f_s \in R\langle x \rangle \right\}$$

from  $\Omega^* R\langle x \rangle$ . Namely, the  $x$ -degree  $p_x(\omega)$ , and the  $dx$ -degree  $p_y(\omega)$ . Thus,  $\Omega^* R\langle x \rangle$  is bigraded,

$$\Omega^* = \oplus \Omega^{p,q}, \quad (2.3)$$

with

$$\Omega^{0,0} = R, \quad \oplus_p \Omega^{p,0} = R\langle x \rangle, \quad \oplus_p \Omega^{p,q} =: \Omega^q. \quad (2.4)$$

We next extend the differential  $d$  to act on the whole ring of differential forms  $\Omega^*$ , by the commutation relations

$$dx_i = x_i d + y_i, \quad i = 1, \dots, n, \quad (2.5a)$$

$$dy_i = -y_i d, \quad i = 1, \dots, n, \quad (2.5b)$$

$$dr = rd, \quad d(r) = 0, \quad \forall r \in R. \quad (2.5c)$$

Thus,  $d$  becomes a graded derivation, of the bi-degree  $(-1, 1)$ , satisfying the relation

$$d(\omega_1 \omega_2) = d(\omega_1) \omega_2 + (-1)^{p_y(\omega_1)} \omega_1 d(\omega_2), \quad \forall \omega_1, \omega_2 \in \Omega^*. \quad (2.6)$$

**Lemma 2.7.**

$$d^2 = 0 \quad \text{on} \quad \Omega^* R\langle x \rangle. \quad (2.8)$$

**Proof.** From formula (2.5) we find that

$$d^2 x_i = d \circ (x_i d + y_i) = (x_i d + y_i) d - y_i d = x_i d^2, \quad (2.9a)$$

$$d^2 y_i = -dy_i d = y_i d^2, \quad (2.9b)$$

$$d^2 r = rd^2, \quad d^2(r) = 0. \quad \blacksquare \quad (2.9c)$$

We now shall examine whether every closed form  $\omega$ ,  $d(\omega) = 0$ , is exact,  $\omega = d(\nu)$  for some  $\nu$ . Let us introduce a new variable  $x_{n+1}$ . Call it  $t$ . Let  $t$  commute with everything. Denote  $dt$  by  $\tau$ . Let  $\tau$  also commute with everything, in the graded-differential sense:

$$\tau\omega = (-1)^{p_y(\omega)}\omega\tau. \quad (2.10)$$

To be a little bit less casual, let us adjoin  $x_{n+1}$  and  $\tau = y_{n+1}$  to  $\Omega^*R\langle x \rangle$  without any assumptions of commutativity apart from the defining relations (2.5), and denote

$$a_i = tx_i - x_it, \quad i = 1, \dots, n, \quad (2.11a)$$

$$b_\alpha = ty_\alpha - y_\alpha t, \quad \alpha = 1, \dots, n+1, \quad (2.11b)$$

$$c_\alpha = x_\alpha\tau - \tau x_\alpha, \quad \alpha = 1, \dots, n+1, \quad (2.11c)$$

$$e_\alpha = \tau y_\alpha + y_\alpha\tau, \quad \alpha = 1, \dots, n+1. \quad (2.11d)$$

Then an easy check shows that

$$da_i = a_id + b_i - c_i, \quad (2.12a)$$

$$db_\alpha = -b_\alpha d + e_\alpha, \quad (2.12b)$$

$$dc_\alpha = -c_\alpha d + e_\alpha, \quad (2.12c)$$

$$e_\alpha d = de_\alpha. \quad (2.12d)$$

Thus, we can indeed self-consistently allow  $t$  and  $\tau$  to commute with everything.

Next, formula (2.10) shows that (when characteristic  $\neq 2$ )

$$\tau^2 = 0. \quad (2.13)$$

Thus,

$$R\langle x, t, y, \tau \rangle = R\langle x, y \rangle[t] \oplus \tau R\langle x, y \rangle[t]. \quad (2.14)$$

In other words, every element  $\omega$  of

$$\overline{\Omega}^* = R\langle x, t, y, \tau \rangle \quad (2.15)$$

can be uniquely decomposed as

$$\omega = \omega_+ + \tau\omega_-, \quad \omega_\pm \in \Omega^*[t]. \quad (2.16)$$

Now, let  $I : \overline{\Omega}^* \rightarrow \Omega^*$  be the following  $R$ -linear map of  $p_y$ -degree  $-1$ :

$$I(\omega) = \int_0^1 dt \omega_-, \quad (2.17)$$

where, for  $\nu \in \Omega^*$ ,

$$\int_0^1 dt (t^m \nu) = \frac{1}{m+1} \nu, \quad \forall m \in \mathbf{Z}_+, \quad (2.18)$$

The map  $I$ , as we shall see presently, satisfies all the properties of a homotopy operator (see, e.g., [3].) Denote by  $A_t : \Omega^* \rightarrow \overline{\Omega}^*$  the ring homomorphism over  $R$ , defined on the polynomial generators of  $\Omega^*$  by the rule:

$$A_t(x_i) = tx_i, \quad i = 1, \dots, n. \quad (2.19a)$$

$$A_t(y_i) = ty_i + \tau x_i, \quad i = 1, \dots, n. \quad (2.19b)$$

Thus,  $A_t$  commutes with the operators  $d$  in  $\Omega^*$  and  $\overline{\Omega}^*$ :

$$dA_t = A_td : \Omega^* \rightarrow \overline{\Omega}^*, \quad (2.20)$$

because formulae (2.19) imply that

$$(dA_t - A_td)x_i = tx_i(dA_t - A_td), \quad (2.21a)$$

$$(dA_t - A_td)y_i = -(ty_i + \tau x_i)(dA_t - A_td). \quad (2.21b)$$

**Homotopy Formula 2.22.** For any  $\omega \in \overline{\Omega}^*$ ,

$$dI(\omega) + Id(\omega) = \omega_+|_{t=1} - \omega_+|_{t=0}. \quad (2.23)$$

**Proof.** By formula (2.16), it's enough to verify the homotopy formula (2.23) for two cases:

$$(A) \quad \omega = t^m \nu, \quad m \in \mathbf{Z}_+, \quad \nu \in \Omega^*; \quad (2.24A)$$

$$(B) \quad \omega = t^m \tau \nu, \quad m \in \mathbf{Z}_+, \quad \nu \in \Omega^*. \quad (2.24B)$$

For the case (A), we have  $\omega = \omega_+$ , so that  $I(\omega) = 0$ , and then

$$Id(\omega) = I(t^m d\nu + mt^{m-1} \tau \nu) = I(mt^{m-1} \tau \nu) = \int_0^1 mt^{m-1} dt \nu = (1 - \delta_m^0) \nu, \quad (2.25\ell)$$

while the *LHS* of formula (2.23) yields

$$t^m \nu|_{t=1} - t^m \nu|_{t=0} = \nu (1 - \delta_m^0). \quad (2.25r)$$

For the case (B), we have  $\omega_+ = 0$ , and then

$$dI(\omega) = d\left(\int_0^1 dt t^m \nu\right) = d\left(\frac{1}{m+1} \nu\right) = \frac{1}{m+1} d(\nu), \quad (2.26a)$$

$$I(d\omega) = I(-t^m \tau d(\nu)) = \int_0^1 dt t^m d(\nu) = -\frac{1}{m+1} d(\nu), \quad (2.26b)$$

so that

$$(Id + dI)(\omega) = 0, \quad (2.26c)$$

while the *RHS* of formula (2.23) vanishes because  $\omega_+ = 0$ . ■

**Corollary 2.27.** *Suppose  $\omega \in \Omega^*$  is a closed form. Then there exists a form  $\nu \in \Omega^*$  such that*

$$(\omega - d(\nu)) \in R. \quad (2.28)$$

*In particular, every closed form of a positive homogeneous  $p_y$ -degree is exact.*

**Proof.** Suppose  $\omega \in \Omega^*$  is closed,  $d(\omega) = 0$ . Then,  $A_t(\omega)$  is also closed, in  $\overline{\Omega}^*$ , in view of formula (2.20). The homotopy formula (2.23) then yields:

$$(A_t(\omega_+))|_{t=1} - dI(A_t(\omega)) = (A_t(\omega))_+|_{t=0}. \quad (2.29)$$

But, by formula (2.19),

$$(A_t(\omega))_+|_{t=1} = \omega, \quad \forall \omega \in \Omega^*, \quad (2.30)$$

$$(A_t(\omega))_+|_{t=0} = pr^{0,0}(\omega), \quad \forall \omega \in \Omega^*, \quad (2.31)$$

where  $pr^{(0,0)}(\omega)$  is the  $x, y$ -independent part of  $\omega$ , its  $R$ -part. Thus,

$$\omega = dIA_t(\omega) + pr^{0,0}(\omega). \quad (2.32)$$

**Remark 2.33.** Everything so far proven remains true if we replace polynomials by formal power series, in any one the combinations

$$R\langle\langle x \rangle\rangle\langle y \rangle, \quad (2.34a)$$

$$R\langle x \rangle\langle\langle y \rangle\rangle, \quad (2.34b)$$

$$R\langle\langle x, y \rangle\rangle. \quad (2.34c)$$

**Example 2.35.** Suppose  $n = 1$  and

$$\omega_1 = y(1 - y)^{-1}, \quad \omega_2 = (1 - y)^{-1}. \quad (2.36)$$

Then both these forms are closed:

$$d(\omega_1) = d(\omega_2) = 0, \quad (2.37)$$

and

$$\omega_1 = d(x(1 - y)^{-1}), \quad (2.38a)$$

$$\omega_2 = 1 + d(x(1 - y)^{-1}). \quad (2.38b)$$

**Remark 2.39.** The emphasis in this Section was on the homotopy operator as the crucial ingredient in establishing the Poincaré Lemma. This is a very efficient route, and it will be followed in other Sections dealing with differential forms, – whenever possible. It won't be *always* possible, as we shall see in Section 5 devoted to Quantum Mechanics proper; we shall have to use other means there.

**Remark 2.40.** The differential forms in this Section appear as independent objects quite apart from their actions on vector fields. The main reason the latter have not been invited

to partake in the feast is that they effectively disappear in various Quantum versions, especially in field theories, by virtue of not being able to preserve the relevant Quantum commutation relations. But interestingly enough, in the universal totally noncommutative framework of this Section, one can develop the formalism of Lie derivatives rather close to the traditional commutative one. This will be done in the next Section.

**Remark 2.41.** The reader will notice that everything in this Section holds true if the number of the  $x$ -generators,  $n$ , is infinite. The same observation applies also to all that follows.

### § 3. Noncommutative Lie derivatives

In the commutative picture, one has the following formulae relating differential forms, vector fields, and the differential  $d$ :

$$X(\omega) = d(X \rfloor \omega) + X \rfloor d(\omega), \quad (3.1)$$

$$X(\omega)(Z_1, \dots, Z_\ell) = X(\omega(Z_1, \dots, Z_\ell)) - \sum_{\alpha=1}^{\ell} \omega(Z_1, \dots, [X, Z_\alpha], \dots, Z_\ell), \quad (3.2)$$

$$X(f) = d(f)(X), \quad (3.3)$$

$$\begin{aligned} d(\omega)(Z_1, \dots, Z_{\ell+1}) &= \sum_{\alpha=1}^{\ell+1} (-1)^{\alpha+1} Z_\alpha(\omega(Z_1, \dots, \hat{Z}_\alpha, \dots, Z_{\ell+1})) \\ &+ \sum_{\alpha < \beta} (-1)^{\alpha+\beta} \omega([Z_\alpha, Z_\beta], Z_1, \dots, \hat{Z}_\alpha, \dots, \hat{Z}_\beta, \dots, Z_{\ell+1}). \end{aligned} \quad (3.4)$$

Here  $X$  and  $Z_i$ 's are vector fields on a (smooth) manifold  $M$ ,  $\omega \in \wedge^\ell(M)$  is a differential  $\ell$ -form on  $M$ ,  $f \in \wedge^0(M)$  is a function on  $M$ ,  $d : \wedge^i(M) \rightarrow \wedge^{i+1}(M)$  is the (exterior) differential,  $X(\omega)$  is the Lie derivative of the form  $\omega$  w.r.t. the vector field  $X$ , the hat  $\hat{\phantom{x}}$  over an argument indicates that it is missing, and  $X \rfloor \omega$  is the interior product:

$$(X \rfloor \omega)(Z_1, \dots, Z_{\ell-1}) = \omega(X, Z_1, \dots, Z_{\ell-1}), \quad \forall \omega \in \wedge^\ell(M). \quad (3.5)$$

In this Section we establish noncommutative analogs of these classical formulae.

We start with the ring  $C = C_x = R\langle x \rangle = R\langle x_1, \dots, x_n \rangle$  of Section 2. Denote by  $\text{Der}(C)$  the Lie algebra of derivations of  $C$  over  $R$ :

$$X(fg) = X(f)g + fX(g), \quad \forall f, g \in C. \quad (3.6)$$

Obviously, every element  $X \in \text{Der}(C)$  is uniquely defined by its (arbitrary) values on the generators of the ring  $C$ :

$$X(x_i) = X_i, \quad X_i \in C, \quad i = 1, \dots, n, \quad (3.7a)$$

$$X(r) = 0, \quad \forall r \in R. \quad (3.7b)$$

We shall find very useful the following device. Instead of requiring  $X$  to be on *a priori* derivation, we simply postulate how an additive map  $X : C \rightarrow C$  commutes with the generators of  $C$ :

$$Xx_i = x_iX + X_i, \quad i = 1, \dots, n, \quad (3.8a)$$

$$Xr = rX, \quad \forall r \in R, \quad (3.8b)$$

$$X(r) = 0, \quad \forall r \in R. \quad (3.8c)$$

**Lemma 3.9.** *An additive map  $X : C \rightarrow C$  satisfying properties (3.8) is in fact a derivation of  $C$ .*

**Proof.** We have to show that

$$X(fg) - X(f)g - fX(g) \quad (3.10)$$

vanishes for all,  $f, g \in C$ . Let us fix  $g$ , and let  $f$  vary. Denote, temporarily,

$$\{X, f\} = X(fg) - X(f)g - fX(g). \quad (3.11)$$

By formulae (3.8b,c),

$$\{X, r\} = 0, \quad \forall r \in R. \quad (3.12)$$

Now,

$$\begin{aligned} \{X, x_if\} &= X(x_ifg) - X(x_if)g - x_ifX(g) \\ &\stackrel{[\text{by (3.8a)}]}{=} (x_iX + X_i)(fg) - (x_iX + X_i)(f) \cdot g - x_ifX(g) = x_i\{X, f\}. \end{aligned} \quad (3.13)$$

Thus, induction on  $\deg_x(f)$  shows that  $\{X, f\} = 0$ . ■

The same device easily proves formula (2.6). Fix  $\omega_2$ , and denote

$$\{\omega\} = d(\omega\omega_2) - d(\omega)\omega_2 - (-1)^p\omega d(\omega_2), \quad p = \deg_y(\omega).$$

Then, by formula (2.5c),

$$\{r\} = 0, \quad \forall r \in R,$$

and then

$$\begin{aligned} \{x_i\omega\} &= d(x_i\omega\omega_2) - d(x_i\omega)\omega_2 - (-1)^p x_i\omega d(\omega_2) \\ &\stackrel{[\text{by (2.5a)}]}{=} (x_id + y_i)(\omega\omega_2) - ((x_id + y_i)(\omega))\omega_2 - (-1)^p x_i\omega d(\omega_2) = x_i\{\omega\}, \\ \{y_i\omega\} &= d(y_i\omega\omega_2) - d(y_i\omega)\omega_2 - (-1)^{p+1} y_i\omega d(\omega_2) \\ &\stackrel{[\text{by (2.5b)}]}{=} -y_id(\omega\omega_2) + y_id(\omega)\omega_2 + (-1)^p y_i\omega d(\omega_2) = -y_i\{\omega\}. \end{aligned}$$

Thus,  $\{\omega\}$  vanishes identically.

Given a derivation  $X \in \text{Der}(C)$ , we now extend its action from  $C$  onto  $\Omega^* = \Omega^*C = R\langle x, y \rangle$ , by adding to the commutation rules (3.8) the relations

$$Xy_i = y_iX + d(X(x_i)), \quad i = 1, \dots, n. \quad (3.14)$$



**Lemma 3.15.** (i)  $X$  is a derivation of the ring  $\Omega^*$ ; (ii) On  $\Omega^*$ ,  $X$  commutes with the differential  $d$ :

$$Xd = dX. \quad (3.16)$$

**Proof.** (i) We proceed exactly as in the Proof of Lemma (3.9), taking  $f$  and  $g$  now not from the ring  $C = R\langle x \rangle$  but from the ring  $\Omega^* = R\langle x, y \rangle$ . We need only to determine what  $\{X, y_i f\}$  is. So,

$$\begin{aligned} \{X, y_i f\} &= X(y_i f g) - X(y_i f)g - y_i f X(g) \\ &\stackrel{[\text{by } 3.14]}{=} (y_i X + d(X_i))(f g) - (y_i X + d(X_i))(f) \cdot g - y_i f X(g) = y_i \{X, f\}; \end{aligned} \quad (3.17)$$

(ii) To prove formula (3.16) we note that

$$(Xd - dX)(r) = 0, \quad \forall r \in R, \quad (3.18)$$

and then verify the relations

$$(Xd - dX)x_i = x_i(Xd - dX), \quad (3.19a)$$

$$(Xd - dX)y_i = -y_i(Xd - dX), \quad (3.19b)$$

Indeed,

$$\begin{aligned} (Xd - dX)x_i &\stackrel{[\text{by } (2.5a), (3.8a)]}{=} X(x_i d + y_i) - d(x_i X + X_i) \\ &\stackrel{[\text{by } (3.14)]}{=} (x_i X + X_i)d + y_i X + d(X_i) - (x_i d + y_i)X - d(X_i) - X_i d \\ &= x_i(Xd - dX), \\ (Xd - dX)y_i &= X(-1)y_i d - d(y_i X + d(X_i)) \\ &= -(y_i X + d(X_i))d + y_i dX + d(X_i)d = -y_i(Xd - dX). \quad \blacksquare \end{aligned}$$

We next define the interior product, inductively:

$$X \rfloor \omega = 0 \quad \text{if} \quad p_y(\omega) = 0, \quad (3.20a)$$

$$X \rfloor \left( \sum_{is} f_{is} y_i g_{is} \right) = \sum_{is} f_{is} X_i g_{is}, \quad f_{is}, g_{is} \in C = R\langle x \rangle, \quad (3.20b)$$

$$X \rfloor x_i \omega = x_i(X \rfloor \omega), \quad \omega \in \Omega^*, \quad i = 1, \dots, n, \quad (3.20c)$$

$$X \rfloor y_i \omega = X_i \omega - y_i(X \rfloor \omega), \quad \omega \in \Omega^*, \quad i = 1, \dots, n, \quad (3.20d)$$

$$X \rfloor r \omega = r(X \rfloor \omega), \quad \omega \in \Omega^*, \quad r \in R. \quad (3.20e)$$

Notice that formulae (3.20c,d,e) agree with (and, together with the relation (3.20a), imply) the formula (3.20b).

**Lemma 3.21.** *For any  $\omega_1, \omega_2 \in \Omega^*$ ,*

$$X \rfloor \omega_1 \omega_2 = (X \rfloor \omega_1) \omega_2 + (-1)^{p_y(\omega_1)} \omega_1 (X \rfloor \omega_2). \quad (3.22)$$

**Proof.** Fix  $\omega_2$ , denote  $p = p_y(\omega_1)$ , and set

$$\{X, \omega_1\} = X \rfloor \omega_1 \omega_2 - (X \rfloor \omega_1) \omega_2 - (-1)^p \omega_1 (X \rfloor \omega_2).$$

Then

$$\begin{aligned} \{X, x_i \omega_1\} &= X \rfloor x_i \omega_1 \omega_2 - (X \rfloor x_i \omega_1) \omega_2 - (-1)^p x_i \omega_1 (X \rfloor \omega_2) \stackrel{[\text{by (3.20c)}]}{=} x_i \{X, \omega_1\}, \\ \{X, y_i \omega_1\} &= X \rfloor y_i \omega_1 \omega_2 - (X \rfloor y_i \omega_1) \omega_2 - (-1)^{p+1} y_i \omega_1 (X \rfloor \omega_2) \\ &\stackrel{[\text{by (3.20d)}]}{=} X_i \omega_1 \omega_2 - y_i (X \rfloor \omega_1 \omega_2) - (X_i \omega_1) \omega_2 + y_i (X \rfloor \omega_1) \omega_2 + (-1)^p y_i \omega_1 (X \rfloor \omega_2) \\ &= -y_i \{X, \omega_1\}. \end{aligned}$$

It remains to notice that, for any  $r \in R$ ,

$$\{X, r\} = X \rfloor r \omega_2 - (X \rfloor r) \omega_2 - r (X \rfloor \omega_2) \stackrel{[\text{by (3.20a,e)}]}{=} 0. \quad \blacksquare$$

We now have all the tools needed to state noncommutative analogs of the classical formulae (3.1)–(3.4). First, formula (3.3):

**Lemma 3.23.** *For any  $X \in \text{Der}(C)$  and  $f \in C$ ,*

$$X(f) = X \rfloor d(f). \quad (3.24)$$

**Proof.** Set

$$\{X, f\} = X(f) - X \rfloor d(f).$$

Obviously,

$$\{X, r\} = 0, \quad \forall r \in R.$$

Now,

$$\begin{aligned} \{X, x_i f\} &= X(x_i f) - X \rfloor d(x_i f) = X_i f + x_i X(f) - X \rfloor (y_i f + x_i d(f)) \\ &= X_i f + x_i X(f) - X_i f - x_i (X \rfloor d(f)) = x_i \{X, f\}. \quad \blacksquare \end{aligned}$$

Next comes formula (3.1):

**Lemma 3.25.** *For any  $X \in \text{Der}(C)$  and  $\omega \in \Omega^*$ ,*

$$X(\omega) = d(X \rfloor \omega) + X \rfloor d(\omega). \quad (3.26)$$

**Proof.** (A) Set

$$\{X, \omega\} = X(\omega) - d(X \rfloor \omega) - X \rfloor d(\omega).$$

By Lemma 3.23 and formula (3.20a),

$$\{X, \omega\} = 0 \quad \text{when} \quad p_y(\omega) = 0.$$

A direct check then shows that

$$\{X, x_i \omega\} = x_i \{X, \omega\}, \quad \{X, y_i \omega\} = y_i \{X, \omega\}.$$

(B) Alternatively, if  $\omega = d(f)$ ,  $f \in C$ , then formula (3.26) becomes

$$Xd(f) = d(X \rfloor d(f))$$

(since  $d^2 = 0$ ), and this is true in view of formula (3.24), since  $Xd = dX$  by formula (3.16). Now, one easily checks that

$$\{X, \omega_1 \omega_2\} = \{X, \omega_1\} \omega_2 + \omega_1 \{X, \omega_2\},$$

and this implies that  $\{X, \omega\}$  vanishes identically, since  $C$  and  $d(C)$  generate the whole ring  $\Omega^*$ . ■

Formula (3.2) is next, but it is a good time to take a skew-symmetric pause. Noncommutative differential forms differ from their commutative counterparts most clearly in not being skewsymmetric; after all, what is skewsymmetric about the expressions

$$(dx_1)^2, \quad \exp(dx_1).$$

and so on? Interestingly enough, the skewsymmetry re-appears when differential forms are considered in their action on the (poly-) vector fields:

**Lemma 3.27.** *For any  $Z_1, Z_2 \in \text{Der}(C)$  and  $\omega \in \Omega^*$ ,*

$$Z_1 \rfloor Z_2 \rfloor \omega = -Z_2 \rfloor Z_1 \rfloor \omega. \quad (3.28)$$

**Proof.** Pick any two elements  $\omega_1, \omega_2 \in \Omega^*$ . By formula (3.22), with  $p = \deg_y(\omega_1)$ ,

$$\begin{aligned} Z_1 \rfloor Z_2 \rfloor \omega_1 \omega_2 &= Z_1 \rfloor ((Z_2 \rfloor \omega_1) \omega_2 + (-1)^p \omega_1 (Z_2 \rfloor \omega_2)) = (Z_1 \rfloor Z_2 \rfloor \omega_1) \omega_2 \\ &\quad - (-1)^p (Z_2 \rfloor \omega_1) (Z_1 \rfloor \omega_2) + (-1)^p (Z_1 \rfloor \omega_1) (Z_1 \rfloor \omega_2) + \omega_1 (Z_1 \rfloor Z_2 \rfloor \omega_2). \end{aligned} \quad (3.29)$$

Thus,

$$\begin{aligned} &Z_1 \rfloor Z_2 \rfloor \omega_1 \omega_2 + Z_2 \rfloor Z_1 \rfloor \omega_1 \omega_2 \\ &= (Z_1 \rfloor Z_2 \rfloor \omega_1 + Z_2 \rfloor Z_1 \rfloor \omega_1) \omega_2 + \omega_1 (Z_1 \rfloor Z_2 \rfloor \omega_2 + Z_2 \rfloor Z_1 \rfloor \omega_2). \quad \blacksquare \end{aligned} \quad (3.30)$$

**Corollary 3.31.** *For any  $Z_1, \dots, Z_\ell \in \text{Der}(C)$  and  $\omega \in \Omega^*$ ,*

$$Z_1 \rfloor Z_2 \rfloor \dots \rfloor Z_\ell \rfloor \omega$$

*is totally skewsymmetric w.r.t. the  $Z$ 's: for any permutation  $\sigma \in S_\ell$ ,*

$$Z_{\sigma(1)} \rfloor \dots \rfloor Z_{\sigma(\ell)} \rfloor \omega = (-1)^{\text{sgn}(\sigma)} Z_1 \rfloor \dots \rfloor Z_\ell \rfloor \omega, \quad \forall \sigma \in S_\ell. \quad (3.32)$$

**Example 3.33.** Denote by  $f\partial_i$  the element of  $\text{Der}(C)$  acting on the generators of  $C$  by the rule

$$(f\partial_i)(x_j) = f\delta_{ij}, \quad \forall f \in C, \quad (3.34)$$

and write simply  $f\partial$  and  $y$  instead of  $f\partial_1$  and  $y_1$  when  $n = 1$ . Then

$$f\partial_i \rfloor g\partial_j \rfloor y_i y_j = [f, g] (= fg - gf), \quad (3.35)$$

$$(a\partial_i + b\partial_j)](cd_i + d\partial_j)]y_i y_j = cb - ad, \quad (3.36)$$

$$f\partial]y = f, \quad (3.37a)$$

$$f\partial]g\partial]y^2 = gf - fg, \quad (3.37b)$$

$$f\partial]g\partial]h\partial]y^3 = hgf + gfh + fhg - hfg - fgh - ghf, \quad (3.37c)$$

$$\partial]y^{2\ell} = 0, \quad \ell \in \mathbf{Z}_+, \quad (3.37d)$$

$$\partial]y^{2\ell+1} = y^{2\ell}, \quad \ell \in \mathbf{Z}_+. \quad (3.37e)$$

Formula (3.2) has the following noncommutative form:

**Lemma 3.38.** *For any  $X, Z_1, \dots, Z_\ell \in \text{Der}(C)$  and  $\omega \in \Omega^*$ ,*

$$Z_\ell] \dots ] Z_1] X(\omega) = X(Z_\ell] \dots ] Z_1] \omega) - \sum_{\alpha=1}^{\ell} Z_\ell] \dots ] [X, Z_\alpha]] \dots ] Z_1] \omega. \quad (3.39)$$

**Remark 3.40.** Notice that, in contradistinction to the commutative case, the differential form  $\omega$  in formula (3.39) does not have to be a  $\ell$ -form.

**Proof.** We first establish formula (3.39) for the case  $\ell = 1$ :

$$Z]X(\omega) = X(Z]\omega) - [X, Z]]\omega. \quad (3.41)$$

We shall prove formula (3.41) in 3 stages:

- 1) The formula is obvious when  $\deg_y(\omega) = 0$ ;
- 2) If  $\omega = y_i$  then

$$Z]X(y_i) = Z]d(X_i) = Z(X_i),$$

while

$$X(Z]y_i) - [X, Z]]y_i = X(Z_i) - [X, Z]_i = X(Z_i) - (X(Z_i) - Z(X_i)) = Z(X_i);$$

- 3) Since  $\Omega^*$  is generated by  $C$  and the  $y_i$ 's, it's enough to check that if formula (3.41) holds for  $\omega_1, \omega_2 \in \Omega^*$  then it also holds for  $\omega = \omega_1 \omega_2$ . Denoting  $p = \deg_y(\omega_1)$ , we find

$$\begin{aligned} Z]X(\omega_1 \omega_2) &= Z](X(\omega_1) \omega_2 + \omega_1 X(\omega_2)) = (Z]X(\omega_1)) \omega_2 \\ &\quad + (-1)^p X(\omega_1)(Z]\omega_2) + (Z]\omega_1) X(\omega_2) + (-1)^p \omega_1 (Z]X(\omega_2)), \end{aligned} \quad (3.42\ell)$$

$$\begin{aligned} X(Z]\omega_1 \omega_2) &= X((Z]\omega_1) \omega_2 + (-1)^p \omega_1 (Z]\omega_2) = X(Z]\omega_1) \omega_2 \\ &\quad + (Z]\omega_1) X(\omega_2) + (-1)^p X(\omega_1)(Z]\omega_2) + (-1)^p \omega_1 X(Z]\omega_2) \\ &\quad - [X, Z]]\omega_1 \omega_2 = -([X, Z]]\omega_1) \omega_2 - (-1)^p \omega_1 ([X, Z]]\omega_2). \end{aligned} \quad (3.42r)$$

Adding all up, we get:

$$\begin{aligned} Z]X(\omega_1 \omega_2) &- X(Z]\omega_1 \omega_2) - [X, Z]]\omega_1 \omega_2 \\ &= (Z]X(\omega_1) - X(Z]\omega_1) - [X, Z]]\omega_1) \omega_2 \\ &\quad + (-1)^p \omega_1 (Z]X(\omega_2) - X(Z]\omega_2) - [X, Z]]\omega_2), \end{aligned} \quad (3.43)$$

as desired.

With formula (3.41) behind us, we could take two routes to the general formula (3.39). The longer route splits  $x_i$  and  $y_i$  from the left of  $\omega$  and uses the formula

$$Z_1] \dots ] Z_\ell] y_i \omega = (-1)^\ell y_i (Z_1] \dots ] Z_\ell] \omega) + \sum_{\alpha=1}^{\ell} (-1)^{\ell-\alpha} y_i (Z_\alpha) (Z_1] \dots ] \hat{Z}_\alpha \dots ] Z_\ell] \omega). \quad (3.44)$$

The shorter route uses induction on  $\ell$ : For  $\ell = 1$ , formula (3.39) turns into already proven formula (3.41), and then

$$\begin{aligned} Z_{\ell+1}] \dots ] Z_1] X(\omega) &\stackrel{[\text{by (3.39)}]}{=} Z_{\ell+1}] \left\{ X(Z_\ell] \dots ] Z_1] \omega) - \sum_{\alpha=1}^{\ell} Z_\ell] \dots ] [X, Z_\alpha]] \dots ] Z_1] \omega \right\} \\ &\stackrel{[\text{by 3.41}]}{=} X(Z_{\ell+1}] \dots ] Z_1] \omega) - [X, Z_{\ell+1}]] Z_\ell] \dots ] Z_1] \omega - \sum_{\alpha=1}^{\ell} Z_{\ell+1}] Z_\ell] \dots ] [X, Z_\alpha]] \dots ] Z_1] \omega \\ &= X(Z_{\ell+1}] \dots ] Z_1] \omega) - \sum_{\alpha=1}^{\ell+1} Z_{\ell+1}] \dots ] [X, Z_\alpha]] \dots ] Z_1] \omega, \end{aligned}$$

which is formula (3.39) with  $\ell$  replaced by  $\ell + 1$ . ■

We are now ready to handle the last of the classical formulae (3.1)–(3.4), E. Cartan's formula (3.4). We start with formula (3.26) rewritten in the form

$$Z_1] d(\omega) = Z_1(\omega) - d(Z_1] \omega). \quad (3.45)$$

Applying the operation  $Z_2]$  to each side of formula (3.45), we find

$$\begin{aligned} Z_2] Z_1] d(\omega) &= Z_2] Z_1(\omega) - Z_2] d(Z_1] \omega) \\ &\stackrel{[\text{by (3.41), (3.45)}]}{=} Z_1(Z_2] \omega) - [Z_1, Z_2]] \omega - Z_2(Z_1] \omega) + d(Z_2] Z_1] \omega). \end{aligned}$$

Thus,

$$Z_2] Z_1] d(\omega) = Z_1(Z_2] \omega) - Z_2(Z_1] \omega) + d(Z_2] Z_1] \omega) - [Z_1, Z_2]] \omega. \quad (3.46)$$

We see that in each of formulae (3.45), (3.46) we get an extra  $d$ -term compared to the classical formula, – because we have not required that the  $d$ -degree of  $\omega$  be equal to the number of vector fields  $Z_i$ 's.

**Lemma 3.47.** *For any  $Z_1, \dots, Z_\ell \in \text{Der}(C)$  and  $\omega \in \Omega^*$ ,*

$$\begin{aligned} Z_\ell] \dots ] Z_1] d(\omega) &= \sum_{\alpha=1}^{\ell} (-1)^{\alpha+1} Z_\alpha(Z_\ell] \dots ] \hat{Z}_\alpha \dots ] Z_1] \omega) + (-1)^\ell d(Z_\ell] \dots ] Z_1] \omega) \\ &\quad + \sum_{\alpha < \beta} (-1)^{\alpha+\beta} (Z_\ell] \dots ] \hat{Z}_\beta \dots ] \hat{Z}_\alpha \dots ] [Z_\alpha, Z_\beta]] \omega). \end{aligned} \quad (3.48)$$

**Proof.** We use induction on  $\ell$ , the cases  $\ell = 1, 2$  having been verified by formulae (3.45) and (3.46) respectively. Applying the operation  $Z_{\ell+1}]$  to each side of formula (3.48), we find:

$$Z_{\ell+1}] \dots ] Z_1] d(\omega)$$

$$= \sum_{\alpha=1}^{\ell} (-1)^{\alpha+1} Z_{\ell+1}] Z_{\alpha}(Z_{\ell}] \dots \hat{Z}_{\alpha} \dots Z_1] \omega) \quad (3.49a)$$

$$+ (-1)^{\ell} Z_{\ell+1}] d(Z_{\ell}] \dots Z_1] \omega) \quad (3.49b)$$

$$+ \sum_{\alpha < \beta \leq \ell} (-1)^{\alpha+\beta} (Z_{\ell+1}] \dots \hat{Z}_{\beta} \dots \hat{Z}_{\alpha} \dots [Z_{\alpha}, Z_{\beta}] ] \omega). \quad (3.49c)$$

By formula (3.41), the sum (3.49a) can be transformed as

$$\sum_{\alpha=1}^{\ell} (-1)^{\alpha+1} Z_{\alpha}(Z_{\ell+1}] \dots \hat{Z}_{\alpha} \dots Z_1] \omega) \quad (3.49a1)$$

$$+ \sum_{\alpha=1}^{\ell} (-1)^{\alpha} ([Z_{\alpha}, Z_{\ell+1}] ] Z_{\ell}] \dots \hat{Z}_{\alpha} \dots Z_1] \omega). \quad (3.49a2)$$

By formula (3.45), the second sum (3.49b) becomes

$$(-1)^{\ell} Z_{\ell+1}(Z_{\ell}] \dots ] Z_1] \omega) \quad (3.49b1)$$

$$+ (-1)^{\ell+1} d(Z_{\ell+1}] \dots ] Z_1] \omega). \quad (3.49b2)$$

Combining the terms in formulae (3.49a1), (3.49b1), we get

$$\sum_{\alpha=1}^{\ell+1} (-1)^{\alpha+1} Z_{\alpha}(Z_{\ell+1}] \dots \hat{Z}_{\alpha} \dots ] Z_1] \omega). \quad (3.50)$$

Rewriting the sum (3.49a2) as

$$\sum_{\alpha=1}^{\ell} (-1)^{\alpha+\ell+1} (Z_{\ell}] \dots \hat{Z}_{\alpha} \dots [Z_{\alpha}, Z_{\ell+1}] ] \omega) \quad (3.51)$$

and combining it with the sum (3.49c), we obtain

$$+ \sum_{\alpha < \beta \leq \ell+1} (-1)^{\alpha+\beta} (Z_{\ell+1}] \dots \hat{Z}_{\beta} \dots \hat{Z}_{\alpha} \dots [Z_{\alpha}, Z_{\beta}] ] \omega). \quad (3.52)$$

Adding up formulae (3.50), (3.49b2), (3.52), we recover the RHS of formula (3.48) with  $\ell$  replaced by  $\ell + 1$ . ■

So far, we have paid no attention to many related subjects lurking in the shadows. No mention has been made of homology (see, e.g., [5]) of the Lie algebra  $\text{Der}(C)$  (or, more generally,  $\text{Der}(\Omega^*, d)$ , see below.) But one shouldn't ignore the  $\mathbf{Z}_2$ -graded nature of the ring  $\Omega^*$ :

$$\Omega^* = \Omega_e^* + \Omega_o^*. \quad (3.53)$$

where  $\omega \in \Omega^*$  is even or odd depending upon  $p(\omega) = p_y(\omega) \bmod 2$  being respectively 0 or 1 in  $\mathbf{Z}_2$ . Consequently, additive maps from  $\Omega^*$  to  $\Omega^*$  are also  $\mathbf{Z}_2$ -graded, and one can talk about  $\mathbf{Z}_2$ -graded derivations  $Y \in \text{Der}(\Omega^*)$ :

$$Y(\omega_1\omega_2) = Y(\omega_1)\omega_2 + (-1)^{p(Y)p(\omega_1)}\omega_1 Y(\omega_2), \quad \forall \omega_1, \omega_2 \in \Omega^*. \quad (3.54)$$

Since we already have the differential  $d$  acting on  $\omega^*$  (as an *old* derivation, see formula (2.6)), the most important subsuperalgebra in the Lie superalgebra  $\text{Der}(\Omega^*)$  is

$$\text{Der}(\Omega^*, d) = \{Y \in \text{Der}(\Omega^*) \mid Yd = (-1)^{p(Y)}dY\}. \quad (3.55)$$

For example,

$$d \in \text{Der}(\omega^*, d), \quad (3.56)$$

and of course

$$\text{Der}(C) \subset \text{Der}(\Omega^*, d). \quad (3.57)$$

$\text{Der}(C)$  is an even subsuperalgebra in  $\text{Der}(\Omega^*, d)$ , but it is by no means *all* of the even part of  $\text{Der}(\Omega^*, d)$ . All noncommutative formulae proved in this Section for elements  $Z_i \in \text{Der}(C)$  remain true for *even* elements  $Z_i \in \text{Der}(\Omega^*, d)_e$ , although this is not immediately obvious in view of the commutators  $[Z_i, Z_j]$  entering our formulae in places. But we can do better still, and consider the vector field arguments  $X$  and  $Z_i$ 's of *arbitrary*  $\mathbf{Z}_2$ -grading, whether even or odd. On the second thought, we could have started with the generators  $x_i$ 's of prescribed arbitrary  $\mathbf{Z}_2$ -grading  $p(i)$ . And on the third thought, we could have taken the coefficient ring  $R$  being  $\mathbf{Z}_2$ -graded as well. This program is realized in the next Section.

#### § 4. $\mathbf{Z}_2$ -graded picture: superdifferential forms

Recall a few basic facts about superobjects. Suppose  $R$  and  $\mathcal{R}$  are  $\mathbf{Z}_2$ -graded associative rings, with  $\mathcal{R}$  being an  $R$ -algebra. The latter means that

$$r\rho = (-1)^{p(r)p(\rho)}\rho r, \quad r \in R, \quad \rho \in \mathcal{R}, \quad (4.1)$$

where  $p(\cdot)$  is the  $\mathbf{Z}_2$ -degree of  $(\cdot)$ . A (left) *derivation* of  $\mathcal{R}$  over  $R$  is an additive map  $Z : \mathcal{R} \rightarrow \mathcal{R}$  satisfying the properties

$$Z(\rho_1\rho_2) = Z(\rho_1)\rho_2 + (-1)^{p(Z)p(\rho_1)}\rho_1 Z(\rho_2), \quad \rho_1, \rho_2 \in \mathcal{R}, \quad (4.2a)$$

$$Z(r\rho) = (-1)^{p(Z)p(r)}rZ(\rho), \quad r \in \mathcal{R}, \quad \rho \in \mathcal{R}, \quad (4.2b)$$

$$Z(r) = 0, \quad r \in R. \quad (4.2c)$$

Property (4.2c) assumes that  $R$  has a unit element. The set of all such derivations is denoted  $\text{Der}(\mathcal{R}) = \text{Der}(\mathcal{R}/R)$ . It is a Lie superalgebra: if  $Z_1, Z_2, Z_3 \in \text{Der}(\mathcal{R})$  then

$$[Z_1, Z_2] := Z_1Z_2 - (-1)^{p(Z_1)p(Z_2)}Z_2Z_1 = -(-1)^{p(Z_1)p(Z_2)}[Z_2, Z_1] \quad (4.2)$$

is also an element of  $\text{Der}(\mathcal{R})$ , and

$$[Z_1, [Z_2, Z_3]] = [[Z_1, Z_2], Z_3] + (-1)^{p(Z_1)p(Z_2)}[Z_2, [Z_1, Z_3]]. \quad (4.3)$$

The reader will notice the convention employed in  $\mathbf{Z}_2$ -graded formulae: they are often written for  $\mathbf{Z}_2$ -homogeneous elements only.

We now take  $\mathcal{R} = R\langle x \rangle = R\langle x_1, \dots, x_n \rangle$ , with the  $x_i$ 's having arbitrarily prescribed  $\mathbf{Z}_2$ -gradings  $p(i)$ :

$$p(x_i) = p(i), \quad i = 1, \dots, n. \quad (4.4)$$

The differential  $d : \mathcal{R} \rightarrow \Omega^* = R\langle x, y \rangle$  is now defined as an odd map satisfying the properties

$$dx_i = (-1)^{p(i)} x_i d + y_i, \quad i = 1, \dots, n, \quad (4.6a)$$

$$dr = (-1)^{p(r)} r d, \quad r \in R, \quad (4.6b)$$

$$d(r) = 0, \quad r \in R. \quad (4.6c)$$

The generators  $y_i$ 's of  $\Omega^*$  have the natural  $\mathbf{Z}_2$ -grading opposite to that of the  $x_i$ 's:

$$p(y_i) = p(x_i) + \bar{1} = p(i) + 1$$

(we write 1 instead of  $\bar{1}$  in  $\mathbf{Z}_2$ ). Extending the action of  $d$  from  $\mathcal{R}$  onto  $\Omega^*$  we add to formulae (4.6) another one:

$$dy_i = -(-1)^{p(i)} y_i d, \quad i = 1, \dots, n. \quad (4.6d)$$

These relations imply that  $d : \Omega^* \rightarrow \Omega^*$  is an odd derivation:

$$d(\omega_1 \omega_2) = d(\omega_1) \omega_2 + (-1)^{p(\omega_1)} \omega_1 d(\omega_2), \quad \omega_1, \omega_2 \in \Omega^*, \quad (4.7)$$

and that  $d^2 = 0$  on  $\Omega^*$ . (This and other easily checked facts in this Section are left to the reader.)

Since  $\Omega^*$  is also an  $R$ -algebra, we have two Lie superalgebras:  $\text{Der}(\mathcal{R})$  and  $\text{Der}(\Omega^*)$ . The latter is too big, and we need only a part of it:

$$\text{Der}(\Omega^*, d) = \{Z \in \text{Der}(\Omega^*) \mid Zd = (-1)^{p(Z)} Z d\}; \quad (4.8)$$

alternatively, we can describe such  $Z$ 's as additive maps  $\Omega^* \rightarrow \Omega^*$  satisfying the relations

$$Zx_i = (-1)^{p(Z)p(i)} x_i Z + Z_i, \quad Z_i = Z(x_i) \in \Omega^*, \quad i = 1, \dots, n, \quad (4.9a)$$

$$Zy_i = (-1)^{p(Z)} ((-1)^{p(Z)p(i)} y_i Z + d(Z_i)), \quad i = 1, \dots, n, \quad (4.9b)$$

$$Zr = (-1)^{p(Z)p(r)} r Z, \quad r \in R, \quad (4.9c)$$

$$Z(r) = 0, \quad r \in R. \quad (4.9d)$$

Let us first dispose of the Poincaré Lemma. As in § 2, we adjoin an *even* variable  $x_{n+1} = t$  and let it commute with everything; its differential  $dt = \tau$  we also let (super) commute with everything:

$$\tau\omega = (-1)^{p(\omega)} \omega\tau, \quad \omega \in \overline{\Omega}^*. \quad (4.10)$$



Using again the unique decomposition

$$\omega = \omega_+ + \tau\omega_-, \quad \omega \in \overline{\Omega}^* = R\langle x, t, y, z \rangle, \quad \omega_{\pm} \in \Omega^*[t], \quad (4.11)$$

we set

$$I(\omega) = \int_0^1 dt \omega_-, \quad (4.12)$$

and define the even ring homomorphism  $A_t : \Omega^* \rightarrow \overline{\Omega}^*$  by the rules

$$A_t(x_i) = tx_i, \quad i = 1, \dots, n, \quad (4.13a)$$

$$A_t(y_i) = ty_i + \tau x_i, \quad i = 1, \dots, n, \quad (4.13b)$$

$$A_t(r) = r, \quad r \in R. \quad (4.13c)$$

These rules imply that

$$(dA_t - A_t d)r = (-1)^{p(r)} r(dA_t - A_t d), \quad r \in R, \quad (4.14a)$$

$$(dA_t - A_t d)x_i = (-1)^{p(i)} tx_i(dA_t - A_t d), \quad i = 1, \dots, n, \quad (4.14b)$$

$$(dA_t - A_t d)y_i = (-1)^{p(i)+1} (ty_i + \tau x_i)(dA_t - A_t d), \quad i = 1, \dots, n, \quad (4.14c)$$

and thus

$$A_t d = dA_t : \Omega^* \rightarrow \overline{\Omega}^*. \quad (4.15)$$

The homotopy formula (2.23):

$$dI(\omega) + Id(\omega) = \omega_+|_{t=1} - \omega_+|_{t=0}, \quad \forall \omega \in \overline{\Omega}^*, \quad (4.16)$$

holds true with the same Proof as in § 2. Therefore, again as in § 2,

$$d(\omega) = 0 \Rightarrow \omega = dIA_t(\omega) + pr^{(0,0)}(\omega), \quad \omega \in \Omega^*. \quad (4.17)$$

Let us now turn to the Lie derivative formulae. First, we define the operation  $X\rfloor$ , for  $X \in \text{Der}(\Omega^*, d)$ , by the rules

$$X\rfloor\omega = 0, \quad p_y(\omega) = 0, \quad (4.18a)$$

$$X\rfloor r\omega = (-1)^{p(r)(p(X)+1)} r(X\rfloor\omega), \quad r \in R, \quad (4.18b)$$

$$X\rfloor x_i\omega = (-1)^{p(i)(p(X)+1)} x_i(X\rfloor\omega), \quad i = 1, \dots, n, \quad (4.18c)$$

$$X\rfloor y_i\omega = X_i\omega + (-1)^{(p(i)+1)(p(X)+1)} y_i(X\rfloor\omega), \quad i = 1, \dots, n, \quad (4.18d)$$

These relations imply, like in § 3, that

$$X\rfloor d(f) = X(f), \quad \forall f \in \mathcal{R} = R\langle x \rangle, \quad (4.19)$$

$$X \rfloor \omega_1 \omega_2 = (X \rfloor \omega_1) \omega_2 + (-1)^{p(\omega_1)(p(X)+1)} \omega_1 (X \rfloor \omega_2), \quad \forall \omega_1, \omega_2 \in \Omega^*, \quad (4.20)$$

$$X(\omega) = (-1)^{p(X)} d(X \rfloor \omega) + X \rfloor d(\omega), \quad \forall \omega \in \Omega^*. \quad (4.21)$$

**Example 4.22.** The differential  $d : \Omega^* \rightarrow \Omega^*$  is an odd derivation, and

$$d \rfloor \omega_1 \omega_2 = (d \rfloor \omega_1) \omega_2 + \omega_1 (d \rfloor \omega_2), \quad \forall \omega_{1,2} \in \Omega^*, \quad (4.23)$$

$$d \rfloor \omega = \deg_y(\omega) \omega, \quad \forall \omega \in \Omega^*. \quad (4.24)$$

Formula (3.28) has the following  $\mathbf{Z}_2$ -graded version:

$$Z_1 \rfloor Z_2 \rfloor \omega = (-1)^{(p(Z_1)+1)(p(Z_2)+1)} (Z_2 \rfloor Z_1 \rfloor \omega), \quad \forall Z_{1,2} \in \text{Der}(\Omega^*, d). \quad (4.25)$$

Formulae (3.41) and (3.39) now become, respectively:

$$Z \rfloor X(\omega) = (-1)^{p(X)(p(Z)+1)} X(Z \rfloor \omega) + (-1)^{p(X)} [Z, X] \rfloor \omega, \quad (4.26)$$

$$\begin{aligned} (-1)^{p(X)(\ell + \sum_{i=1}^{\ell} p(Z_i))} (Z_{\ell} \rfloor \dots \rfloor Z_1 \rfloor X(\omega)) &= X(Z_{\ell} \rfloor \dots \rfloor Z_1 \rfloor \omega) \\ &+ \sum_{\alpha=1}^{\ell} (-1)^{p(X)(\ell - \alpha + \sum_{j \geq \alpha} p(Z_j))} (Z_{\ell} \rfloor \dots \rfloor \hat{Z}_{\alpha} [Z_{\alpha}, X] \rfloor \dots \rfloor Z_1 \rfloor \omega). \end{aligned} \quad (4.27)$$

Finally, formula (3.48) turns into

$$\begin{aligned} Z_{\ell} \rfloor \dots \rfloor Z_1 \rfloor d(\omega) &= (-1)^{\sum_{j=1}^{\ell} (p(Z_j)+1)} d(Z_{\ell} \rfloor \dots \rfloor Z_1 \rfloor \omega) \\ &+ \sum_{\alpha=1}^{\ell} (-1)^{u(\alpha)} Z_{\alpha} (Z_{\ell} \rfloor \dots \rfloor \hat{Z}_{\alpha} \dots \rfloor Z_1 \rfloor \omega) \\ &+ \sum_{\alpha < \beta \leq \ell} (-1)^{v(\alpha, \beta)} (Z_{\ell} \rfloor \dots \rfloor \hat{Z}_{\beta} [Z_{\beta}, Z_{\alpha}] \rfloor \dots \rfloor \hat{Z}_{\alpha} \dots \rfloor Z_1 \rfloor \omega), \end{aligned} \quad (4.28)$$

where

$$u(\alpha) = \sum_{s < \alpha} (p(Z_s) + 1) + p(Z_{\alpha}) \sum_{j > \alpha} (p(Z_j) + 1), \quad (4.29a)$$

$$v(\alpha, \beta) = \sum_{s < \alpha} (p(Z_s) + 1) + p(Z_{\alpha}) \left( 1 + \sum_{\alpha < j < \beta} (p(Z_j) + 1) \right), \quad (4.29b)$$

with the understanding that empty sums contribute nothing, and that for  $\ell = 1$  formula (4.28) becomes simply

$$Z_1 \rfloor d(\omega) = Z_1(\omega) + (-1)^{p(Z_1)+1} d(Z_1 \rfloor \omega), \quad (4.30)$$

which is just the formula (4.21).

**Remark 4.31.** Nothing is sacred about the  $\mathbf{Z}_2$ -grading. We could easily replace the grading group  $\mathbf{Z}_2$  by an arbitrary abelian group  $\Gamma$ . In the *commutative* case, related calculations can be found in [9].

## § 5. $\mathbf{h}$ -Quantum spaces

Let  $R$  continue as the coefficient ring. It is not important what  $R$  really is as long as it is a  $\mathbf{Q}$ -algebra. Let  $h$  be a formal parameter commuting with everything. We shall denote by  $R_h$  either  $R[h]$  or  $R[[h]]$ , depending upon the circumstances. Let

$$R\langle p, q \rangle = R_h\langle p_1, \dots, p_n, q_1, \dots, q_n \rangle \quad (5.1)$$

be the ring of polynomials subject to the relations

$$[p_i, p_j] = [q_i, q_j] = 0, \quad [p_i, q_j] = h\delta_{ij}, \quad 1 \leq i, j \leq n. \quad (5.2)$$

This is our quantum algebra, — or space on which this algebra serves as the algebra of functions. Let us consider differential forms on this space.

Let  $H \in R\langle p, q \rangle$  be a Hamiltonian. We have seen in the preceding paper [11] that even though the  $p$ 's and the  $q$ 's do not commute, there exist the well-defined objects

$$\frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n, \quad (5.3)$$

and that the corresponding partial derivatives commute:

$$\frac{\partial^2 H}{\partial u_\alpha \partial u_\beta} = \frac{\partial^2 H}{\partial u_\beta \partial u_\alpha}, \quad u_{\alpha, \beta} \in \{p_1, \dots, p_n, q_1, \dots, q_n\}. \quad (5.4)$$

Thus, we can define the differential  $d$  on  $R\langle p, q \rangle$  by setting

$$d(H) = \sum_i \left( dp_i \frac{\partial H}{\partial p_i} + dq_i \frac{\partial H}{\partial q_i} \right). \quad (5.5)$$

Alternatively, we can proceed in the spirit of § 2, and define the differential  $d$  to be a derivation of  $R\langle p, q \rangle$  with values in

$$\Omega^* = R_h\langle p, q, \bar{p}, \bar{q} \rangle, \quad (5.6)$$

with

$$d(p_i) = \bar{p}_i, \quad d(q_i) = \bar{q}_i, \quad i = 1, \dots, n. \quad (5.7)$$

Finally, to set the  $d$ -complex in  $\Omega^*$ , we can use the device of § 3 and set the commutation relations

$$dp_i = p_i d + \bar{p}_i, \quad d\bar{p}_i = -\bar{p}_i d, \quad i = 1, \dots, n, \quad (5.8a)$$

$$dq_i = q_i d + \bar{q}_i, \quad d\bar{q}_i = -\bar{q}_i d, \quad i = 1, \dots, n, \quad (5.8b)$$

$$dr = rd, \quad d(r) = 0, \quad r \in R. \quad (5.8c)$$

These are previously the commutation relations (2.5). Since our ring  $R\langle p, q \rangle$  is not free noncommutative anymore, having the quantum commutation relations (5.2) imposed upon it, we have to add the corresponding commutation relations on the differential  $\bar{p}_i$ 's and  $\bar{q}_i$ 's. In view of formulae (5.3)–(5.5), we set

$$[\bar{p}_i, p_j] = [\bar{p}_i, q_j] = [\bar{q}_i, p_j] = [\bar{q}_i, q_j] = 0, \quad 1 \leq i, j \leq n, \quad (5.9)$$

$$[\bar{p}_i, \bar{p}_j]_+ = [\bar{p}_i, \bar{q}_j]_+ = [\bar{q}_i, \bar{q}_j]_+ = 0, \quad 1 \leq i, j \leq n, \quad (5.10)$$

where

$$[u, v]_+ = uv + vu \quad (5.11)$$

is the anti-commutator. We need only to make sure that the old relations (5.2) in  $R\langle p, q \rangle$  and the new ones (5.9), (5.10) in  $\Omega^*$  are compatible, but this is obvious once we apply the differential  $d$  to the relations (5.2).

If we now try to establish a homotopy formula, we quickly discover that this can't be done, since some of the relations (5.2) are not homogeneous and thus preclude the definition of the dual contraction  $A_t$ . What to do?

Consider the ring of *symbols*  $R_h[p, q] = R_h[p_1, \dots, p_n, q_1, \dots, q_n]$ , where the  $p$ 's and the  $q$ 's *commute*. Let us agree to write every polynomial in this ring in the *normal form*, with every monomial written as

$$r q_1^{\cdot\cdot} q_2^{\cdot\cdot} \dots p_1^{\cdot\cdot} \dots p_n^{\cdot\cdot}, \quad r \in R_h. \quad (5.12)$$

We can also agree to use the same arrangement of “*normal quantization*” in the quantum ring  $R_h\langle p, q \rangle$ . Upon this agreement, we see that

- (A) The quantum ring  $R_h\langle p, q \rangle$  and the classical ring  $R_h[p, q]$  are *isomorphic* as filtered *vector spaces* over  $R_h$ ; and,
- (B) With such vector-space isomorphism at hand, the differential  $d$  acts in an identical way on both  $R_h\langle p, q \rangle$  and  $R_h[p, q]$ ; therefore,
- (C) If we also arrange the  $\mathbf{Z}_2$ -graded rings  $\Omega^* R_h\langle p, q \rangle$  and  $\Omega^* R_h[p, q]$  into normal forms, the differential  $d$  will act in an identical way on both of these rings; and thus,
- (D) The de Rham cohomologies of the quantum space are exactly the same as those of the classical one.

But the quantum ring  $R_h\langle p, q \rangle$  has its uses as the fundamental building object possessing quantum differential forms. This will be seen in § 7.

**Remark 5.13.** The same rigidity of the cohomologies can be seen in the more general situation outlined in [11] where the quantum commutation relations (5.2) are replaced by the commutation relations

$$[u_i, u_j] = h c_{ij}, \quad c_{ij} = -c_{ji} \in \mathcal{Z}(R)_h, \quad 1 \leq i, j \leq m, \quad (5.14)$$

in the ring  $R_h\langle u_1, \dots, u_m \rangle$ ; here  $\mathcal{Z}(R)$  is the center of the ring  $R$ . The commutation relations (5.9), (5.10) on the differentials are replaced by the commutation relations

$$[du_i, u_j] = 0, \quad 1 \leq i, j \leq m, \quad (5.15)$$

$$[du_i, du_j]_+ = 0, \quad 1 \leq i, j \leq m. \quad (5.16)$$

## § 6. Quantum Clebsch representations

Let  $\mathcal{G}$  be a Lie algebra and  $\chi : \mathcal{G} \rightarrow \text{End}(V)$  its representation. In Classical mechanics, the symplectic space  $V \oplus V^*$  serves as a symplectic model for the Poisson spaces  $C^\infty(\mathcal{G}^*)$  and  $C^\infty((\mathcal{G} \ltimes V)^*)$ , where  $\mathcal{G} \ltimes V$  is the semidirect sum of  $\mathcal{G}$  and  $V$  w.r.t. the representation  $\chi : \mathcal{G} \ltimes V$  is the vector space  $\mathcal{G} \oplus V$  with the commutator

$$\left[ \begin{pmatrix} g_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} g_2 \\ v_2 \end{pmatrix} \right] = \begin{pmatrix} [g_1, g_2] \\ \chi(g_1)(v_2) - \chi(g_2)(v_1) \end{pmatrix}, \quad g_{1,2} \in \mathcal{G}, \quad v_{1,2} \in V. \quad (6.1)$$

With suitable modifications, the similar picture persists in Classical fluid dynamics, with vector spaces being replaced by differential algebras (see [10].) A close look at the Poisson map  $C^\infty((\mathcal{G} \ltimes V)^*) \rightarrow C^\infty(V \oplus V^*)$ , called nowadays the *Clebsch representation*, shows that it is linear and quadratic in its arguments, and is thus likely to represent the Classical remnant of a more general Quantum map. This is indeed the case, at least for systems with finite number of degrees of freedom. Let us see the details.

Let  $\{e_i\}$  be a basis of  $\mathcal{G}$ , and  $\{f_\alpha\}$  be a basis of  $V$ . Let  $(A_{i\alpha}^\beta)$  be the set of the matrix elements of the representation  $\chi$  on  $V$ :

$$\chi(e_i)(f_\alpha) = \sum_\beta A_{i\alpha}^\beta f_\beta. \quad (6.2)$$

The condition on  $\chi$  to be a representation,

$$\chi([g_1, g_2]) = [\chi(g_1), \chi(g_2)], \quad \forall g_1, g_2 \in \mathcal{G}, \quad (6.3)$$

translates into the set of equalities

$$\sum_k c_{ij}^k A_{k\alpha}^\gamma = \sum_\beta (A_{i\beta}^\gamma A_{j\alpha}^\beta - A_{j\beta}^\gamma A_{i\alpha}^\beta), \quad (6.4)$$

where  $\{c_{ij}^k\}$  are the structure constants of  $\mathcal{G}$  in the basis  $\{e_i\}$ :

$$[e_i, e_j] = \sum_k c_{ij}^k e_k. \quad (6.5)$$

All our constants are from  $R$  which is now assumed to be *commutative*. Let  $\{g^\beta\}$  be the dual basis in  $V^*$ . Let  $\nabla : V \otimes V^* \rightarrow \mathcal{G}^*$  be the basic Clebsch map of Chapter 8 in [10], defined by the formula

$$\langle \nabla(v \otimes v^*), g \rangle = \langle v^*, \chi(g)(v) \rangle, \quad (6.6)$$

so that, in components,

$$f_\alpha \nabla g^\beta = \nabla(f_\alpha \otimes g^\beta) = \sum_i A_{i\alpha}^\beta e^i \quad \Leftrightarrow \quad (6.7a)$$

$$(f_\alpha \nabla g^\beta)_i = A_{i\alpha}^\beta. \quad (6.7b)$$

**Lemma 6.8 (Quantum Clebsch representation.)** *Let  $\{F^\alpha; G_\alpha\}$  be the generators of the Quantum algebra  $R_h\langle F, G \rangle$ , with the commutation relations*

$$[F^\alpha, F^\beta] = [G_\alpha, G_\beta] = 0, \quad [F^\alpha, G_\beta] = h\delta_\beta^\alpha. \quad (6.9)$$

Set

$$e_i = \sum_{\alpha\beta} A_{i\alpha}^\beta F^\alpha G_\beta h^{-1}, \quad (6.10)$$

$$f_\alpha = k G_\alpha h^{-1}, \quad k \in R. \quad (6.11)$$

Then the thus defined elements satisfy the commutation relations of the basis in  $\mathcal{G}$  and in  $\mathcal{G} \ltimes V$  :

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad (6.12)$$

$$[e_i, f_\alpha] = \sum_\beta A_{i\alpha}^\beta f_\beta. \quad (6.13)$$

$$[f_\alpha, f_\beta] = 0. \quad (6.14)$$

**Proof.** We have,

$$[e_i, e_j] \stackrel{[\text{by (6.10)}]}{=} \sum h^{-2} A_{i\alpha}^\beta A_{j\alpha}^\nu [F^\alpha G_\beta, F^\mu G_\nu]. \quad (6.15)$$

Now,

$$[F^\alpha G_\beta, F^\mu G_\nu] = h \left( -\delta_\beta^\mu F^\alpha G_\nu + \delta_\nu^\alpha F^\mu G_\beta \right). \quad (6.16)$$

Indeed,

$$\begin{aligned} [F^\alpha G_\beta, F^\mu G_\nu] &= F^\alpha [G_\beta, F^\mu G_\nu] + [F^\alpha, F^\mu G_\nu] G_\beta \\ &= F^\alpha [G_\beta, F^\mu] G_\nu + F^\mu [F^\alpha, G_\nu] G_\beta \stackrel{[\text{by (6.9)}]}{=} \left( -\delta_\beta^\mu F^\alpha G_\nu + \delta_\nu^\alpha F^\mu G_\beta \right) h. \end{aligned}$$

Substituting (6.16) into (6.15), we find:

$$\begin{aligned} [e_i, e_j] &= \sum h^{-2} A_{i\alpha}^\beta A_{j\alpha}^\nu h \left( -\delta_\beta^\mu F^\alpha G_\nu + \delta_\nu^\alpha F^\mu G_\beta \right) \\ &= -h^{-1} \sum A_{j\gamma}^\nu A_{i\alpha}^\gamma F^\alpha G_\nu + h^{-1} \sum A_{i\gamma}^\nu A_{j\alpha}^\gamma F^\alpha G_\nu \\ &= h^{-1} \sum_{\alpha\nu} F^\alpha G_\nu \sum_\gamma \left( A_{i\gamma}^\nu A_{j\alpha}^\gamma - A_{j\gamma}^\nu A_{i\alpha}^\gamma \right) \\ &\stackrel{[\text{by (6.4)}]}{=} h^{-1} \sum_{\alpha\nu} F^\alpha G_\nu \sum_k c_{ij}^k A_{k\alpha}^\nu = \sum_k c_{ij}^k \sum_{\alpha\nu} h^{-1} A_{k\alpha}^\nu F^\alpha G_\nu \stackrel{[\text{by (6.10)}]}{=} \sum_k c_{ij}^k e_k, \end{aligned}$$

and this is formula (6.12).

Next,

$$\begin{aligned} [e_i, f_\alpha] &\stackrel{[\text{by (6.10), (6.11)}]}{=} \sum h^{-2} A_{i\mu}^\nu k [F^\mu G_\nu, G_\alpha] \\ &= \sum h^{-2} A_{i\mu}^\nu k \delta_\alpha^\nu h G_\nu = \sum A_{i\alpha}^\nu k G_\nu h^{-1} \stackrel{[\text{by (6.11)}]}{=} \sum A_{i\alpha}^\nu f_\nu, \end{aligned}$$

and this is formula (6.13).

Formula (6.14) is obvious.  $\blacksquare$

**Remark 6.17.** The Quantum Clebsch formulae (6.10), (6.11) are *singular* in  $\hbar$  and thus do not allow the passage to the quasiclassical limit. To make sure such passage is possible, we should rescale these formulae into the form:

$$\bar{e}_i = \sum A_{i\alpha}^\beta F^\alpha G_\beta, \quad (6.18)$$

$$\bar{f}_\alpha = k G_\alpha, \quad (6.19)$$

$$[\bar{e}_i, \bar{e}_j] = \hbar \sum c_{ij}^k \bar{e}_k, \quad (6.20)$$

$$[\bar{e}_i, \bar{f}_\alpha] = \hbar \sum A_{i\alpha}^\beta \bar{f}_\beta, \quad (6.21)$$

$$[\bar{f}_\alpha, \bar{f}_\beta] = 0. \quad (6.22)$$

In the limit  $\hbar \rightarrow 0$ ,  $\bar{e}_i$ 's and  $\bar{f}_\alpha$ 's form the polynomial generators of the Poisson function rings  $R[\mathcal{G}^*]$  and  $R[(\mathcal{G} \ltimes V)^*]$ .

**Remark 6.23.** In the older literature, the Quantum Clebsch representations, considered primarily for real and complex semisimple Lie algebras, have been called “canonical realizations” of Lie algebras (see , e.g., [13, 7, 4, 2]); a more recent terminology is “boson representations”. Since there exist also the so-called “boson-fermion representations”, one can suspect that Quantum Clebsch representations can be generalized to include fermions. This is indeed the case. Formulae (6.10)–(6.15) and (6.16) remain unchanged if formulae (6.9) are replaced by the formulae

$$F^\alpha F^\beta - (-1)^{p(\alpha)p(\beta)} F^\beta F^\alpha = 0, \quad (6.24a)$$

$$G_\alpha G_\beta - (-1)^{p(\alpha)p(\beta)} G_\beta G_\alpha = 0, \quad (6.24b)$$

$$F^\alpha G_\beta - (-1)^{p(\alpha)p(\beta)} G_\beta F^\alpha = (-1)^{p(\alpha)} \delta_\beta^\alpha \hbar, \quad (6.24c)$$

and formula (6.14) is replaced by formula

$$f_\alpha f_\beta - (-1)^{p(\alpha)p(\beta)} f_\beta f_\alpha = 0; \quad (6.25)$$

here

$$p(\alpha) = p(F^\alpha) = p(G_\alpha) \quad (6.26)$$

are arbitrary  $\mathbf{Z}_2$ -gradings on the space of Quantum variables  $\{F^\alpha\}$  and  $\{G_\beta\}$ , distinguishing bosons (with  $p(\alpha) = 0$ ) from fermions (with  $p(\alpha) = 1$ ). The details are left to the reader.

The Quantum Clebsch representations will be used in the next Section to construct a complex of differential forms on the Universal enveloping algebra  $U(\mathcal{G})$ .

**Remark 6.27.** The Quantum Clebsch representation constructed in this Section is *general*, i.e., not dependent upon any particular properties of the Lie algebra  $\mathcal{G}$ . When one considers

some *special* Lie algebras, one can naturally expect some extra effects. For example, for the quantum group  $GL(V)$ , acting on a pair of vector spaces  $V$  and  $V^*$  by the rule

$$\mathbf{x}' = M\mathbf{x}, \quad x'_i = \sum_{\alpha} M_{i\alpha} x_{\alpha}, \quad (6.28)$$

$$\mathbf{p}'^t = \mathbf{p}^t M, \quad p'_i = \sum_{\alpha} p_{\alpha} M_{\alpha i}, \quad (6.29)$$

with the commutation relation on  $V$  and  $V^*$  given by the generalized commutation relations of the form

$$\sum_{k\ell} R_{\alpha}^{k\ell} x_k x_{\ell} = 0, \quad \alpha \in \mathcal{A}, \quad (6.30)$$

$$\sum_{\beta} \bar{R}_{\beta}^{k\ell} p_k p_{\ell} = 0, \quad \beta \in \mathcal{B}, \quad (6.31)$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are some index sets, the induced quantum group structure on  $GL(V)$  is easily seen to allow the representation

$$M_{i\alpha} = u_i v_{\alpha}, \quad (6.32)$$

where the  $u$ 's and the  $v$ 's satisfy the commutation relations

$$\sum_{k\ell} R_{\alpha}^{k\ell} u_k u_{\ell} = 0, \quad \alpha \in \mathcal{A}, \quad (6.33)$$

$$\sum_{k\ell} \bar{R}_{\beta}^{k\ell} v_k v_{\ell} = 0, \quad \beta \in \mathcal{B}, \quad (6.34)$$

$$[u_k, v_{\ell}] = 0, \quad \forall k, \ell, \quad (6.35)$$

We shan't pursue this avenue further.

## § 7. Differential forms on Lie algebras

Continuing with the notation of the preceding Section, let  $U(\mathcal{G})$  be the universal enveloping algebra of the Lie algebra  $\mathcal{G}$ . This is simply the noncommutative ring  $R\langle e_1, \dots \rangle$ , subject to the relations

$$e_i e_j - e_j e_i = \sum_k c_{ij}^k e_k, \quad \forall i, j. \quad (7.1)$$

We wish to construct an analog of the ring of differential forms  $\Omega^*$  for  $U(\mathcal{G})$ , preferably on the lines of § 2. In order to achieve, this, we need to determine the commutation relations between the  $e_i$ 's, and  $de_j$ 's, of the form

$$[e_i, de_j] = \sum_k \theta_{ij}^k de_k, \quad \forall i, j. \quad (7.2)$$



To be consistent with the Lie algebra structures (7.1), the relations (7.2) have to be compatible with the relations

$$[de_i, e_j] + [e_i, de_j] = \sum_k c_{ij}^k de_k. \quad (7.3)$$

This amounts to the series of the identities

$$\theta_{ij}^k - \theta_{ji}^k = c_{ij}^k, \quad \forall i, j, k. \quad (7.4)$$

Also, formule (7.2) must define a *representation* of the Lie algebra  $\mathcal{G}$  on the vector space of differentials  $\{de_i\}$ ; by formula (6.4), this amounts to the series of identities

$$\sum_s \left( \theta_{is}^k \theta_{jl}^s - \theta_{js}^k \theta_{il}^s \right) = \sum_s c_{ij}^s \theta_{sl}^k, \quad \forall i, j, k, \ell. \quad (7.5)$$

These are to be compared with the Jacobi identity for the structure constants  $c_{ij}^k$ 's:

$$\sum_i \left( c_{is}^k c_{jl}^s - c_{js}^k c_{il}^s \right) = \sum_s c_{ij}^s c_{sl}^k, \quad \forall i, j, k, \ell. \quad (7.6)$$

Clearly, such structure consists  $\theta_{ij}^k$ 's do not exist in *general*, although they may and do in fact exist in *particular* (see §§ 8, 9). Let us bring in the Quantum Clebsch representation of the preceding Section. Thus, we abandon our initial goal to have a differential-forms-complex solely in terms of the Lie algebra  $\mathcal{G}$  and use the additional data in the form of a representation  $\chi$  of  $\mathcal{G}$  on a vector space  $V$ . By formula (6.10),

$$e_i = h^{-1} \sum_{\alpha\beta} A_{i\alpha}^\beta F^\alpha G_\beta. \quad (7.7)$$

Hence, we can set

$$de_i = h^{-1} \sum A_{i\alpha}^\beta (dF^\alpha G_\beta + F^\alpha dG_\beta). \quad (7.8)$$

Denoting

$$\omega_\beta^\alpha = h^{-1} dF^\alpha G_\beta, \quad \Omega_\beta^\alpha = h^{-1} F^\alpha dG_\beta, \quad (7.9)$$

we get

$$de_i = \sum A_{i\alpha}^\beta (\omega_\beta^\alpha + \Omega_\beta^\alpha). \quad (7.10)$$

Now,

$$\begin{aligned} [e_i, \omega_\alpha^\beta] &= \left[ h^{-1} \sum_{\mu\nu} A_{i\mu}^\nu F^\mu G_\nu, h^{-1} dF^\alpha G_\beta \right] \stackrel{[\text{by (5.9)}]}{=} h^{-2} \sum A_{i\mu}^\nu dF^\alpha [F^\mu, G_\beta] G_\nu \\ &= h^{-1} \sum_\nu A_{i\beta}^\nu dF^\alpha G_\nu = \sum_\nu A_{i\beta}^\nu \omega_\nu^\alpha \Rightarrow \\ [e_i, \omega_\beta^\alpha] &= \sum_\nu A_{i\beta}^\nu \omega_\nu^\alpha. \end{aligned} \quad (7.11)$$

Similarly,

$$\begin{aligned}
[e_i, \Omega_\beta^\alpha] &= \sum h^{-2} A_{i\mu}^\nu [F^\mu G_\nu, F^\alpha dG_\beta] = \sum h^{-2} A_{i\mu}^\nu F^\nu [G_\nu, F^\alpha] dG_\beta \\
&= -h^{-1} \sum A_{i\mu}^\alpha F^\mu dG_\beta = - \sum_\mu A_{i\mu}^\alpha \Omega_\beta^\mu \quad \Rightarrow \\
[e_i, \Omega_\beta^\alpha] &= - \sum_\mu A_{i\mu}^\alpha \Omega_\beta^\mu.
\end{aligned} \tag{7.12}$$

Combining formulae (7.10)–(7.12), we find that

$$[e_i, de_j] = \sum_{\alpha\beta} \left\langle \left( \sum_\gamma A_{i\gamma}^\beta A_{j\alpha}^\gamma \right) \omega_\beta^\alpha - \left( \sum_\gamma A_{j\gamma}^\beta A_{i\alpha}^\gamma \right) \Omega_\beta^\alpha \right\rangle, \tag{7.13}$$

still another indication that our original goal of constructing the differential complex on  $\Omega^*U(\mathcal{G})$  was ill-posed. (If  $\mathcal{G}$  is semisimple or reductive, we can choose some special representation: adjoint, coadjoint, fundamental, etc. But these fall under “special” category. On the other hand, we are interested in a general construction.)

Let us verify that the differential relations (7.10)–(7.12) are compatible with the Lie algebra relations (7.1). We have to check the identity

$$[de_i, e_j] + [e_i, de_j] = \sum_k c_{ij}^k de_k. \tag{7.14}$$

By formula (7.13), for the LHS of formula (7.14) we get

$$\begin{aligned}
[de_i, e_j] + [e_i, de_j] &= \sum_{\alpha\beta} \left\langle \sum_\gamma \left( A_{j\gamma}^\beta A_{i\alpha}^\gamma - A_{i\gamma}^\beta A_{j\alpha}^\gamma \right) \omega_\beta^\alpha - \sum_\gamma \left( A_{j\gamma}^\beta A_{i\alpha}^\gamma - A_{i\gamma}^\beta A_{j\alpha}^\gamma \right) \Omega_\beta^\alpha \right\rangle \\
&= \sum_{\alpha\beta} \left\langle \sum_\gamma \left( A_{i\gamma}^\beta A_{j\alpha}^\gamma - A_{j\gamma}^\beta A_{i\alpha}^\gamma \right) (\omega_\beta^\alpha + \Omega_\beta^\alpha) \right\rangle \stackrel{[\text{by (6.4)}]}{=} \sum_{\alpha\beta} c_{ij}^k A_{k\alpha}^\beta (\omega_\beta^\alpha + \Omega_\beta^\alpha) \\
&\stackrel{[\text{by (7.4)}]}{=} \sum_k c_{ij}^k de_k,
\end{aligned}$$

and this is the RHS of formula (7.14).

The construction of our differential complex is not complete yet, for we have to define the action of the differential  $d$  on the generators  $\omega_\beta^\alpha$  and  $\Omega_\beta^\alpha$ . Keeping our deep-background formulae (7.9) in mind, we see that in the Quantum Clebsch representation we should take

$$d(\omega_\beta^\alpha) = -h^{-1} dF^\alpha dC_\beta, \quad d(\Omega_\beta^\alpha) = h^{-1} dF^\alpha dG_\beta. \tag{7.15}$$

Accordingly, we introduce *new generators*  $\rho_\beta^\alpha$  into  $\Omega^*$ , and set

$$d(\omega_\beta^\alpha) = -\rho_\beta^\alpha, \tag{7.16}$$

$$d(\Omega_\beta^\alpha) = \rho_\beta^\alpha, \tag{7.17}$$

$$d(\rho_\beta^\alpha) = 0. \tag{7.18}$$

To keep track of the differential degrees, let us set

$$p_y(e_i) = p_y(R) = 0, \quad (7.19a)$$

$$p_y(\omega_\beta^\alpha) = p_y(\Omega_\beta^\alpha) = 1, \quad (7.19b)$$

$$p_y(\rho_\beta^\alpha) = 2. \quad (7.19c)$$

These gradings make the differential  $d$  into a homogeneous operator of  $p_y$ -degree 1; the  $\mathbf{Z}_2$ -grading on  $\Omega^*$  is given, as usual, by the elements  $p_y \pmod{2}$ . Formulae (7.10) and (7.16)–(7.18) show that  $d^2 = 0$  on  $\Omega^*$ . However, we still have to verify that our operator  $d$  preserves the commutation relations (7.11), (7.12), and some relations still to come, such as

$$[e_i, \rho_\beta^\alpha] = 0. \quad (7.20)$$

Applying the differential  $d$  to the relation (7.11), rewritten as

$$e_i \omega_\beta^\alpha - \omega_\beta^\alpha e_i = \sum_\nu A_{i\beta}^\nu \omega_\nu^\alpha,$$

we find

$$\sum_{\mu\nu} A_{i\mu}^\nu (\omega_\nu^\mu + \Omega_\nu^\mu) \omega_\beta^\alpha - e_i \rho_\beta^\alpha + \rho_\beta^\alpha e_i + \omega_\beta^\alpha \sum_{\mu\nu} A_{i\mu}^\nu (\omega_\nu^\mu + \Omega_\nu^\mu) = - \sum_\nu A_{i\beta}^\nu \rho_\nu^\alpha. \quad (7.21)$$

Remembering the nature of  $\rho_\beta^\alpha$  as  $h^{-1}dF^\alpha G_\beta$ , let us postulate that the  $\rho_\beta^\alpha$ 's commute with everything:

$$[\rho_\beta^\alpha, e_i] = 0, \quad (7.22a)$$

$$[\rho_\beta^\alpha, \omega_\nu^\mu] = 0, \quad (7.22b)$$

$$[\rho_\beta^\alpha, \Omega_\nu^\mu] = 0, \quad (7.22c)$$

$$[\rho_\beta^\alpha, \rho_\nu^\mu] = 0. \quad (7.22d)$$

Obviously, these relations remain consistent when acted upon by the differential  $d$ . And while we are at it, we can make use of the defining background relations (7.9) and postulate the commutation relations

$$[\omega_\beta^\alpha, \omega_\nu^\mu]_+ = 0, \quad (7.23a)$$

$$[\Omega_\beta^\alpha, \Omega_\nu^\mu]_+ = 0, \quad (7.23b)$$

$$[\omega_\beta^\alpha, \Omega_\nu^\mu]_+ = -\delta_\beta^\mu \rho_\nu^\alpha. \quad (7.23c)$$

The latter formula is suggested by the following background calculation:

$$\begin{aligned} [\omega_\beta^\alpha, \Omega_\nu^\mu]_+ &= h^{-2}(dF^\alpha G_\beta F^\mu dG_\nu + F^\mu dG_\nu dF^\alpha G_\beta) \\ &= h^{-2}dF^\alpha dG_\nu (G_\beta F^\mu - F^\mu G_\beta) = -h^{-1}dF^\alpha dG_\nu \delta_\beta^\mu = -\delta_\beta^\mu \rho_\nu^\alpha. \end{aligned} \quad (7.24)$$

Now, substituting formulae (7.22a), (7.23a) into the identity to be verified, (7.21), we get

$$\sum_{\mu\nu} A_{i\mu}^\nu (\Omega_\nu^\mu \omega_\beta^\alpha + \omega_\beta^\alpha \Omega_\nu^\mu) = - \sum_\nu A_{i\beta}^\nu \rho_\nu^\alpha,$$

which is true in view of formula (7.23c). Similarly, applying the differential  $d$  to formula (7.12), we get

$$[e_i, \rho_\beta^\alpha] + \left[ \sum_{\mu\nu} A_{i\mu}^\nu (\omega_\nu^\mu + \Omega_\nu^\mu), \Omega_\beta^\alpha \right]_+ = - \sum_\mu A_{i\mu}^\alpha \rho_\beta^\mu,$$

which is true in view of formulae (7.22a), (7.23b), (7.23c).

Finally, applying the differential  $d$  to the remaining relations (7.23) and using the formula

$$d([\varphi, \psi]_+) = [d(\varphi), \psi] - [\varphi, d(\psi)], \quad p(\varphi) = p(\psi) = 1 \in \mathbf{Z}_2, \quad (7.25)$$

we see that the resulting relations are satisfied in view of formulae (7.22), (7.16)–(7.18). The end result is the  $d$ -complex  $\Omega^*$ , with the generators  $\{e_i\}$ ,  $\{\omega_\beta^\alpha\}$ ,  $\{\Omega_\beta^\alpha\}$ ,  $\{\rho_\beta^\alpha\}$ , the relations (7.1), (7.11), (7.12), (7.22), (7.23), and the action of the differential  $d$  given by the formulae (7.10), (7.16)–(7.18). The Quantum generators  $\{F^\alpha\}$  and  $\{G_\alpha\}$ , having served their suggestive purpose, do not enter into the picture anymore. But they still can be of some use: note that our complex  $\Omega^*$  is not finite-dimensional over  $U(\mathcal{G})$ . To make it so, we can use formulae (7.9), (7.15) and impose the additional relations

$$\omega_\mu^\alpha \omega_\nu^\alpha = 0, \quad (7.26a)$$

$$\Omega_\beta^\mu \Omega_\beta^\nu = 0, \quad (7.26b)$$

$$\omega_\mu^\alpha \rho_\nu^\alpha = 0, \quad (7.26c)$$

$$\Omega_\beta^\mu \rho_\beta^\nu = 0, \quad (7.26d)$$

$$\rho_\mu^\alpha \rho_\nu^\alpha = 0, \quad \rho_\beta^\mu \rho_\beta^\nu = 0. \quad (7.26e)$$

These relations are obviously preserved under the action of the differential  $d$ . The resulting complex  $\{\Omega^{*fin}, d\}$ , and the bigger complex  $\{\Omega^*, d\}$ , are easily seen to be natural in the category of  $\mathcal{G}$ -modules. Both these complexes, suggested by Quantum mechanical considerations, are quite different from the usual Lie-algebraic ones (see [5, 6]), and the low-dimensional cohomologies of the new complexes should have a different interpretation as well.

**Remark 7.27.** The Quantum Clebsch map (7.7) which is serving as a motivator of the  $\{\Omega^*, d\}$ -complex, is constructed from elements of both  $V$  and  $V^*$ . Accordingly, nothing is gained if we replace the representation  $\chi$  on  $V$  by the dual representation  $\chi^d$  on  $V^*$ .

In the next three Sections we shall look at the special Lie algebras  $aff(1)$ ,  $gl(V)$ , and  $so(V)$ , where the size of the differential complex  $\{\Omega^*, d\}$  constructed in this Section can be substantially reduced.

### § 8. The Lie algebra $\mathfrak{aff}(1)$ and its generalizations

Let  $G = \text{Aff}(1)$  be the Lie group of affine transformations of the line,

$$\{x \mapsto x' = ax + b, \quad a \text{ is invertible.}\} \quad (8.1)$$

From the matrix representation

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \quad (8.2)$$

of this Lie group, we can represent the Lie algebra  $\mathcal{G} = \mathfrak{aff}(1)$  as the subspace in  $\mathfrak{gl}(2)$  of the form

$$\mathfrak{aff}(1) = \left\{ \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \right\}. \quad (8.3)$$

Setting

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (8.4)$$

we get the commutator in  $\mathcal{G}$

$$[e_1, e_2] = e_2. \quad (8.5)$$

The same commutator relation (8.5) is afforded by the following generators in the Quantum algebra  $R_h\langle p, q \rangle$ :

$$E_1 = p, \quad E_2 = e^{q/h}. \quad (8.6)$$

Since

$$d(E_1) = dp, \quad d(E_2) = h^{-1}E_2 dq, \quad (8.7)$$

we find

$$[d(E_1), E_1] = [d(E_1), E_2] = 0, \quad (8.8a)$$

$$[E_1, d(E_2)] = d(E_2), \quad [E_2, d(E_2)] = 0. \quad (8.8b)$$

Thus, we can take the relations (8.8) as defining the commutation relations in  $\Omega^*(U(\mathcal{G}))$ :

$$[e_1, de_1] = [e_2, de_1] = 0, \quad (8.9a)$$

$$[e_1, de_2] = de_2, \quad [e_2, de_2] = 0. \quad (8.9b)$$

To make the combined relations (8.5), (8.9) in  $\Omega^*$  self-consistent, we have to apply the differential  $d$  to the commutation relations (8.9). We thus obtain:

$$(de_1)^2 = (de_2)^2 = 0, \quad (8.10a)$$

$$(de_1)(de_2) + (de_2)(de_1) = 0. \quad (8.10b)$$

To show that the cohomologies of the constructed complex  $\{\Omega^*, d\}$  are trivial, we could in principle embed the complex  $\Omega^*(U(\mathcal{G}))$  into  $\Omega^*(R_h\langle p, q \rangle)$ ; the latter has been proven to be trivial in § 3, but only in the *polynomial* setting, and formula (8.6) contains the exponential function; the latter, in addition, is *singular* in  $h$ . This is not fatal for the argument, but it's more efficient to use the *method* of § 3 instead of the final result. Namely, let's identify  $U(\mathcal{G})$ , as a vector space, with the polynomial ring  $R[e_1, e_2]$  via the *normal* ordering of monomials in the form

$$\{re_1^{\cdots} e_2^{\cdots}\} \quad (8.11)$$

Then, by formulae (8.9)

$$d(re_1^n e_2^m) = r(ne_1^{n-1} e_2^m de_1 + me_1^n e_2^{m-1} de_2), \quad (8.12)$$

so that  $\{\Omega^*(U(\mathcal{G})), d\}$  is isomorphic, as a vector space, to  $\{\Omega^*(R[e_1, e_2]), d\}$ . Thus, the cohomologies of both are identical, and trivial.

The example of the Lie algebra  $aff(1)$  suggests that one may have a similar result for more general solvable Lie algebras. Another generalization, less sweeping, is to replace our Lie algebra  $aff(1)$  by the Lie algebra  $\mathcal{G} = \mathcal{G}(A)$ , where  $A$  is an arbitrary constant  $n \times n$  matrix, and  $\mathcal{G}(A)$  has the generators

$$e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n, \quad (8.13)$$

with the relations

$$[e_i, e_j] = [\bar{e}_i, \bar{e}_j] = 0, \quad 1 \leq i, j \leq n, \quad (8.14a)$$

$$[e_i, \bar{e}_j] = A_{ji} \bar{e}_j, \quad 1 \leq i, j \leq n. \quad (8.14b)$$

This Lie algebra has the Quantum model  $R_h\langle p_1, \dots, p_n, q_1, \dots, q_n \rangle$ , with

$$E_i = p_i, \quad \bar{E}_i = \exp\left(h^{-1} \sum_s A_{is} q_s\right), \quad 1 \leq i \leq n. \quad (8.15)$$

Accordingly, we impose the following commutation relations in  $\Omega^*(U(\mathcal{G}))$ :

$$[de_i, e_j] = [de_i, \bar{e}_j] = 0 = [d\bar{e}_i, \bar{e}_j], \quad 1 \leq i, j \leq n, \quad (8.16a)$$

$$[e_i, d\bar{e}_j] = A_{ji} d\bar{e}_j, \quad 1 \leq i, j \leq n. \quad (8.16b)$$

To insure self-consistency between relations (8.14), (8.16), we have to adjoin the skewsymmetry relations

$$[de_i, de_j]_+ = [de_i, d\bar{e}_j]_+ = [d\bar{e}_i, d\bar{e}_j]_+ = 0, \quad 1 \leq i, j \leq n. \quad (8.17)$$

To see that cohomologies of the constructed complex are trivial, we identify  $U(\mathcal{G})$  with  $R[e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n]$  (again, as vector spaces) via the normal ordering

$$\{re_1^{\cdots} \dots e_n^{\cdots} \bar{e}_1^{\cdots} \dots \bar{e}_n^{\cdots}\}, \quad (8.18)$$

and agree to write the differentials  $d(\dots)$  on the *right* from the normalized monomials (8.18). The complex  $\{\Omega^*(U(\mathcal{G})), d\}$  then looks exactly as the one for the commutative polynomial algebra  $R[e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n]$ .

It remains to give the algebra  $\mathcal{G}(A)$  a suitable monicker. The traditional method to chose such is to attach to the nameless subject of attention the adjective Schrödinger, Heisenberg, Dirac, etc., but these worthies have already everything under the sun named after them. Accordingly, I shall call the algebra  $\mathcal{G}(A)$  the *Ehrenfest algebra*.

## § 9. The Lie algebra $gl(V)$

The Lie algebra  $gl(V)$  has two most natural representations: the natural actions on  $V$  and  $V^*$ . If we chose a basis  $\{f_\alpha\}$  in  $V$  and the corresponding basis of elementary matrices  $\{e_{ij}\}$  in  $End(V)$ , then

$$e_{ij}(f_\alpha) = \delta_{j\alpha} f_i. \quad (9.1)$$

Thus, the structure constants entering formulae (6.2) and (7.10) are

$$A_{ij|\alpha}^\beta = \delta_{j\alpha} \delta_i^\beta. \quad (9.2)$$

Accordingly, formulae (7.10)–(7.12) become

$$d(e_{ij}) = \omega_i^j + \Omega_i^j, \quad (9.3a)$$

$$[e_{ij}, \omega_\beta^\alpha] = \delta_{j\beta} \omega_i^\alpha, \quad (9.3b)$$

$$[e_{ij}, \Omega_\beta^j] = -\delta_{i\alpha} \Omega_\beta^j. \quad (9.3c)$$

The defining relations on  $gl(V)$ ,

$$[e_{i\alpha}, e_{j\beta}] = \delta_{j\alpha} e_{i\beta} - \delta_{i\beta} e_{j\alpha}, \quad (9.5)$$

and the remaining unchanged relations (7.16)–(7.18), (7.22), (7.23), complete the picture. We don't have to bother with the dual representation, since it's accounted for simply by replacing in the background picture each monomial  $p_\alpha x_\beta$  by the monomial  $-x_\alpha p_\beta$ , the result of the canonical transformation

$$p_\alpha \mapsto -x_\alpha, \quad x_\alpha \mapsto p_\alpha. \quad (9.6)$$

However, the Lie algebra  $gl(V)$  is special in that it is a Lie algebra generated by the associative algebra  $End(V)$ . So, in this case,  $U(\mathcal{G}) \approx \mathcal{G}$  as a vector space. Accordingly, we can break up the differential of the commutation relations (9.4) either as

$$[e_{i\alpha}, de_{j\beta}] = \delta_{j\alpha} de_{i\beta} \quad (9.7)$$

or as

$$[e_{i\alpha}, de_{j\beta}] = -\delta_{i\beta} de_{j\alpha}. \quad (9.8)$$

Obviously, each one of these formulae agrees with the differential  $d$  applied to the commutation relations (9.4). It's easy to see that formulae (9.6) and (9.7) correspond to the representations  $\hat{L}$  and  $-\hat{R}$  of the Lie algebra  $Lie(R)$  of an associative ring  $R$  ( $= End(V)$  in our case) on the ring  $R$  itself. Here

$$\hat{L}_X(r) = Xr, \quad X \in Lie(R), \quad r \in R, \quad (9.8a)$$

$$\hat{R}_X(r) = rX, \quad (9.8b)$$

are the left and the right multiplication operators.

Alternatively, we can treat the commutation relations (9.6) and (9.7) in the spirit of the Diamond Lemma [1], as the rules allowing us to move the differentials  $de_{j\beta}$ 's to the left, say, from the elements  $e_{i\alpha}$ 's:

$$e_{i\alpha}M_{j\beta} = M_{j\beta}e_{i\alpha} + \delta_{j\alpha}M_{i\beta}, \quad (9.9)$$

$$e_{i\alpha}M_{j\beta} = M_{j\beta}e_{i\alpha} - \delta_{i\beta}M_{j\alpha}, \quad (9.10)$$

where  $M_{j\beta}$  stands temporarily instead of the more cumbersome  $de_{j\beta}$ ; formulae (9.9) and (9.10) are reformulations of formulae (9.6) and (9.7) respectively. To see that the moving rules (9.9) and (9.10) are consistent with the commutators (9.4), we calculate, first, for the rule (9.9):

$$\begin{aligned} e_{i\alpha}e_{j\beta}M_{k\gamma} &= e_{i\alpha}(M_{k\gamma}e_{j\beta} + \delta_{k\beta}M_{j\gamma}) \\ &= (M_{k\gamma}e_{i\alpha} + \delta_{k\alpha}M_{i\gamma})e_{j\beta} + \delta_{k\beta}(M_{j\gamma}e_{i\alpha} + \delta_{j\alpha}M_{i\gamma}). \end{aligned} \quad (9.11a)$$

Interchanging  $e_{i\alpha}$  and  $e_{j\beta}$ , we get

$$e_{j\beta}e_{i\alpha}M_{k\gamma} = (M_{k\gamma}e_{j\beta} + \delta_{k\beta}M_{j\gamma})e_{i\alpha} + \delta_{k\alpha}(M_{i\gamma}e_{j\beta} + \delta_{i\beta}M_{j\gamma}). \quad (9.11b)$$

Subtracting (9.11b) from (9.11a), we find

$$(e_{i\alpha}e_{j\beta} - e_{j\beta}e_{i\alpha})M_{k\gamma} = M_{k\gamma}(e_{i\alpha}e_{j\beta} - e_{j\beta}e_{i\alpha}) + \delta_{k\beta}\delta_{j\alpha}M_{i\gamma} - \delta_{k\alpha}\delta_{i\beta}M_{j\gamma}. \quad (9.12)$$

On the other hand,

$$\begin{aligned} [e_{i\alpha}, e_{j\beta}]M_{k\gamma} &= (\delta_{j\alpha}e_{i\beta} - \delta_{i\beta}e_{j\alpha})M_{k\gamma} \\ &= \delta_{j\alpha}(M_{k\gamma}e_{i\beta} + \delta_{k\beta}M_{i\gamma}) - \delta_{i\beta}(M_{k\gamma}e_{j\alpha} + \delta_{k\alpha}M_{j\gamma}), \end{aligned} \quad (9.13)$$

and this is the same as formula (9.12).

The relation (5.10) can be handled in the same way:

$$\begin{aligned} e_{i\alpha}e_{j\beta}M_{k\gamma} &= e_{i\alpha}(M_{k\gamma}e_{j\beta} - \delta_{j\gamma}M_{k\beta}) \\ &= (M_{k\gamma}e_{i\alpha} - \delta_{i\gamma}M_{k\alpha})e_{j\beta} - \delta_{j\gamma}(M_{k\beta}e_{i\alpha} - \delta_{i\beta}M_{k\alpha}) \quad \Rightarrow \end{aligned} \quad (9.14a)$$

$$e_{j\beta}e_{i\alpha}M_{k\gamma} = (M_{k\gamma}e_{j\beta} - \delta_{j\gamma}M_{k\beta})e_{i\alpha} - \delta_{i\gamma}(M_{k\alpha}e_{j\beta} - \delta_{j\alpha}M_{k\beta}) \quad \Rightarrow \quad (9.14b)$$

$$(e_{i\alpha}e_{j\beta} - e_{j\beta}e_{i\alpha})M_{k\gamma} = M_{k\gamma}(e_{i\alpha}e_{j\beta} - e_{j\beta}e_{i\alpha}) + \delta_{j\gamma}\delta_{i\beta}M_{k\alpha} - \delta_{i\gamma}\delta_{j\alpha}M_{k\beta}, \quad (9.15)$$

$$(\delta_{j\alpha}e_{i\beta} - \delta_{i\beta}e_{j\alpha})M_{k\gamma} = M_{k\gamma}(\delta_{j\alpha}e_{i\beta} - \delta_{i\beta}e_{j\alpha}) - \delta_{j\alpha}\delta_{i\gamma}M_{k\beta} + \delta_{i\beta}\delta_{j\gamma}M_{k\alpha}, \quad (9.16)$$

and the last two formulae are identical. To complete our differential complex, we have to apply the differential  $d$  to the relations (9.6) or (9.7). In each of these two cases the result is the same:

$$[de_{i\alpha}, de_{j\beta}]_+ = 0. \quad (9.17)$$

Thus, we have two differential complexes on  $gl(V)$ : (9.4), (9.6), (9.17) and (9.4), (9.7), (9.17). It would be interesting to calculate the corresponding cohomologies.



**Remark 9.18.** The Lie algebra  $gl(V)$  has the Cartan involution  $\theta$ ,

$$\theta(g) = -g^t, \quad \theta(e_{i\alpha}) = -e_{\alpha i}. \quad (9.18a)$$

Extended naturally to the differentials,

$$\theta(de_{i\beta}) = -de_{\beta j}, \quad (9.18b)$$

the isomorphism  $\theta$  interchanges the commutation rules (9.6) and (9.7). Therefore, the two differential complexes on  $gl(V)$  are isomorphic.

**Remark 9.19.** Neither of the formulae (9.6), (9.7) would allow the reduction from  $gl$  to  $sl$ . (See also Remark 10.14.)

**Remark 9.20.** Each one of the two differential complexes constructed above on the Lie algebra  $gl(V)$  can be further reduced onto 4 subalgebras: upper-triangular; lower-triangular; upper-nilpotent (upper-triangular with zeroes on the diagonal); lower-nilpotent (lower-triangular with zeroes on the diagonal). Some of these subalgebras appear useful in many different circumstances. For example, the Lie algebra  $\Delta^+$  of upper-triangular matrices underwrites the 1<sup>st</sup> Hamiltonian structure of the lattice KP hierarchy [8]. Moreover, since that hierarchy is universal w.r.t. to its finite-components cut-outs, one immediately sees that cutting-off in  $\Delta^+$  all diagonals above a fixed one results in a Lie algebra as nice, w.r.t. to the differential-forms complex, as  $\Delta^+$  itself. Moreover still, since the KP hierarchy can be considered either on infinite or periodic lattice, the same conclusion applies to  $\Delta^+$  and all its cut-offs. Similar considerations are pertinent for the other 3 Lie subalgebras of this Remark.

Notice that the Lie subalgebra  $\Delta^+$  provides *another* generalization of the differential complex constructed in the preceding Section for the Lie algebra  $aff(1)$ . How different is it? Consider, in each of the 2 complexes, a subcomplex generated by

$$x = e_{11}, \quad y = e_{12}, \quad dx = de_{11}, \quad dy = de_{12}. \quad (9.21)$$

In the 1<sup>st</sup> complex (9.6), we have

$$[x, y] = y, \quad (9.22)$$

$$[x, dx] = dx, \quad [x, dy] = dy, \quad (9.23a)$$

$$[y, dx] = [y, dy] = 0, \quad (9.23b)$$

$$(dx)^2 = (dy)^2 = dx dy + dy dx = 0. \quad (9.23c)$$

In the 2<sup>nd</sup> complex (9.7) we have

$$[dx, x] = dx, \quad [dx, y] = dy, \quad (9.24a)$$

$$[dy, x] = 0, \quad [dy, y] = 0, \quad (9.24b)$$

$$(dx)^2 = (dy)^2 = dx dy + dy dx = 0. \quad (9.24c)$$

Since in each case, (9.23) or (9.24),  $[x, dx] \neq 0$ , we got something quite different from the formulae in § 8. Let us look more closely at the new formulae. Starting with the 1<sup>st</sup>

complex (9.22)–(9.23), let us agree to write elements of  $U(\mathcal{G})$ ,  $\mathcal{G} = aff(1)$ , in the normal form

$$ry^{\cdots}x^{\cdots}, \quad (9.25a)$$

and elements of  $\Omega^1$  in the form

$$dya + dx b, \quad (9.25b)$$

with  $a, b \in U(\mathcal{G})$  written in the normal form (9.25a). Since, as can be easily seen by induction,

$$d(f(x)) = dx(f(x+1) - f(x)), \quad (9.26)$$

we arrive at the following conclusion: identifying  $U(\mathcal{G})$  with  $R[x, y]$  via formula (9.25a), the differential  $d$  on  $U(\mathcal{G})$  acts by the rule

$$d(f(y, x)) = dy f_y + dx(f(y, x+1) - f(y, x)), \quad f \in R[x, y] \quad (9.27)$$

It follows that every closed 1-form is exact. Indeed, let the form

$$\omega = dya + dx b, \quad a, b \in R[x, y], \quad (9.28)$$

be closed:

$$a(y, x+1) - a(y, x) = b_y(y, x). \quad (9.29)$$

Find  $F \in R[x, y]$  such that

$$a = F_y, \quad (9.30)$$

and set

$$G = b - (F(y, x+1) - F(y, x)). \quad (9.31)$$

Then the closedness condition (9.29) becomes

$$G_y = 0 \quad \Rightarrow \quad G = G(x), \quad (9.32)$$

and thus

$$\omega = dy F_y + dx(F(y, x+1) - F(y, x)) + dx G(x) = d(F) + dx G(x). \quad (9.33)$$

Since the map

$$R[x] \ni G \mapsto G(x+1) - G(x) \in R[x] \quad (9.34)$$

is an epimorphism,

$$\{\text{every 1-form } G(x)dx \text{ is exact}\}. \quad (9.35)$$

Thus, the 1-form  $\omega$  (9.33) is exact as well.

In the second complex, (9.22), (9.24), the situation is similar. Taking the normal form in  $U(\mathcal{G})$  to be

$$rx^{\cdots}y^{\cdots}, \quad (9.36)$$

and writing elements of  $\Omega^1$  as

$$\omega = dxa + dyb, \quad a, b \in R[x, y], \quad (9.37)$$

we see that

$$d(f(x, y)) = dx(f(x, y) - f(x - 1, y)) + dyf_y. \quad (9.38)$$

Thus, if the 1-form  $\omega$  (9.37) is closed,

$$a_y = b(x, y) - b(x - 1, y), \quad (9.39)$$

we first find  $F \in R[x, y]$  such that

$$b = F_y, \quad (9.40)$$

and set

$$G = a - (F(x, y) - F(x - 1, y)). \quad (9.41)$$

The closedness condition (9.39) then becomes

$$G_y = 0 \quad \Rightarrow \quad G = G(x),$$

and thus

$$\omega = dx(F(x, y) - F(x - 1, y)) + dyF_y + dxG(x) = d(F) + dxG(x), \quad (9.42)$$

and, again, since the map

$$R[x] \ni G(x) \mapsto G(x) - G(x - 1) \in R[x] \quad (9.43)$$

is onto, the closed 1-form  $\omega$  is exact. Thus, we have 3 different complexes for the Lie algebra  $\mathcal{G} = aff(1)$ , all with identically trivial cohomologies.

## § 10. The Lie algebra $so(n)$

Consider the Lie subalgebra  $so(V)$  of  $gl(V)$  consisting of skewsymmetric matrices (recall that we have fixed a basis on  $V$ ), with the basis

$$M_{ij} = -M_{ji} = e_{ij} - e_{ji}, \quad i \neq j. \quad (10.1)$$

The commutation relations (9.4) for  $gl(V)$  imply the following commutation relations for  $so(V)$ :

$$[M_{i\alpha}, M_{j\beta}] = \delta_{j\alpha}M_{i\beta} - \delta_{i\beta}M_{j\alpha} - \delta_{\alpha\beta}M_{ij} - \delta_{ij}M_{\alpha\beta}; \quad (10.2)$$

it is understood that  $M_{ij}$  vanishes whenever  $i = j$ . It's easy to see that neither of the special  $gl(V)$  relations, (9.6) or (9.7), reduces onto  $so(V)$ . Hence, we have to start from scratch.

The Quantum Clebsch representation for  $gl(V)$ , (6.10), (9.2),

$$e_{i\alpha} = h^{-1}p_{\alpha}x_i, \quad (10.3)$$

induces the corresponding representation on  $so(V)$ :

$$M_{i\alpha} = -h^{-1}(p_i x_\alpha - p_\alpha x_i). \quad (10.4)$$

Hence,

$$d(M_{i\alpha}) = h^{-1}((dp_\alpha \cdot x_i - dx_\alpha \cdot p_i) - (dp_i \cdot x_\alpha - dx_i \cdot p_\alpha)) = \theta_{i\alpha} - \theta_{\alpha i}, \quad (10.5)$$

where

$$\theta_{i\alpha} = h^{-1}(p_\alpha dx_i - x_\alpha dp_i). \quad (10.6)$$

Let us next determine the commutation relations between the  $M_{i\alpha}$ 's and the  $\theta_{j\beta}$ 's. By formulae (10.4) and (10.6),

$$\begin{aligned} [M_{i\alpha}, \theta_{j\beta}] &= (-h)^{-2}[p_i x_\alpha - p_\alpha x_i, x_\beta dp_j - p_\beta dx_j] \\ &= h^{-2}dp_j(x_\alpha h\delta_{i\beta} - x_i h\delta_{\alpha\beta}) + h^{-2}dx_j(p_j h\delta_{\beta\alpha} - p_\alpha h\delta_{\beta i}) \\ &= h^{-1}\delta_{\alpha\beta}(p_i dx_j - x_i dp_j) - h^{-1}\delta_{i\beta}(p_\alpha dx_j - x_\alpha dp_j) \stackrel{[\text{by (10.6)}]}{=} \delta_{\alpha\beta}\theta_{ji} - \delta_{i\beta}\theta_{j\alpha} : \\ [M_{i\alpha}, \theta_{j\beta}] &= \delta_{\alpha\beta}\theta_{ji} - \delta_{i\beta}\theta_{j\alpha}. \end{aligned} \quad (10.7)$$

These have been suggestive background calculations. We now have to check the consistency of formulae (10.2), (10.5), (10.7). Applying the differential  $d$  to the LHS of formula (10.2), we get

$$\begin{aligned} d([M_{i\alpha}, M_{j\beta}]) &= [M_{i\alpha}, d(M_{j\beta})] - [M_{j\beta}, d(M_{i\alpha})] \\ &\stackrel{[\text{by (10.5)}]}{=} [M_{i\alpha}, \theta_{j\beta} - \theta_{\beta j}] - [M_{j\beta}, \theta_{i\alpha} - \theta_{\alpha i}] \stackrel{[\text{by (10.7)}]}{=} (\delta_{\alpha\beta}\theta_{ji} - \delta_{i\beta}\theta_{j\alpha}) \\ &\quad - (\delta_{\alpha j}\theta_{\beta i} - \delta_{ij}\theta_{\beta\alpha}) - (\delta_{\beta\alpha}\theta_{ij} - \delta_{j\alpha}\theta_{i\beta}) + (\delta_{\beta i}\theta_{\alpha j} - \delta_{ji}\theta_{\alpha\beta}) \\ &= \delta_{j\alpha}(\theta_{i\beta} - \theta_{\beta i}) - \delta_{i\beta}(\theta_{j\alpha} - \theta_{\alpha j}) - \delta_{\alpha\beta}(\theta_{ij} - \theta_{ji}) - \delta_{ij}(\theta_{\alpha\beta} - \theta_{\beta\alpha}) \\ &\stackrel{[\text{by (10.5)}]}{=} \delta_{j\alpha}d(M_{i\beta}) - \delta_{i\beta}d(M_{j\alpha}) - \delta_{\alpha\beta}d(M_{ij}) - \delta_{ij}d(M_{\alpha\beta}), \end{aligned}$$

and this is the differential of the *RHS* of formula (10.2). Now, set

$$d(\theta_{i\alpha}) = \rho_{i\alpha} = \rho_{\alpha i}, \quad (10.8)$$

$$d(\rho_{i\alpha}) = 0. \quad (10.9)$$

By formula (10.6),

$$\rho_{i\alpha} = d(\theta_{i\alpha}) = h^{-1}(dp_\alpha dx_i - dx_\alpha dp_i); \quad (10.10)$$

we thus impose the commutation relations

$$[\rho_{i\alpha}, M_{j\beta}] = 0, \quad (10.11a)$$

$$[\rho_{i\alpha}, \theta_{j\beta}] = 0, \quad (10.11b)$$

$$[\rho_{i\alpha}, \rho_{j\beta}] = 0. \quad (10.11c)$$

Applying the differential  $d$  to the relations (10.7), we get

$$[\theta_{i\alpha} - \theta_{\alpha i}, \theta_{j\beta}]_+ + [M_{i\alpha}, \rho_{j\beta}] = \delta_{\alpha\beta}\rho_{ji} - \delta_{i\beta}\rho_{j\alpha}. \quad (10.12)$$

Thus, we need to determine  $[\theta_{i\alpha}, \theta_{j\beta}]_+$ 's. Using the background formulae (10.6), we find

$$\begin{aligned} [\theta_{i\alpha}, \theta_{j\beta}]_+ &= h^{-2}((p_\alpha dx_i - x_\alpha dp_i)(p_\beta dx_j - x_\beta dp_j) + (p_\beta dx_j - x_\beta dp_j)(\rho_\alpha dx_i - x_\alpha dp_i)) \\ &= -h^{-2}(x_\alpha p_\beta dp_i dx_j + p_\alpha x_\beta dx_i dp_j + x_\beta p_\alpha dp_j dx_i + p_\beta x_\alpha dx_j dp_i) \\ &= h^{-2}dp_i dx_j (-x_\alpha p_\beta + p_\beta x_\alpha) - h^{-2}dx_i dp_j (p_\alpha x_\beta - x_\beta p_\alpha) \\ &= \delta_{\alpha\beta}h^{-1}(dp_j dx_j - dx_i dp_j) \stackrel{[\text{by (10.10)}]}{=} \delta_{\alpha\beta}\rho_{ji}; \\ [\theta_{i\alpha}, \theta_{j\beta}]_+ &= \delta_{\alpha\beta}\rho_{ji}. \end{aligned} \quad (10.13)$$

Taking this formula as a new relation, substituting it into the *LHS* of formula (10.12), and remembering formula (10.11a), we find

$$[\theta_{i\alpha}, \theta_{j\beta}]_+ - [\theta_{\alpha i}, \theta_{j\beta}]_+ = \delta_{\alpha\beta}\rho_{ji} - \delta_{i\beta}\rho_{j\alpha},$$

and this is the *RHS* of formula (10.12). It remains to apply the differential  $d$  to each of the relations (10.11a,b,c), (10.13), and in each case we get an identitically satisfied relation.

Thus, the differential complex  $\Omega^*$  on  $so(V)$  has: 1) the generators  $\{M_{ij} = -M_{ji}, i \neq j\}$ ,  $\{\theta_{ij}\}$ ,  $\{\rho_{ij} = \rho_{ji}\}$ ; 2) the action of the differential  $d$ , (10.5), (10.8) (10.9); 3) and the relations (10.2), (10.7), (10.11), (10.13). We see that in addition to the generators  $M_{ij}$ 's of  $\mathcal{G}$ , we had to introduce some extra generators,  $\theta_{ij}$ 's and  $\rho_{ij}$ 's, to complete the complex  $\Omega^*U(so(n))$ ; however, the number of extra generators has turned out to be smaller than what one would have expected from the general formulae of § 7.

**Remark 10.14.** Is it possible to construct a differential forms complex on the Lie algebra  $so(n)$  (or other semi-simple Lie algebras) without introducing Quantum ghosts? It seems unlikely. Let us look, for example, at the first nontrivial case,  $\mathcal{G} = so(3) \approx sl(2)$ . According to formulae (7.4), (7.5), we need to choose a 3-dimensional representation of  $\mathcal{G}$ . So, it's either the direct sum of 1-dimensional trivial and 2-dimensional fundamental, or the adjoint representation. The 1<sup>st</sup> alternative can be ruled out in view of the 1-dimensional representation being trivial. This leaves the adjoint representation. In the standard basis  $e, f, h$  of  $sl(2)$ , we thus must have

$$\begin{pmatrix} de \\ df \\ dh \end{pmatrix} = \mathcal{M} \begin{pmatrix} e \\ f \\ h \end{pmatrix}, \quad (10.15)$$

with some constant nondegenerate matrix  $\mathcal{M}$ ; this formulae is to be understood not literally, but only as describing the action of  $\mathcal{G}$  on  $d(\mathcal{G})$ . Now, the consistency conditions (7.4) imply that the matrix  $\mathcal{M}$  has the form

$$\mathcal{M} = \begin{pmatrix} \lambda & 0 & -\nu \\ 0 & -\lambda & \mu \\ -2\mu & 2\nu & 0 \end{pmatrix}. \quad (10.16)$$

But  $\det(\mathcal{M}) = 0$ , so the ghostless complex of differential forms on  $\mathcal{G} = sl(2)$  doesn't exist.

## § 11. $Q$ -Quantum spaces

In the associative ring  $R\langle x \rangle$ , consider the commutation relations

$$x_i x_j = Q_{ij} x_j x_i, \quad \forall i, j, \quad (11.1)$$

where  $Q_{ij}$ 's are arbitrary invertible constants,

$$Q_{ij} = Q_{ji}^{-1}, \quad Q_{ii} = 1; \quad (11.2)$$

if the  $Q_{ij}$ 's do not belong initially to the ring  $R$ , we can always adjoin them. Let us construct a complex of differential forms over our ring  $R_Q\langle x \rangle$ . To do that, we need to postulate the commutation relations between  $x_i$ 's and  $dx_j$ 's. From the experience of Quantum Groups, one knows that there is no canonical way to extend relations from a ring into the corresponding differential-forms ring; such extensions may vary with the situation at hand and with the imagination of the extender. With this in mind, let us proceed in the engineering spirit of this paper, taking the view that  $dx_i$  is "a very small increment in the variable  $x_i$ ", and thus  $dx_i$  should have the same commutation relations as  $x_i$  does, to wit:

$$(dx_i)x_j = Q_{ij}x_jdx_i, \quad \forall i, j. \quad (11.3)$$

In particular,

$$[dx_i, x_i] = 0, \quad \forall i. \quad (11.4)$$

Before proceeding further, we have to verify that the commutation rules (11.1) and (11.3) are compatible. Applying the differential  $d$  to the relation (11.1) and keeping in mind that  $d$  is a derivation, we find

$$\begin{aligned} d(x_i x_j - Q_{ij} x_j x_i) &= (dx_i)x_j + x_i dx_j - Q_{ij}(dx_j)x_j - Q_{ij}x_j dx_i \\ &= ((dx_i)x_j - Q_{ij}x_j dx_i) - Q_{ij}((dx_j)x_i - Q_{ji}x_i dx_j), \end{aligned}$$

and each of these 2 summands vanishes by formulae (11.3). Finally, applying the differential  $d$  to the relation (11.3) and remembering that  $d$  is a  $\mathbf{Z}_2$ -graded derivations, we get

$$dx_i dx_j = -Q_{ij} dx_j dx_i, \quad \forall x, j. \quad (11.5)$$

The differential complex  $\{\Omega^*, d\}$  results thereby. (In this and subsequent Sections, all the variables are considered bosonic. A more general case, on the lines of § 4, is left to the reader.) Let us ascertain whether the cohomologies of our complex are trivial or not. Proceeding as in § 2, we extend  $R_Q\langle x \rangle$  and  $\Omega^*$  by adjoining a new variable  $t$  *commuting with everything*, with its differential  $\tau = dt$  behaving accordingly, i.e., commuting with everything in the  $\mathbf{Z}_2$ -graded sense. Denoting the extended differential-forms ring by  $\bar{\Omega}^*$ , we again have: the unique decomposition

$$\omega = \omega_+ + \tau \omega_-, \quad \forall \omega \in \bar{\Omega}^*, \quad \omega_{\pm} \in \Omega^*[t]; \quad (11.6)$$

the homotopy operator

$$I : \bar{\Omega}^* \rightarrow \Omega^*, \quad (11.7)$$

$$I(\omega) = \int_0^1 dt \omega_-; \quad (11.8)$$

and the ring homomorphism  $A_t : \Omega^* \rightarrow \bar{\Omega}^*$  (over  $R$ ),

$$A_t(x_i) = tx_i, \quad \forall i, \quad (11.9a)$$

$$A_t(dx_i) = tdx_i + \tau x_i, \quad \forall i, \quad (11.9b)$$

so that

$$A_t d = d A_t. \quad (11.10)$$

To make sure that the homomorphism  $A_t$  is well-defined, we have to verify that the relations (11.1), (11.3), (11.5) are preserved when acted upon by  $A_t$ . So,

$$A_t(x_i x_j - Q_{ij} x_j x_i) = t^2(x_i x_j - Q_{ij} x_j x_i), \quad (11.11a)$$

$$\begin{aligned} A_t((dx_i)x_j - Q_{ij}x_j dx_i) &= (tdx_i + \tau x_i)tx_j - Q_{ij}tx_j(tdx_i + \tau x_i) \\ &= t^2((dx_i)x_j - Q_{ij}x_j dx_i) + t\tau(x_i x_j - Q_{ij}x_j x_i), \end{aligned} \quad (11.11b)$$

$$\begin{aligned} A_t(dx_i dx_j + Q_{ij} dx_j dx_i) &= (tdx_i + \tau x_i)(tdx_j + \tau x_j) \\ &\quad + Q_{ij}(tdx_j + \tau x_j)(tdx_i + \tau x_i) = t^2(dx_i dx_j + Q_{ij} dx_j dx_i) \\ &\quad - t\tau \langle ((dx_i)x_j - Q_{ij}x_j dx_i) + Q_{ij}((dx_j)x_i - Q_{ji}x_i dx_j) \rangle, \end{aligned} \quad (11.11c)$$

and we can now proceed to establish the homotopy formula:

**Lemma 11.12.** *For any  $\omega \in \bar{\Omega}^*$ ,*

$$Id(\omega) + dI(\omega) = \omega_+|_{t=1} - \omega_+|_{t=0}, \quad (11.13)$$

**Proof.** It's enough to consider two separate cases:  $\omega = t^n \nu$  and  $\omega = \tau t^n \nu$ ,  $\nu \in \Omega^*$ ,  $n \in \mathbf{Z}_+$ .

(A) If  $\omega = t^n \nu$  then  $\omega_- = 0$ , so that  $I(\omega) = 0$ , and hence

$$Id(\omega) = Id(t^n \nu) = I(nt^{n-1} \tau \nu) = \int_0^1 nt^{n-1} dt \nu = \nu (1 - \delta_n^0) = t^n \nu|_{t=1} - t^n \nu|_{t=0};$$

(B) If  $\omega = \tau t^n$ , then  $\omega_+ = 0$ , and

$$Id(\omega) + dI(\omega) = I(-\tau t^n d(\nu)) + d\left(\int_0^1 t^n dt \nu\right) = -\int_0^1 t^n dt d(\nu) + \int_0^1 t^n dt d(\nu) = 0. \quad \blacksquare$$

**Corollary 11.14.** *Every closed form  $\omega \in \Omega^*$  differs from an exact one by an element from  $R$ .*

**Proof.** If  $\omega$  is closed,  $d(\omega) = 0$ , then so is  $A_t(\omega)$ . Therefore, by formula (11.13) applied to  $A_t(\omega)$ ,

$$dIA_t(\omega) = \omega - pr^{0,0}(\omega). \quad \blacksquare \quad (11.15)$$

So far we have treated differential forms as self-important entities, without any reference to vector fields. The reason for this reticence is a common bane of Quantum mathematics: there exist very few vector fields, and whenever they do exist, their values on the generators  $x_i$ 's are far from arbitrary. It's easy to understand why this is so: any Quantum derivation has to preserve all the defining commutation relations (11.1) (or similar ones in more general Quantum circumstances), and this is,

in general, close to impossible. This is the chief reason the traditional approach to the variational calculus, either commutative [10] or noncommutative one [12], has to be abandoned in the Quantum framework. But some useful things can be salvaged.

Among the latter are (left) partial derivatives  $\frac{\partial}{\partial x_k}$ 's. They are *not* derivatives any more, but are instead additive maps over  $R$ , satisfying the properties

$$\frac{\partial}{\partial x_k}(r) = 0, \quad \frac{\partial}{\partial x_k}r = r \frac{\partial}{\partial x_k}, \quad \forall r \in R, \quad (11.16)$$

$$\frac{\partial}{\partial x_k}x_i = \delta_{ik} + Q_{ik}x_i \frac{\partial}{\partial x_k}, \quad \forall i, k. \quad (11.17)$$

Denote

$$P_{ij} = x_i x_j - Q_{ij} x_j x_i. \quad (11.18)$$

By formula (11.17), we have

$$\begin{aligned} \frac{\partial}{\partial x_k}x_i x_j &= \left( \delta_{ik} + Q_{ik}x_i \frac{\partial}{\partial x_k} \right) x_j = \delta_{ik}x_j + Q_{ik}x_i \left( \delta_{kj} + Q_{jk}x_j \frac{\partial}{\partial x_k} \right) \\ &= \delta_{ik}x_j + \delta_{jk}Q_{ik}x_i + Q_{ik}Q_{jk}x_i x_j \frac{\partial}{\partial x_k}. \end{aligned} \quad (11.19)$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial x_k}P_{ij} &= \left( \delta_{ik}x_j + \delta_{jk}Q_{ik}x_i + Q_{ik}Q_{jk}x_i x_j \frac{\partial}{\partial x_k} \right) \\ &\quad - Q_{ij} \left( \delta_{jk}x_i + \delta_{ik}Q_{jk}x_j + Q_{jk}Q_{ik}x_j x_i \frac{\partial}{\partial x_k} \right) = Q_{ik}Q_{jk}P_{ij} \frac{\partial}{\partial x_k} \\ &\quad + \delta_{ik}x_j(1 - Q_{jk}Q_{ij}) + \delta_{jk}x_i(Q_{ik} - Q_{ij}) = Q_{ik}Q_{jk}P_{ij} \frac{\partial}{\partial x_k}. \end{aligned} \quad (11.20)$$

Thus, the partial derivatives  $\frac{\partial}{\partial x_k}$ 's are well-defined. Their connection with differential forms is described by the following

**Lemma 11.21.** *Denote by*

$$X = \sum_k dx_k \frac{\partial}{\partial x_k} \quad (11.22)$$



the additive map (over  $R$ ) from  $R_Q\langle x \rangle$  into  $\Omega^1$ . (The sum is well-defined even if the number of generators  $x_k$ 's is infinite.) Then:

(i)  $X$  is a derivation:

$$X(HF) = X(H)F + HX(F), \quad \forall H, F \in R_Q\langle x \rangle; \quad (11.23)$$

(ii)  $X = d$ :

$$d(H) = \sum_k dx_k \frac{\partial H}{\partial x_k}, \quad \forall H \in R_Q\langle x \rangle. \quad (11.24)$$

**Proof.** (i) We have,

$$\begin{aligned} Xx_s &= \sum dx_k \frac{\partial}{\partial x_k} x_s \stackrel{[\text{by (11.17)}]}{=} \sum dx_k \left( \delta_{ks} + Q_{sk} x_s \frac{\partial}{\partial x_k} \right) \\ &= [\text{by (11.3)}] dx_s + \sum_k x_s dx_k \frac{\partial}{\partial x_k} = X(x_s) + x_s X. \end{aligned} \quad (11.25)$$

By induction on  $\deg_x(H)$ , it follows that

$$XH = X(H) + HX, \quad \forall H, \quad (11.26)$$

and this is equivalent to the derivation property (i);

(ii) Both  $X$  and  $d$  are derivations over  $R$ , sending  $x_s$  into  $dx_s$  for all  $s$ . Hence,  $X = d$ . ■

**Remark 11.27.** Denote by  $\Omega^\ell$  the  $R\langle x \rangle$ -bimodule of  $\ell$ -forms in  $\Omega^*$ . The previous Lemma shows that instead of the general associative definition

$$\Omega^1 = \left\{ \sum_{ks} f_{ks} dx_k g_{ks} \mid f_{ks}, g_{ks} \in R\langle x \rangle \right\}, \quad (11.28)$$

in the  $Q$ -picture we can take  $\Omega^1$  as

$$\Omega^1 = \left\{ \sum dx_k f_k \mid f_k \in R_Q\langle x \rangle \right\}. \quad (11.29)$$

Similar observation applies to  $\Omega^\ell$ : we can move all  $\ell$   $dx$ 's to the left in each monomial in a  $\ell$ -form  $\omega \in \Omega^\ell$ .

**Remark 11.30.** The partial derivatives  $\frac{\partial}{\partial x_k}$ 's are no longer derivations, as their defining formula (11.17) shows; they should be called  $Q$ -derivations instead. Nevertheless, these partial derivatives almost commute between themselves:

**Lemma 11.31.**

$$\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_\ell} = Q_{k\ell} \frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x_k}, \quad \forall k, \ell. \quad (11.32)$$

**Proof.** Denote

$$\mathcal{O}_{k\ell} = \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_\ell} - Q_{k\ell} \frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x_k}. \quad (11.33)$$

Then  $\mathcal{O}_{k\ell}$  is an additive map over  $R$  which annihilates  $R$  and the  $x_s$ 's. Further,

$$\begin{aligned}
\mathcal{O}_{k\ell}x_s &= \left( \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_\ell} - Q_{k\ell} \frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x_k} \right) x_s \\
&\stackrel{[by\ 11.17]}{=} \frac{\partial}{\partial x_k} \circ \left( \delta_{\ell s} + Q_{s\ell} x_s \frac{\partial}{\partial x_\ell} \right) - Q_{k\ell} \frac{\partial}{\partial x_\ell} \circ \left( \delta_{ks} + Q_{sk} x_s \frac{\partial}{\partial x_k} \right) \\
&= \delta_{\ell s} \frac{\partial}{\partial x_k} + Q_{s\ell} \left( \delta_{ks} + Q_{sk} x_s \frac{\partial}{\partial x_k} \right) \frac{\partial}{\partial x_\ell} - Q_{k\ell} \delta_{ks} \frac{\partial}{\partial x_\ell} \\
&\quad - Q_{k\ell} Q_{sk} \left( \delta_{\ell s} + Q_{s\ell} x_s \frac{\partial}{\partial x_\ell} \right) \frac{\partial}{\partial x_k} = \delta_{\ell s} \frac{\partial}{\partial x_k} (1 - Q_{k\ell} Q_{sk}) + \delta_{ks} \frac{\partial}{\partial x_\ell} (Q_{s\ell} - Q_{k\ell}) \\
&\quad + Q_{sk} Q_{s\ell} x_s \left( \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_\ell} - Q_{k\ell} \frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x_k} \right) = Q_{sk} Q_{s\ell} x_s \left( \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_\ell} - Q_{k\ell} \frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x_k} \right).
\end{aligned}$$

Thus,

$$\mathcal{O}_{k\ell}x_s = Q_{sk} Q_{s\ell} x_s \mathcal{O}_{k\ell}, \quad \forall s. \quad (11.34)$$

Therefore,  $\mathcal{O}_{k\ell} = 0$ .  $\blacksquare$

## § 12. $Q$ -Quantum spaces and discrete groups

When one considers a discrete version of a physical or mathematical picture, the basic variables acquire discrete indices, either of a discrete group  $G$  or its homogeneous space. Most often one has  $\mathbf{Z}$ ,  $\mathbf{Z}_N$ , and their products as the underlying group, but in certain constructions it is easier to work with an arbitrary unspecified group. This is what we shall do in this Section. Suppose, in the language of the preceding Section, that our variables carry two indices,  $i$  and  $g$ :  $x_i^{(g)}$ , where letters  $f, g, h$  in this Section are reserved for typical elements of the fixed discrete group  $G$ . The group  $G$  acts on  $R_Q\langle x \rangle$  by automorphisms, with the action on the generators by the rule

$$\hat{h}(x_i^{(g)}) = x_i^{(hg)}, \quad \forall i, \quad \forall h, g \in G. \quad (12.1)$$

Further, the commutation relations between the  $x_i^{(g)}$ 's are assumed to be  $G$ -invariant:

$$x_i^{(g)} x_j^{(h)} = Q_{ij}^{g^{-1}h} x_j^{(h)} x_i^{(g)}, \quad \forall i, j, \quad \forall g, h \in G. \quad (12.2)$$

Also, the actions of  $G$  and  $d$  on  $\Omega^*$  commute:

$$\hat{g}d = d\hat{g}, \quad \forall g \in G. \quad (12.3)$$

If nothing else intervenes, the results of § 11 remain true as there stated: every closed  $\ell$ -form is exact for  $\ell > 0$ . But suppose we introduce into  $\Omega^*$  the equivalence relation of equivariance:

$$\omega_1 \sim \omega_2 \quad \Leftrightarrow \quad \exists g \in G: \quad \hat{g}(\omega_1) = \omega_2. \quad (12.4)$$

**Lemma 12.5.** *Suppose  $\omega \in \Omega^\ell$ ,  $\ell > 0$ , and  $d(\omega) \sim 0$ . Then there exists  $\nu \in \Omega^{\ell-1}$  such that  $\omega \sim d(\nu)$ .*

**Proof.** We proceed as in the preceding Section, by adding one more variable  $t$  on which  $G$  acts trivially:

$$\hat{g}(t) = t, \quad \forall g \in G. \quad (12.6)$$

Then we again get the homotopy formula

$$Id(\bar{\omega}) + dI(\bar{\omega}) = \bar{\omega}_+|_{t=1} - \bar{\omega}_+|_{t=0}, \quad \forall \bar{\omega} \in \bar{\Omega}^*. \quad (12.7)$$

Taking

$$\bar{\omega} = A_t(\omega), \quad (12.8)$$

and noticing that

$$A_t \hat{g} = \hat{g} A_t, \quad \forall g \in G, \quad (12.9)$$

$$I \hat{g} = \hat{g} I, \quad \forall g \in G, \quad (12.10)$$

we find that

$$\omega = d(IA_t(\omega)) + IA_t d(\omega). \quad (12.11)$$

Thus, if  $d(\omega)$  is trivial, i.e.,  $d(\omega) \sim 0$ , then so is  $\omega - d(\nu)$ ,  $\nu = IA_t(\omega)$ . ■

It is an entirely different matter to describe by *differential equations* not simply exact differential forms, as in the Poincaré Lemma, but just the trivial ones (w.r.t. the action of the group  $G$ .) The machinery to perform such feats is customarily called the *Variational Calculus*. This will be developed in the 4<sup>th</sup> Act.

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