Journal of Nonlinear Mathematical Physics 1998, V.5, N 3, 289–293. Article

A Solvable Many-Body Problem in the Plane

F. CALOGERO

[*] *Dipartimento di Fisica, Università di Roma "La Sapienza", 00153 Roma, Italy*
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy

Received May 1, 1998

Abstract

A solvable many-body problem in the plane is exhibited. It is characterized by rotation-invariant Newtonian ("acceleration equal force") equations of motion, featuring one-body ("external") and pair ("interparticle") forces. The former depend quadratically on the velocity, and nonlinearly on the coordinate, of the moving particle. The latter depend linearly on the coordinate of the moving particle, and linearly respectively nonlinearly on the velocity respectively the coordinate of the other particle. The model contains $2n^2$ arbitrary coupling constants, n being the number of particles. The behaviour of the solutions is outlined; special cases in which the motion is confined (multiply periodic), or even completely periodic, are identified.

1 Introduction

Recently several solvable and/or integrable many-body problems in the plane have been introduced [1]–[4]. (We use here the heuristic – imprecise but useful – distinction among *solvable* models, for which the solutions can be obtained in relatively explicit form, and *integrable* models, which possess a sufficiently large number of independent globally defined constants of the motion). In this paper we report one more such model, which seems worthy of special notice because of its simplicity and generality ($2n^2$ arbitrary coupling constants, see below). Indeed its equations of motion read as follows:

$$\ddot{\vec{r}}_j = \frac{2\dot{\vec{r}}_j(\dot{\vec{r}}_j \cdot \vec{r}_j) - \vec{r}_j(\dot{\vec{r}}_j \cdot \dot{\vec{r}}_j)}{r_j^2} + \sum_{k=1}^{n} (\beta_{jk} + \gamma_{jk}\hat{z}\wedge) \frac{\dot{\vec{r}}_k(\vec{r}_k \cdot \vec{r}_j) + \vec{r}_j(\dot{\vec{r}}_k \cdot \vec{r}_k) - \vec{r}_k(\dot{\vec{r}}_k \cdot \vec{r}_j)}{r_k^2}. \tag{1.1}$$

Notation. Superimposed arrows identify 2-vectors, for which we use the notation

$$\vec{r} \equiv (x, y), \qquad \hat{z} \wedge \vec{r} \equiv (-y, x), \qquad r^2 = x^2 + y^2. \tag{1.2}$$

[*]On leave while serving as Chairman of the Council, Pugwash Conferences on Science and World Affairs

(The second formula corresponds to the 3-dimensional notation $\vec{r} \equiv (x, y, 0)$, $\hat{z} \equiv (1, 0, 0)$, with the symbol $\wedge$ denoting the standard 3-dimensional vector product). The index j always takes integer values from 1 to n, the total number of interacting particles. The $2n^2$ "coupling constants" β_{jk} and γ_{jk} are arbitrary. Dots denote of course differentiation with respect to the time t.

The equations of motion (1.1) are of Newtonian type ("acceleration equal force"), with one- and two-body forces. The one-body ("external") force is quadratic in the particle velocity $\dot{\vec{r}}_j$, and depends nonlinearly on the particle coordinate $\vec{r}_j$. The two-body ("interparticle") force depends linearly (only) on the coordinate of the moving particle, and linearly respectively nonlinearly on the velocity respectively the coordinate of the other particle. The model is *rotation-invariant*, but not *translation-invariant* (see however below for a *translation-invariant* version). It is invariant under the *scale transformation* $\vec{r}_j \to c\vec{r}_j$, with c an arbitrary constant.

This model is *solvable*, as explained in the following Section. In the subsequent Section 3 a brief discussion is given of the particle motions it entails, and the following generalized version of (1.1) is also introduced:

$$\ddot{\vec{r}}_j = (\lambda + \omega\hat{z}\wedge)\dot{\vec{r}}_j + \frac{2\dot{\vec{r}}_j(\dot{\vec{r}}_j \cdot \vec{r}_j) - \vec{r}_j(\dot{\vec{r}}_j \cdot \dot{\vec{r}}_j)}{r_j^2} + \sum_{k=1}^{n} (\widetilde{\beta}_{jk}(t) + \widetilde{\gamma}_{jk}(t)\hat{z}\wedge) \frac{\dot{\vec{r}}_k(\vec{r}_k \cdot \vec{r}_j) + \vec{r}_j(\dot{\vec{r}}_k \cdot \vec{r}_k) - \vec{r}_k(\dot{\vec{r}}_k \cdot \vec{r}_j)}{r_k^2}, \tag{1.3a}$$

$$\widetilde{\beta}_{jk}(t) = [\beta_{jk}\cos(\omega t) - \gamma_{jk}\sin(\omega t)]\exp(\lambda t),$$
$$\widetilde{\gamma}_{jk}(t) = [\gamma_{jk}\cos(\omega t) + \beta_{jk}\sin(\omega t)]\exp(\lambda t). \tag{1.3b}$$

For $\lambda = \omega = 0$, this generalized model reduces to the previous one; for $\lambda = 0$, $\omega \neq 0$ (and of course *real*), *all* its solutions are completely periodic, with period $T = 2\pi/\omega$ (see Section 3). In Section 3 we also indicate how to manufacture analogous solvable models, which are also *translation-invariant*.

Let us finally emphasize that many-body models *in the plane* generally exhibit a much more interesting behavior than models *on the line*; this is confirmed by the results reported below, as well as by those displayed in previous papers [1]–[4].

2 The solution

In this Section we indicated how to solve the system (1.1).

Let the n complex quantities $f_j(t)$ evolve in time according to the *linear firstorder* evolution equations

$$\dot{f}_j = \sum_{k=1}^{n} \alpha_{jk} f_k, \tag{2.1}$$

whose general solution reads

$$f_j(t) = \sum_{k}^{n} \varphi_j^{(k)} \exp(a_k t), \tag{2.2}$$

where the n quantities a_k are the n eigenvalues of the $(n \times n)$-matrix A with elements α_{jk}, and the n n-vectors $\Phi^{(k)}$, with components $\varphi^{(k)}$, are the corresponding eigenvectors:

$$\sum_{k=1}^{n} \alpha_{jk} \varphi_k^{(m)} = a_m \varphi_j^{(m)}. \tag{2.3}$$

Here we are implicitly assuming, for simplicity, that the matrix A is diagonalizable, and that all its eigenvalues a_k are distinct; the diligent reader will have no difficulty in detailing the modifications in the treatment which are required if these assumptions do not hold.

Introduce now the n quantities $z_j(t)$ via the position

$$f_j = \dot{z}_j / z_j, \tag{2.4}$$

which clearly entails

$$\dot{f}_j = \ddot{z}_j / z_j - \dot{z}_j^2 / z_j^2, \tag{2.5}$$

hence, via (2.1) and (2.4),

$$\ddot{z}_j = \dot{z}_j^2 / z_j + \sum_{k=1}^{n} \alpha_{jk} z_j \dot{z}_k / z_k. \tag{2.6}$$

On the other hand, from (2.4) and (2.2), one gets

$$z_j(t) = z_j(0) \exp \left[\sum_{k=1}^{n} \varphi_j^{(k)} \left[\exp(a_k t) - 1 \right] / a_k \right], \tag{2.7}$$

while the normalization of the n n-vectors $\Phi^{(k)}$, with components $\varphi_j^{(k)}$, is fixed by the conditions

$$\sum_{k=1}^{n} \varphi_j^{(k)} = \dot{z}_j(0) / z_j(0), \tag{2.8}$$

also implied by (2.2) and (2.4).

Hence (2.6) is explicitly solvable, for any choice of the n^2 quantities α_{jk}, and $2n$ initial data $z_j(0)$ and $\dot{z}_j(0)$. And this is true as well if all these quantities are *complex*, so that we can put

$$z_j = x_j + i y_j, \tag{2.9}$$

$$\alpha_{jk} = \beta_{jk} + i \gamma_{jk}, \tag{2.10}$$

of course now with x_j, y_j, β_{jk} and γ_{jk} all *real*.

But it is now a trivial exercise to verify that, via these positions, the *complex* evolution equations (2.6) coincide with the 2-*dimensional rotation-invariant real* equations of motion (1.1), whose solvability is thereby demonstrated.

3 Discussion of the motions, and other models

In this Section we briefly discuss the motions yielded by the many-body problem (1.1), using the findings of the preceding Section(see in particular (2.7)); and we then indicate how to obtain other solvable models *in the plane*, including the many-body problem (1.3) and *translation-invariant* generalizations of the many-body problems mentioned above.

Clearly the most important element that characterizes the behavior of the system (1.1) are the n eigenvalues of the matrix A. If *all* these eigenvalues have negative real parts, from *any* initial conditions the system will tend to a standstill configuration; *all* particle velocities vanish exponentially as $t \to \infty$. Such an outcome may also occur, but only for *special* initial conditions, if only *some* of these eigenvalues have negative real parts. A necessary and sufficient condition for the system to possess periodic trajectories is that *at least one* of these n eigenvalues be imaginary. If *all* these eigenvalues are imaginary, the motion remains confined, and is in fact multiply periodic, for *any* initial conditions. If moreover *all* these eigenvalues are rational multiples of the *same* imaginary quantity, then for *any* initial conditions the system behaves completely periodically. If one of these eigenvalues vanishes, the system (2.6) possesses the *similarity solutions*

$$z_j(t) = z_j(0)\exp(\eta t), \tag{3.1a}$$

with $\eta = \lambda + i\omega$ an arbitrary (complex) constant. The corresponding *similarity solution* of (1.1) reads

$$\vec{r}_j(t) = \exp(\lambda t)\left[\cos(\omega t) + \sin(\omega t)\hat{z}\wedge\right]\vec{r}_j(0). \tag{3.1b}$$

If some of the n eigenvalues a_k have positive real parts, the generic solution may feature *some* (from 0 to n) particles which escape, doubly exponentially fast, to infinity, while the others converge, doubly exponentially fast, to the origin.

Let us now derive the system (1.3a) with (1.3b). To this end let us start from the system (2.6), but with the quantities $z_j(t)$ formally replaced by, say, $\zeta_j(\tau)$ (and accordingly with dots replaced by, say, primes, the latter indicating derivatives with respect to τ). We then introduce new quantities $z_j(t)$ via the position $z_j(t) = \zeta_j(\tau)$, so that (2.6) gets replaced by

$$\ddot{z}_j = \dot{z}_j(\ddot{\tau}/\dot{\tau}) + \dot{z}_j^2/z_j + \dot{\tau}\sum_{k=1}^{n}\alpha_{jk}z_j\dot{z}_k/z_k. \tag{3.2}$$

Now we make again the transition from the *complex* dependent variables $z_j(t)$ to the *real* 2-*vectors* $\vec{r}_j(t)$ (as above, see (2.9) and (1.2)). It is then easily seen that the new particle coordinates $\vec{r}_j(t)$ satisfy precisely (1.3a) with (1.3b), provided

$$\tau = \left\{\exp[(\lambda + i\omega)t] - 1\right\}/(\lambda + i\omega). \tag{3.3}$$

The statement made after (1.3b) is thereby proven.

Finally, let us manufacture a generalized version of (1.1), which is *translation-invariant*. To this end we introduce a model with $2n$ particles of two types, whose coordinates we denote respectively by $\vec{r}_j^{\,(+)}(t)$ and $\vec{r}_j^{\,(-)}(t)$, and we set

$$\vec{r}_j(t) = \vec{r}_j^{\,(+)}(t) - \vec{r}_j^{\,(-)}(t), \qquad \vec{R}_j(t) = \vec{r}_j^{\,(+)}(t) + \vec{r}_j^{\,(-)}(t), \tag{3.4}$$

with $\vec{r}_j(t)$ satisfying the (solvable) equations of motion (1.1), and $\vec{R}_j(t)$ satisfying the (clearly, also solvable; see below) equations of motion

$$\ddot{\vec{R}}_j = (\Lambda_j + \Omega_j \hat{z}\wedge)\dot{\vec{R}}_j. \tag{3.5}$$

Here λ_j and Ω_j are $2n$ arbitrary (of course *real*) coupling constants.

We thereby see that the new "particle coordinates" $\vec{r}_j^{(\pm)}(t)$ evolve according to the *translation-invariant* equations of motion

$$\begin{aligned}\ddot{\vec{r}}_j^{(\pm)} = \frac{1}{2}\Big\{(\Lambda_j + \Omega_j\hat{z}\wedge)\dot{\vec{R}}_j \pm \frac{2\dot{\vec{r}}_j(\dot{\vec{r}}_j\cdot\vec{r}_j) - \vec{r}_j(\dot{\vec{r}}_j\cdot\dot{\vec{r}}_j)}{r_j^2} \\ \pm \sum_{k=1}^{n}(\beta_{jk} + \gamma_{jk}\hat{z}\wedge)\frac{\dot{\vec{r}}_k(\vec{r}_k\cdot\vec{r}_j) + \vec{r}_j(\dot{\vec{r}}_k\cdot\vec{r}_k) - \vec{r}_k(\dot{\vec{r}}_k\cdot\vec{r}_j)}{r_k^2}\Big\},\end{aligned} \tag{3.6}$$

of course with (3.4).

As for (3.5), a convenient way to exhibit its solution is via the formula

$$Z_j(t) = Z_j(0) + \dot{Z}_j(0)\left\{\exp\left[(\Lambda_j + i\Omega_j)t\right]\right\}/(\Lambda_j + i\Omega_j), \tag{3.7}$$

which clearly provides the general solution of (3.5) via the identification of the *complex* dependent variables $Z_j \equiv X_j + iY_j$, with the *real* 2-vectors $\vec{R}_j \equiv (X_j, Y_j)$, since this identification entails that, to (3.5), there corresponds

$$\ddot{Z}_j = (\Lambda_j + i\Omega_j)\dot{Z}_j. \tag{3.8}$$

Clearly the same trick may be directly applied to (1.3a) with (1.3b), yielding a *translation-invariant* generalization of these equations of motion. An alternative possibility, generally yielding a different model, is to apply the change of independent variable (3.3) to the complex scalar version of (3.6), and *subsequently* to change the dependent variables (from scalar complex numbers to real 2-vectors). We do not exhibit the corresponding results, which the interested reader may immediately write down. Nor do we report an analysis of the actual behavior of the solutions of (3.6) with (3.4), or of the new models just mentioned, since the above results are so transparent not to require elaborations. But we end by reemphasizing the reachness of the motions entailed by these formulas, which is now enhanced by the additional freedom connected with the choice of the $2n$ constants Λ_j and Ω_j.

References

[1] Calogero F., A Solvable n-Body Problem in the Plane. I, *J. Math. Phys.*, 1996, V.37, 1735–1759.

[2] Calogero F., Three Solvable Many-Body Problems in the Plane, *Acta Applicandae Mathematicae*, 1998, V.51, 93–111.

[3] Calogero F., Tricks of the Trade: Relating and Deriving Solvable and Integrable Dynamical Systems, in: Proceedings of the Workshop on Calogero-Moser-Sutherland Models, Montreal, 10–15 March, 1997 (in press).

[4] Calogero F., Integrable and Solvable Many-Body Problems in the Plane via Complexification, *J. Math. Phys.* (in press).