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Lie Symmetries, Kac-Moody-Virasoro Algebras and Integrability of Certain (2+1)-Dimensional Nonlinear Evolution Equations

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Abstract

In this paper we study Lie symmetries, Kac-Moody-Virasoro algebras, similarity reductions and particular solutions of two different recently introduced (2+1)-dimensional nonlinear evolution equations, namely (i) (2+1)-dimensional breaking soliton equation and (ii) (2+1)-dimensional nonlinear Schrödinger type equation introduced by Zakharov and studied later by Strachan. Interestingly our studies show that not all integrable higher dimensional systems admit Kac-Moody-Virasoro type sub-algebras. Particularly the two integrable systems mentioned above do not admit Virasoro type subalgebras, eventhough the other integrable higher dimensional systems do admit such algebras which we have also reviewed in the Appendix. Further, we bring out physically interesting solutions for special choices of the symmetry parameters in both the systems.

1 Introduction

In recent years important progress has been made in the understanding of (2+1)-dimensional nonlinear evolution equations (NLEEs) and their methods of solution [1, 2]. In this direction it is well realized that the Lie group method [3–8], originally introduced by Sophus Lie, can play a crucial role, since in most of the problems it not only explores the intrinsic geometric properties but also brings out interesting physical solutions in a straightforward manner. Eventhough the last decade has witnessed a veritable explosion on the applications of this method to explore the invariance and integrability properties of a large class of problems in (1+1)-dimensions [6, 9–11] very few systems in higher dimensions have been explored in this way.

Recently the invariance properties of some of the physically important integrable NLEEs in (2+1)-dimensions, such as the Kadomtsev-Petvishilli equation [12], the Davey-Stewartson equation [13], the three wave interaction problem [14], cylindrical Kadomtsev-Petviashvili equations [15] and stimulated Raman scattering equation [16] have been studied through Lie symmetry analysis and it has been shown that all these equations admit infinite dimensional Lie point symmetry groups with a specific Kac-Moody-Virasoro structure. Further, the present authors have also carried out a detailed study on the invariance properties of certain higher-dimensional nonlinear evolution equations, namely, (i) Nizhnik-Novikov-Veselov equation, (ii) breaking soliton equation, (iii) nonlinear Schrödinger type equation studied by Fokas recently, (iv) sine-Gordon equation and (v) (2+1)-dimensional long dispersive wave equation introduced by Chakravarthy, Kent and Newman and explored possible similarity reductions and Kac-Moody-Virasoro algebras and also particular solutions associated with them [17, 18]. While all the above mentioned equations except the breaking soliton equation admit Virasoro type subalgebras the later one does not admit such subalgebras.

In contradistinction to the above integrable NLEEs, the (2+1)-dimensional nonintegrable partial differential equations (PDEs) do not admit Virasoro type subalgebras. Typical examples are Infeld-Rowlands equation [19], and a nonintegrable dispersive long-wave equation [20]. Thus it is commonly believed in the current literature that all integrable higher dimensional NLEEs will admit Virasoro type subalgebras while nonintegrable equations do not. However in this paper we wish to point out that not all the integrable (2+1)-dimensional NLEEs admit Virasoro type algebras. For example, we have carried out a detailed investigation on the invariance properties of two different classes of NLEEs, namely, (i) breaking soliton equation [21] which is an asymmetric generalization of the Korteweg de Vries (KdV) equation in (2+1)-dimensions and (ii) (2+1)-dimensional nonlinear Schrödinger equation (NLS) studied recently by Strachan [22]. We have found that both the systems do not admit Kac-Moody-Virasoro type subalgebras. However it has been shown in the literature [21–24] that both the systems are in fact integrable.

Another interesting feature of our study is that both the equations, namely, the breaking soliton equation and the Zakharov-Strachan equation, eventhough belong to two different categories, namely KdV and NLS type respectively, admit a specific type of symmetries (see eqs.(2.2) and (3.4) given below). For example, both the equations allow the infinitesimals upto quadratic power in t explicitly. Also they do not admit any arbitrary function in the infinitesimal variations in t which inturn leads to the absence of Kac-Moody-Virasoro type subalgebras in both the systems. However, the other integrable nonlinear evolution equations, given in the Appendix, admit arbitrary functions in the infinitesimal transformation of t.

Further, we also bring out the unexplored invariance properties of the above two NLEEs through Lie group method. First we obtain the appropriate point transformation groups and generators, which are infinite in both the systems, which leave the above two nonlinear systems invariant. By solving the characteristic equation associated with the infinitesimals we obtain the similarity variables interms of which the original system with three independent variables reduce to a PDE with two independent variables. For the latter, again another set of similarity variables are found interms of which the PDE reduces to an ordinary differential equation (ODE). We have used the symbolic manipulation program LIE [25] to find out the Lie symmetries.

The plan of the paper is as follows. In Appendix A, we briefly summarize the Lie symmetries and Kac-Moody-Virasoro algebras of certain NLEEs discussed in the literature. In Sec. 2, we present the symmetry algebra of the breaking soliton equation and its similarity reductions. In Sec. 3, we report the Lie symmetries and Kac-Moody-Virasoro algebras of the nonlinear Schrödinger equation studied by Strachan. Further we have also explored the possible similarity reductions and particular solutions. In Sec. 4 we present our conclusions.

2 Lie symmetries and Kac-Moody-Virasoro algebras of the breaking soliton equation

An asymmetric generalization of the KdV equation in (2+1)-dimensions [21] is

$$\begin{aligned} &u_t + Bu_{xxy} + 4Buv_x + 2Bvu_x = 0, \\ &u_y = v_x, \end{aligned} \tag{2.1}$$

which describes the interaction of a Riemann wave propagating along the y axis with a long wave propagating along the x axis. Eq.(2.1) can also be written as the single fourth order nonlinear PDE of the form,

$$\rho_{xt} + \rho_{xxxy} + 4B\rho_x\rho_{xy} + 2B\rho_y\rho_{xx} = 0,$$

by introducing the transformation, $u = \rho_x$, $v = \rho_y$. Eq.(2.1) admits Lax representation [21]. One can also easily verify that eq.(2.1) admits Painlevé property. Special features of eq.(2.1) have also been studied extensively in ref. [21].

To study the invariance properties we have considered the equation of the form (2.1). The invariance of eq.(2.1) under the infinitesimal point transformations

$$\begin{aligned} &x \longrightarrow X = x + \varepsilon\xi_1(t,x,y,u,v), \\ &y \longrightarrow Y = y + \varepsilon\xi_2(t,x,y,u,v), \\ &t \longrightarrow T = t + \varepsilon\xi_3(t,x,y,u,v), \\ &u \longrightarrow U = u + \varepsilon\phi_1(t,x,y,u,v), \\ &v \longrightarrow V = v + \varepsilon\phi_2(t,x,y,u,v), \quad \varepsilon \ll 1 \end{aligned}$$

leads to the expressions for the infinitesimals

$$\begin{aligned} &\xi_1 = -\frac{c_1}{3}xt - \frac{c_2}{2}x + f(t), \\ &\xi_2 = -\frac{2c_1}{3}yt + \left(\frac{c_2}{2} - c_4\right)y - 4Bc_3t + c_5, \\ &\xi_3 = -\frac{2c_1}{3}t^2 - \left(\frac{c_2}{2} + c_4\right)t - c_6, \\ &\phi_1 = \frac{2c_1}{3}ut + c_2u - \frac{c_1}{6B}y - c_3, \\ &\phi_2 = c_1vt + c_4v - \frac{c_1}{6B}x + \frac{\dot{f}(t)}{2B}, \end{aligned} \tag{2.2}$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants and $f(t)$ is an arbitrary function of t and dot denotes differentiation with respect to t.

2.1 Lie algebra of symmetry vector fields

The presence of the arbitrary function f of t leads to an infinite dimensional Lie algebra of symmetries. We can write a general element of this Lie algebra as

$$V = V_1(f) + V_2 + V_3 + V_4 + V_5 + V_6,$$

where

$$V_1(f) = f(t)\frac{\partial}{\partial x} + \frac{\dot{f}(t)}{2B}\frac{\partial}{\partial v},$$
$$V_2 = -\frac{1}{3}xt\frac{\partial}{\partial x} - \frac{2}{3}yt\frac{\partial}{\partial y} - \frac{2}{3}t^2\frac{\partial}{\partial t} + \left(\frac{2}{3}ut - \frac{y}{6B}\right)\frac{\partial}{\partial u} + \left(vt - \frac{x}{6B}\right)\frac{\partial}{\partial v},$$
$$V_3 = -\frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} - \frac{1}{2}t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad V_4 = -4Bt\frac{\partial}{\partial y} - \frac{\partial}{\partial u},$$
$$V_5 = -y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}, \quad V_6 = \frac{\partial}{\partial y}, \quad V_7 = \frac{\partial}{\partial t}.$$

The associated Lie algebra between these vector fields becomes

$$[V_1, V_2] = V_1\left(-\frac{1}{3}tf + \frac{2}{3}t^2\dot{f}\right), \quad [V_1, V_3] = V_1\left(\frac{f}{2} + \frac{1}{2}t\dot{f}\right), \quad [V_1, V_4] = 0,$$
$$[V_1, V_5] = V_1(t\dot{f}), \quad [V_1, V_6] = 0, \quad [V_1, V_7] = -V_1(\dot{f}), \quad [V_2, V_3] = \frac{V_2}{2},$$
$$[V_2, V_4] = 0, \quad [V_2, V_5] = V_2, \quad [V_2, V_6] = -\frac{1}{6B}V_4, \quad [V_2, V_7] = -\frac{2}{3}V_3 - V_5,$$
$$[V_3, V_4] = -V_4, \quad [V_3, V_5] = 0, \quad [V_3, V_6] = -\frac{1}{2}V_6, \quad [V_3, V_7] = \frac{1}{2}V_7,$$
$$[V_4, V_5] = 0, \quad [V_4, V_6] = 0, \quad [V_4, V_7] = 4BV_6, \quad [V_5, V_6] = V_6,$$
$$[V_5, V_7] = V_7, \quad [V_6, V_7] = 0,$$

whereas the commutation relation between $V_1(f_1), V_1(f_2)$ turns out to be

$$[V_1(f_1), V_1(f_2)] = 0,$$

which is not of *Virasoro type* which typically exists in most of the integrable systems mentioned in the introduction and also pointed out in the Appendix.

2.2 Similarity variables and similarity reductions

The similarity variables associated with the infinitesimal symmetries (2.2) can be obtained by solving the associated invariant surface condition or the related characteristic equation. The latter reads

$$\begin{aligned}\frac{dx}{-\frac{c_1}{3}xt - \frac{c_2}{2}x + f(t)} &= \frac{dy}{-\frac{2}{3}c_1yt + \left(\frac{c_2}{2} - c_4\right)y - 4Bc_3t + c_5}\\ &= \frac{dt}{-\frac{2}{3}c_1t^2 - \left(\frac{c_2}{2} + c_4\right)t - c_6} = \frac{du}{\frac{2}{3}c_1ut + c_2u - \frac{1}{6B}c_1y - c_3}\\ &= \frac{dv}{-\frac{c_1}{6B}x + \frac{1}{2B}\dot{f}(t) + c_1vt + c_4v}.\end{aligned} \tag{2.3}$$

Integrating eq.(2.3) with the condition $c_1 \neq 0$ we get the following similarity variables:

$$\tau_1 = \frac{x(t+k_1+k_2)^{n-(1/4)}}{(t+k_1-k_2)^{n+(1/4)}} + \frac{3}{2c_1}\int_0^t \frac{f(t')(t'+k_1+k_2)^{n-(5/4)}}{(t'+k_1-k_2)^{n+(5/4)}}dt',$$

$$\tau_2 = \frac{y(t+k_1+k_2)^{-(2n+(1/2))}}{(t+k_1-k_2)^{-2n+(1/2)}} - \frac{3}{2c_1}\int_0^t \frac{(4Bc_3t'-c_5)(t'+k_1+k_2)^{-(2n+(3/2))}}{(t'+k_1-k_2)^{-2n+(3/2)}}dt',$$

$$\begin{aligned} F = {} & \frac{u(t+k_1+k_2)^{2n+(1/2)}}{(t+k_1-k_2)^{2n-(1/2)}} \\ & -\frac{3}{8Bc_1}\int_0^t\left[\int_0^{t'}\frac{(4Bc_3t''-c_5)(t''+k_1+k_2)^{-(2n+(3/2))}}{(t''+k_1-k_2)^{-2n+(3/2)}}dt''\right]dt' \\ & -\frac{\tau_2 t}{4B} - \frac{3c_3}{2c_1}\int_0^t \frac{(t'+k_1-k_2)^{2n-(1/2)}}{(t'+k_1+k_2)^{2n+(1/2)}}dt', \end{aligned}$$

$$\begin{aligned} G = {} & \frac{v(t+k_1+k_2)^{n+(3/4)}}{(t+k_1-k_2)^{n-(3/4)}} + \frac{3}{8Bc_1}\int_0^t\left[\int_0^{t'}\frac{f(t'')(t''+k_1+k_2)^{n-(5/4)}}{(t''+k_1-k_2)^{n+(5/4)}}dt''\right]dt' \\ & -\frac{\tau_1 t}{4B} + \frac{3}{4Bc_1}\int_0^t \frac{\dot{f}(t')(t'+k_1-k_2)^{-(n+(1/4))}}{(t'+k_1+k_2)^{-n+(1/4)}}dt', \end{aligned}$$

where F and G are functions of τ_1 and τ_2 and

$$k_1 = (3c_2+6c_4)/8c_1, \qquad k_2 = \sqrt{(3c_2+6c_4)^2 - 96c_1c_6}/8c_1,$$
$$n = (9c_2-6c_4)/4\sqrt{(3c_2+6c_4)^2-96c_1c_6}.$$

Under the above similarity transformations, eq.(2.1) gets reduced to a system of PDEs in two independent variables τ_1 and τ_2:

$$\begin{aligned} & BF_{\tau_1\tau_1\tau_2} + 4BFG_{\tau_1} + 2BGF_{\tau_1} + \frac{(3c_2-6c_4)}{4c_1}\tau_2 F_{\tau_2} \\ & \quad -\frac{3c_2}{4c_1}\tau_1 F_{\tau_1} - \frac{3c_2}{2c_1}F + \frac{3c_6}{8Bc_1}\tau_2 = 0, \qquad F_{\tau_2} = G_{\tau_1}. \end{aligned} \tag{2.4}$$

Since the original (2+1)-dimensional PDE (2.1) satisfies the Painlevé property for a general manifold, the (1+1)-dimensional similarity reduced PDE (3.5) will also naturally satisfy the P-property and so is a candidate for a completely integrable system in (1+1)-dimensions.

2.3 Subcases

In addition to the above general similarity reduction one can also look into the subcases by assuming one or more of the vector fields to be zero. We have considered all the subcases and in the following we report only the distinct nontrivial cases.

Case 1: $c_1 = 0$. The similarity variables are

$$\tau_1 = \frac{x}{(k_1t+c_6)^{c_2/(c_2+2c_4)}} + \int_0^t \frac{f(t')dt'}{(k_1t'+c_6)^{2(c_2+c_4)/(c_2+2c_4)}},$$

$$\tau_2 = y(k_1 t + c_6)^{(c_2-2c_4)/(c_2+2c_4)} - \int_0^t (4Bc_3 t' - c_5)(k_1 t' + c_6)^{-4c_4/(c_2+2c_4)} dt',$$

$$F = (c_2 u - c_3)(k_1 t + c_6)^{2c_2/(c_2+2c_4)},$$

$$G = v(k_1 t + c_6)^{2c_4/(c_2+2c_4)} + \int_0^t \frac{\dot{f}(t')dt'}{2B(k_1 t' + c_6)^{c_2/(c_2+2c_4)}},$$

where $k_1 = (c_2/2) + c_4$.

The reduced PDE takes the form

$$BF_{\tau_1\tau_1\tau_2} + 4BFG_{\tau_1} + 2BGF_{\tau_1} + \frac{(c_2 - 2c_4)}{2}\tau_2 F_{\tau_2} - \frac{c_2}{2}\tau_1 F_{\tau_1} - c_2 F = 0,$$

$$F_{\tau_2} = c_2 G_{\tau_1}.$$

Case 2: $c_1, c_2 = 0$. The similarity variables are

$$\tau_1 = x + \int_0^t \frac{f(t')dt'}{(c_4 t' + c_6)},$$

$$\tau_2 = \frac{y}{(c_4 t + c_6)} - \frac{4Bc_3}{c_4^2}\log(c_4 t + c_6) - \frac{(4Bc_3 c_6 + c_4 c_5)}{c_4^2(c_4 t + c_6)},$$

$$F = u - \frac{c_3}{c_4}\log(c_4 t + c_6), \qquad G = v(c_4 t + c_6) + \frac{f(t)}{2B}.$$

In this case the reduced PDE turns out to be

$$\begin{aligned} &BF_{\tau_1\tau_1\tau_2} + 4BFG_{\tau_1} + 2BGF_{\tau_1} - \tau_2 c_4 F_{\tau_2} \\ &\qquad - \left(4Bc_3 + \frac{4Bc_3 c_6}{c_4} + c_5\right) F_{\tau_2} + c_3 = 0, \qquad F_{\tau_2} = G_{\tau_1}. \end{aligned} \tag{2.5}$$

Case 3: $c_1, c_4 = 0$. The similarity variables are

$$\tau_1 = \frac{x}{(c_2/2)t + c_6} + \int_0^t \frac{f(t')dt'}{((c_2/2)t' + c_6)^2}, \quad \tau_2 = y((c_2/2)t + c_6) - 2Bc_3 t^2 + c_5 t,$$

$$F = (c_2 u - c_3)((c_2/2)t + c_6)^2, \quad G = v + \int_0^t \frac{\dot{f}(t')dt'}{2B((c_2/2)t' + c_6)}.$$

The reduced PDE takes the form

$$\begin{aligned} &BF_{\tau_1\tau_1\tau_2} + \frac{4BFG_{\tau_1}}{c_2} + \frac{2BGF_{\tau_1}}{c_2} + \left(c_2 c_5 c_6 + 4Bc_3 c_6^2\right) G_{\tau_1} \\ &\qquad -\frac{\tau_1}{2}F_{\tau_1} + \frac{\tau_2}{2}F_{\tau_2} - 2F = 0, \qquad F_{\tau_2} = c_2 G_{\tau_1}. \end{aligned}$$

Case 4: $c_1, c_2, c_3 = 0$. The similarity variables are

$$\tau_1 = x + \int_0^t \frac{f(t')dt'}{c_4 t' + c_6}, \qquad \tau_2 = \frac{c_4 y}{c_4 t + c_6} - \frac{c_5}{c_4 t + c_6},$$

$$F = u, \qquad G = v(c_4 t + c_6) + \frac{f(t)}{2B}.$$

The reduced PDE takes the form

$$BF_{\tau_1\tau_1\tau_2} + \frac{4B}{c_4} FG_{\tau_1} + \frac{2B}{c_4} GF_{\tau_1} - \tau_2 F_{\tau_2} = 0,$$
$$G_{\tau_1} = c_4 F_{\tau_2}.$$

Case 5: $c_1, c_2, c_4 = 0$. The similarity variables are

$$\tau_1 = x + \frac{1}{c_6}\int_0^t f(t')dt', \qquad \tau_2 = y - \frac{2Bc_3}{c_6}t^2 + \frac{c_5}{c_6}t,$$
$$F = u - \frac{c_3}{c_6}t, \qquad G = v + \frac{f(t)}{2Bc_6}.$$

The reduced PDE takes the form

$$BF_{\tau_1\tau_1\tau_2} + 4BFG_{\tau_1} + 2BGF_{\tau_1} + \frac{c_5}{c_6}F_{\tau_2} + \frac{c_3}{c_6} = 0,$$
$$F_{\tau_2} = G_{\tau_1}.$$

Case 6: $c_1, c_2, c_6 = 0$. The similarity variables are

$$\tau_1 = x + \frac{1}{c_4}\int_0^t \frac{f(t')dt'}{t'}, \qquad \tau_2 = \frac{y}{t} - \frac{4Bc_3}{c_4}\log t - \frac{c_5}{c_4 t},$$
$$F = u - \frac{c_3}{c_4}\log t, \qquad G = vt + \frac{f(t)}{2Bc_4}.$$

The reduced PDE takes the form

$$BF_{\tau_1\tau_1\tau_2} + 4BFG_{\tau_1} + 2BGF_{\tau_1} - \frac{4Bc_3}{c_4}F_{\tau_2} - \tau_2 F_{\tau_2} - \frac{c_3}{c_4} = 0,$$
$$F_{\tau_2} = G_{\tau_1}.$$

Case 7: $c_1, c_3, c_4 = 0$. The similarity variables are

$$\tau_1 = \frac{x}{(c_2/2)t + c_6} + \int_0^t \frac{f(t')dt'}{((c_2/2)t' + c_6)^2}, \quad \tau_2 = y((c_2/2)t + c_5) + c_5 t,$$
$$F = u((c_2/2)t + c_6)^2, \qquad G = v + \int_0^t \frac{\dot{f}(t')dt'}{2B((c_2/2)t' + c_6)}.$$

The reduced PDE takes the form

$$BF_{\tau_1\tau_1\tau_2} + 4BFG_{\tau_1} + 2BGF_{\tau_1} + \left[\frac{c_2}{2}\tau_2 + c_5^2\right]F_{\tau_2} - \frac{\tau_1 c_2}{2}F_{\tau_1} - c_2 F = 0,$$
$$F_{\tau_2} = G_{\tau_1}.$$

2.4 Lie symmetries and similarity reduction of eqs.(2.4)

Now the reduced PDE (2.4) in two independent variables can itself be further analyzed for its symmetry properties by looking at its own invariance property under the classical Lie algorithm again. The invariance of the eq.(2.4) leads to the following infinitesimal symmetries

$$\xi_1 = c_7\tau_1 + c_8, \quad \xi_2 = -2c_7\tau_2, \quad \eta_1 = -2c_7 F, \quad \eta_2 = c_7 G + \frac{3c_2c_8}{8Bc_1},$$

where c_7 and c_8 are arbitrary constants. The associated Lie vector fields are

$$V_1 = \tau_1 \frac{\partial}{\partial \tau_1} - 2\tau_2 \frac{\partial}{\partial \tau_2} - 2F\frac{\partial}{\partial F} + G\frac{\partial}{\partial G}, \quad V_2 = \frac{\partial}{\partial \tau_1} + \frac{3c_2}{8Bc_1}\frac{\partial}{\partial G},$$

leading to the Lie algebra

$$[V_1, V_2] = V_1.$$

Solving the associated characteristic equation

$$\frac{d\tau_1}{c_7\tau_1 + c_8} = \frac{d\tau_2}{-2c_7\tau_2} = \frac{dF}{-2c_7F} = \frac{dG}{c_7G + (3c_2c_8/8Bc_1)},$$

we obtain the similarity variables

$$z = (c_7\tau_1 + c_8)\tau_2^{(1/2)}, \quad w_1 = F(c_7\tau_1 + c_8)^2, \quad w_2 = \frac{c_7G + (3c_2c_8/8Bc_1)}{(c_7\tau_1 + c_8)}. \tag{2.6}$$

The associated similarity reduced ODE follows from eqs.(2.6) and (2.4) as

$$\begin{aligned} &w_1''' - \frac{2}{z}w_1'' + \frac{2}{z^2}w_1' + \frac{8}{c_1^2}w_1w_2' + \frac{4}{c_1^2}w_1'w_2 - \frac{3c_2 + 6c_4}{4Bc_1^3}w_1' + \frac{3c_6}{2B^2c_1^3}z = 0, \\ &w_1' = 2z(w_2 + zw_2'). \end{aligned} \tag{2.7}$$

While the exact solution for eq.(2.7) has not been found for the general two parameter case, particular solutions can be obtained for the special one parameter choice. For example, by choosing $c_7 = 0$ and redoing the calculations one gets the following solution

$$F = \frac{3c_2}{8Bc_1}\tau_2 + I_1, \qquad G = \frac{3c_2}{8Bc_1}\tau_1 + w_2(\tau_2), \tag{2.8}$$

where I_1 is an integration constant and $w_2(\tau_2)$ is an arbitrary function of τ_2. Now rewriting eq.(2.8) interms of old variables one can get a solution for the PDE (2.1). However, the other possibility $c_8 = 0$ leads to the same similarity reduction (2.7).

Similarly one can analyse each one of the other equations given in Sec. 2.3. For example let us consider Case 2, eq.(2.5). Now applying the invariance condition to eq.(2.5) one gets the following infinitesimals

$$\xi_1 = c_7, \qquad \xi_2 = \frac{4Bc_8}{c_4}, \qquad \eta_1 = c_8, \qquad \eta_2 = 0.$$

Solving the characteristic equation we get the following similarity variables:

$$z = \tau_1 - \frac{c_4c_7}{4Bc_8}\tau_2, \qquad w_1 = F - \frac{c_8}{c_7}\tau_1, \qquad w_2 = G.$$

Under this similarity transformation the reduced ODE takes the form

$$w_1^{'''} + 6w_1w_1^{'} + \frac{4c_8}{c_7}zw_1^{'} + k_1w_1^{'} + \frac{2c_8}{c_7^2}w_1 + k_2 = 0, \tag{2.9}$$

where

$$k_1 = \left(\frac{8Bc_8I_1}{c_4c_7} - 4c_3 - \frac{4c_3c_6}{c_4} - \frac{c_5}{B}\right), \qquad k_2 = -\left(\frac{4c_3c_8}{c_7} + \frac{8Bc_8^2I_1}{c_4c_7^2}\right),$$

and

$$w_2 = -\frac{c_4 c_7}{4Bc_8} w_1 + I_1.$$

Solving eq.(2.9) one gets the solution for the PDE (2.5).

Similarly by choosing $c_7 = 0$, we get the solution

$$F = \frac{c_4}{4B}\tau_2 + \frac{1}{2}\left(\frac{I_2}{\tau_1 + (4BI_1/c_4)}\right)^2 - \left(\frac{c_3}{c_4} - c_3 - \frac{c_3 c_6}{c_4} - \frac{c_5}{4B}\right),$$
$$G = \frac{c_4}{4B}\tau_1 + I_1,$$

where I_1, I_2 are integration constants. Substituting the expressions for τ_2 and τ_1 from eq.(2.5) one gets a particular solution for eq.(2.1).

Similarly one can also bring out other particular solutions for all the other sub-cases in the same manner.

3 Lie symmetries and Infinite Dimensional Lie Algebras of the Zakharov-Strachan equation

In this section we investigate the symmetries and similarity reductions associated with another important (2+1)-dimensional generalization of the NLS equation, introduced originally by Zakharov and studied recently by Strachan of the form [22]. Its form reads

$$\begin{aligned} 2kq_t &= q_{xy} - 2q\int \partial_y[p.q]dx, \\ -2kp_t &= p_{xy} - 2p\int \partial_y[p.q]dx. \end{aligned} \tag{3.1}$$

By introducing a potential $v(p,q)$ defined by

$$v_x(p,q) = 2\partial_y[p.q],$$

and imposing the algebraic constraints on the fields p and q such that $q = p^* = \psi$ and choosing $k = i/2$ eq.(3.1) reduces to

$$i\psi_t = \psi_{xy} + v\psi, \qquad v_x = 2\partial_y|\psi|^2. \tag{3.2}$$

When $\partial_x = \partial_y$ eq.(3.2) reduces to the NLS equation and when $\partial_t = 0$, it reduces to a complicated sine-Gordon equation.

It has been shown that in ref. [23] that eq.(3.2) admits P-property. Further, the authors have also constructed a new class of localized solutions called "induced localized structures" [24].

To study the invariance properties of the the eq.(3.2) we introduce the transformation $\psi = a + ib$ so that eq.(3.2) becomes

$$a_t - b_{xy} - bv = 0, \qquad b_t + a_{xy} + av = 0, \qquad v_x - 4aa_y - 4bb_y = 0. \tag{3.3}$$

We will investigate the Lie symmetries of eq.(3.3) under the one parameter (ε) group of transformations

$$
\begin{aligned}
x &\longrightarrow X = x + \varepsilon\xi_1(t,x,y,a,b,v),\\
y &\longrightarrow Y = y + \varepsilon\xi_2(t,x,y,a,b,v),\\
t &\longrightarrow T = t + \varepsilon\xi_3(t,x,y,a,b,v),\\
a &\longrightarrow A = a + \varepsilon\phi_1(t,x,y,a,b,v),\\
b &\longrightarrow B = b + \varepsilon\phi_2(t,x,y,a,b,v),\\
v &\longrightarrow V = v + \varepsilon\phi_3(t,x,y,a,b,v).
\end{aligned}
$$

The infinitesimal transformations can be worked out to be

$$
\begin{aligned}
\xi_1 &= -\left(\frac{c_1}{2}xt + c_2 x - f(t)\right),\\
\xi_2 &= -\frac{c_1}{2}yt + (c_2 - c_3)y - c_5 t + c_6,\\
\xi_3 &= -\left(\frac{c_1}{2}t^2 + c_3 t + c_4\right),\\
\phi_1 &= -\frac{c_1}{2}bxy + \frac{c_1}{2}at + c_2 a - c_5 bx + \dot{f}(t)by - g(t)b,\\
\phi_2 &= \frac{c_1}{2}axy + \frac{c_1}{2}bt + c_2 b + c_5 ax - \dot{f}(t)ay + g(t)a,\\
\phi_3 &= (c_1 t + c_3)v + \ddot{f}(t)y - \dot{g}(t),
\end{aligned}
\tag{3.4}
$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants and $f(t)$ and $g(t)$ are arbitrary functions of t and dot denotes differentiation with respect to t.

3.1 Lie algebra

The Lie vector fields associated with the infinitesimal transformations can be written as

$$
V = V_1(f) + V_2(g) + V_3 + V_4 + V_5 + V_6 + V_7 + V_8,
$$

where f and g are arbitrary functions of t and

$$
\begin{aligned}
V_1(f) &= f(t)\frac{\partial}{\partial x} + \dot{f}(t)by\frac{\partial}{\partial a} - \dot{f}(t)ay\frac{\partial}{\partial b} + \ddot{f}(t)y\frac{\partial}{\partial v},\\
V_2(g) &= -g(t)b\frac{\partial}{\partial a} + g(t)a\frac{\partial}{\partial b} - \dot{g}(t)\frac{\partial}{\partial v},\\
V_3 &= -\frac{1}{2}xt\frac{\partial}{\partial x} - \frac{1}{2}yt\frac{\partial}{\partial y} - \frac{t^2}{2}\frac{\partial}{\partial t} + \frac{1}{2}(at - bxy)\frac{\partial}{\partial a} + \frac{1}{2}(bt + axy)\frac{\partial}{\partial b} + vt\frac{\partial}{\partial v},\\
V_4 &= -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}, \qquad V_5 = -y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v},\\
V_6 &= -t\frac{\partial}{\partial y} - bx\frac{\partial}{\partial a} + ax\frac{\partial}{\partial b}, \qquad V_7 = \frac{\partial}{\partial y}, \qquad V_8 = \frac{\partial}{\partial t}.
\end{aligned}
$$

The corresponding Lie algebra between the vector fields becomes

$$[V_1(f_1), V_2(f_2)] = 0, \quad [V_2(g_1), V_2(g_2)] = 0, \quad [V_1(f), V_2(g)] = 0,$$
$$[V_1, V_3] = V_1(-f + 3t\dot{f}), \quad [V_1, V_4] = V_2(f - t\dot{f}), \quad [V_1, V_5] = \frac{1}{4B} V_2(\dot{f}),$$
$$[V_1, V_6] = V_1(\dot{f}), \quad [V_2, V_3] = V_2(g + 3t\dot{g}), \quad [V_2, V_4] = 0, \quad [V_2, V_5] = 0,$$
$$[V_2, V_6] = V_2(\dot{g}), \quad [V_3, V_4] = -2V_4, \quad [V_3, V_5] = V_5, \quad [V_3, V_6] = 3V_6,$$
$$[V_4, V_5] = -\frac{3A}{2Bg(t)} V_2, \quad [V_4, V_6] = 4BV_6, \quad [V_5, V_6] = 0.$$

It is interesting to note that the the above algebra does not contain a *Virasoro algebra*, which is typical of integrable (2+1)-dimensional systems such as the NLS equation of Fokas type and other integrable systems quoted in the Appendix.

3.2 Similarity variables and reductions

The similarity variables can be found by integrating the following characteristic equation

$$\frac{dx}{-(\frac{c_1}{2}xt + c_2 x - f(t))} = \frac{dy}{-\frac{c_1}{2}yt + (c_2 - c_3)y - c_5 t + c_6}$$
$$= \frac{dt}{-(\frac{c_1}{2}t^2 + c_3 t + c_4)} = \frac{da}{-\frac{c_1}{2}bxy + \frac{c_1}{2}at + c_2 a - c_5 bx + \dot{f}(t)by - g(t)b}$$
$$= \frac{db}{\frac{c_1}{2}axy + \frac{c_1}{2}bt + c_2 b + c_5 ax - \dot{f}(t)ay + g(t)a} = \frac{dv}{c_1 vt + c_3 v + \ddot{f}(t)y - \dot{g}(t)}.$$

Solving the characteristic equation, we obtain the following similarity transformations:

$$\tau_1 = \frac{x(t + k_1 + k_2)^{n-(1/2)}}{(t + k_1 - k_2)^{n+(1/2)}} + \int_0^t \frac{2}{c_1} \frac{f(t')(t' + k_1 + k_2)^{n-(3/2)}}{(t' + k_1 - k_2)^{n+(3/2)}} dt',$$

$$\tau_2 = \frac{y(t + k_1 + k_2)^{-(n+(1/2))}}{(t + k_1 - k_2)^{-n+(1/2)}} - \int_0^t \frac{2}{c_1} \frac{(c_5 t' - c_6)(t' + k_1 + k_2)^{-(n+(3/2))} dt'}{(t' + k_1 - k_2)^{-n+(3/2)}},$$

$$a = \frac{F_2 \sin U (t + k_1 + k_2)^{n-(1/2)}}{(t + k_1 - k_2)^{n+(1/2)}}, \qquad b = \frac{F_2 \cos U (t + k_1 + k_2)^{n-(1/2)}}{(t + k_1 - k_2)^{n+(1/2)}},$$

$$F_3 = \frac{c_1}{2}(t + k_1 + k_2)(t + k_1 - k_2)v \tag{3.5}$$
$$+ \frac{\dot{f}(t)(t + k_1 - k_2)^{-n+(1/2)}}{(t + k_1 + k_2)^{-n-(1/2)}} \int_0^t \frac{2(c_5 t' - c_6)(t' + k_1 + k_2)^{-n-(3/2)} dt'}{c_1(t' + k_1 - k_2)^{-n+(3/2)}}$$
$$- \frac{f(t)(t + k_1 - 2nk_2)(t + k_1 - k_2)^{-n-(1/2)}}{(t + k_1 + k_2)^{-n+(1/2)}} \int_0^t \frac{2(c_5 t' - c_6)(t' + k_1 + k_2)^{-n-(3/2)} dt'}{c_1(t' + k_1 - k_2)^{-n+(3/2)}}$$
$$+ \int_0^t \frac{f(t)(t' + k_1 - k_2)^{-n-(1/2)} dt'}{(t' + k_1 + k_2)^{-n+(1/2)}} \int_0^{t'} \frac{2(c_5 t'' - c_6)(t'' + k_1 + k_2)^{-n-(3/2)} dt''}{c_1(t'' + k_1 - k_2)^{-n+(3/2)}}$$

$$
\begin{aligned}
&-\int_0^t \frac{f(t')(t'+k_1+2nk_2)(t'+k_1-2nk_2)(t'+k_1-k_2)^{-n-(3/2)}dt'}{(t'+k_1+k_2)^{-n+(3/2)}} \\
&\times \int_0^{t'} \frac{2(c_5t''-c_6)(t''+k_1+k_2)^{-n-(3/2)}dt''}{c_1(t''+k_1-k_2)^{-n+(3/2)}} \\
&+\int_0^t \frac{2f(t')(t'+k_1-2nk_2)(c_5t'-c_6)dt'}{c_1(t'+k_1+k_2)^2(t'+k_1-k_2)^2} - \frac{2f(t)(c_5t-c_6)}{c_1(t+k_1+k_2)(t+k_1-k_2)} \\
&+\int_0^t \frac{2c_5f(t')dt'}{c_1(t'+k_1+k_2)(t'+k_1-k_2)} - \int_0^t \frac{4f(t')(t'+k_1)(c_5t'-c_6)dt'}{c_1(t'+k_1+k_2)^2(t'+k_1-k_2)^2} - g(t),
\end{aligned}
$$

where

$$
k_1 = \frac{c_3}{c_1}, \qquad k_2 = \frac{\sqrt{c_3^2-2c_1c_4}}{c_1}, \qquad n = \frac{2c_2-c_3}{2\sqrt{c_3^2-2c_1c_4}}
$$

and

$$
\begin{aligned}
U = &-\int_0^t \left[\int_0^{t'} \frac{2f(t'')(t''+k_1+k_2)^{n-(3/2)}}{c_1(t''+k_1-k_2)^{n+(3/2)}}dt''\right] \\
&\times \left[\int_0^{t'} \frac{2(c_5t''-c_6)(t''+k_1+k_2)^{-(n+(3/2))}}{c_1(t''+k_1-k_2)^{-n+(3/2)}}dt''\right] dt' \\
&-\tau_2 \int_0^t \left[\int_0^{t'} \frac{2f(t'')(t''+k_1+k_2)^{n-(3/2)}}{c_1(t''+k_1-k_2)^{n+(3/2)}}dt''\right] dt' + \tau_1\tau_2 t \\
&+\tau_1 \int_0^t \left[\int_0^{t'} \frac{2(c_5t''-c_6)(t''+k_1+k_2)^{-(n+(3/2))}}{c_1(t''+k_1-k_2)^{-n+(3/2)}}dt''\right] dt' \\
&-\int_0^t \frac{2c_5(t'+k_1-k_2)^{n-(1/2)}}{c_1(t'+k_1+k_2)^{n+(1/2)}} \left[\int_0^{t'} \frac{2f(t'')(t''+k_1+k_2)^{n-(3/2)}dt''}{c_1(t''+k_1-k_2)^{n+(3/2)}}\right] dt' \\
&+\tau_1 \int_0^t \frac{2c_5(t'+k_1-k_2)^{n-(1/2)}}{c_1(t'+k_1+k_2)^{n+(1/2)}}dt' + \int_0^t \frac{4f(t')(c_5t'-c_6)}{c_1^2(t'+k_1+k_2)^2(t'+k_1-k_2)^2}dt' \\
&-\frac{f(t)(t+k_1-k_2)^{-(n+(1/2))}}{(t+k_1+k_2)^{-n+(1/2)}} \int_0^t \frac{4(c_5t'-c_6)(t'+k_1+k_2)^{-(n+(3/2))}}{c_1^2(t'+k_1-k_2)^{-n+(3/2)}}dt' \\
&-\int_0^t \frac{2f(t')(t'+k_1+2nq)(t'+k_1-k_2)^{-(n+(3/2))}}{c_1(t'+k_1+k_2)^{-n+(1/2)}} \\
&\times \left[\int_0^{t'} \frac{2(c_5t''-c_6)(t''+k_1+k_2)^{-(n+(3/2))}}{c_1(t''+k_1-k_2)^{-n+(3/2)}}dt''\right] dt'
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2\tau_2 f(t)(t+k_1-k_2)^{-(n+(1/2))}}{c_1(t+k_1+k_2)^{-n+(1/2)}}\\
&-\tau_2\int_0^t \frac{2f(t')(t'+k_1+2nq)(t'+k_1-k_2)^{-(n+(3/2))}}{c_1(t'+k_1+k_2)^{-n+(3/2)}}dt'\\
&+\int_0^t \frac{2g(t')dt'}{c_1(t'+k_1+k_2)(t'+k_1-k_2)}+F_1,
\end{aligned}
$$

where F_1, F_2 and F_3 are arbitrary functions of τ_1 and τ_2. Under this set of similarity transformations eq.(3.3) takes the form

$$
\begin{aligned}
&F_{2\tau_1\tau_2}+\frac{2c_2}{c_1}\tau_1F_2F_{1\tau_1}+\frac{2(c_3-c_2)}{c_1}\tau_2F_2F_{1\tau_2}-F_2F_{1\tau_1}F_{1\tau_2}-\frac{2c_4}{c_1}\tau_1\tau_2F_2+2F_2F_3=0,\\
&F_2F_{1\tau_1\tau_2}+F_{2\tau_1}F_{1\tau_2}+F_{1\tau_1}F_{2\tau_2}-\frac{2c_2}{c_1}\tau_1F_{2\tau_1}-\frac{2(c_3-c_2)}{c_1}\tau_2F_{2\tau_2}-\frac{2c_2}{c_1}F_2=0,\\
&F_{3\tau_1}-2c_1F_2F_{2\tau_2}=0.
\end{aligned}
\tag{3.6}
$$

3.3 Subcases

Besides the above general similarity reductions, one can find a number of special reductions corresponding to lesser parameter symmetries by choosing some of the arbitrary parameter c_1, c_2, c_3, c_4 and c_5 and arbitrary functions $f(t)$ and $g(t)$ to be zero. Important nontrivial cases are given below.

Case 1: $c_1=0$.

$$
\begin{aligned}
&\tau_1=\frac{x}{(c_3t+c_4)^{c_2/c_3}}+\int_0^t\frac{f(t')dt'}{(c_3t'+c_4)^{(c_2+c_3)/c_3}},\\
&\tau_2=y(c_3t+c_4)^{(c_2-c_3)/c_3}-\int_0^t(c_5t'-c_6)(c_3t'+c_4)^{(c_2-2c_3)/c_3}dt',\\
&a=F_2\sin U(c_3t+c_4)^{-c_2/c_3},\qquad b=F_2\cos U(c_3t+c_4)^{-c_2/c_3},\\
&U=-c_5\int_0^t(c_3t'+c_4)^{(c_2-c_3)/c_3}\left[\int_0^{t'}\frac{f(t'')dt''}{(c_3t''+c_4)^{(c_2+c_3)/c_3}}\right]dt'\\
&\quad+c_5\tau_1\int_0^t(c_3t'+c_4)^{(c_2-c_3)/c_3}dt'\\
&\quad-\int_0^t\dot f(t')(c_3t'+c_4)^{-c_2/c_3}\left[\int_0^{t'}(c_5t''-c_6)(c_3t''+c_4)^{(c_2-2c_3)/c_3}dt''\right]dt'\\
&\quad-\tau_2\int_0^t\dot f(t')(c_3t'+c_4)^{-c_2/c_3}dt'+\int_0^t\frac{g(t')dt'}{(c_3t'+c_4)}+F_1.
\end{aligned}
$$

Under this similarity transformation the reduced PDE takes the form

$$F_{2\tau_1\tau_2} + c_2\tau_1 F_2 F_{1\tau_1} + (c_3 - c_2)\tau_2 F_2 F_{1\tau_2} - F_2 F_{1\tau_1} F_{1\tau_2} - 2F_2F_3 = 0,$$
$$F_2 F_{1\tau_1\tau_2} + F_{2\tau_1} F_{1\tau_2} + F_{1\tau_1} F_{2\tau_2} - c_2 F_2 - c_2\tau_1 F_{2\tau_1} + (c_2 - c_3)\tau_2 F_{2\tau_2} = 0,$$
$$F_{3\tau_1} - F_2 F_{2\tau_2} = 0.$$

Case 2: $c_1, c_2, c_3 = 0$.

$$\tau_1 = x + \frac{1}{c_4}\int_0^t f(t')dt', \quad \tau_2 = y - \frac{c_5 t^2}{2c_4} + \frac{c_6 t}{c_4}, \quad a = F_2 \sin U, \quad b = F_2 \cos U,$$

$$v = -\frac{c_5}{2c_4} t^2 \dot{f}(t) + \frac{c_5}{c_4} t f(t) - \frac{c_5}{c_4}\int_0^t f(t')dt' + \frac{c_6}{c_4} t\dot{f}(t) - \frac{c_6}{c_4} f(t) - \tau_2 \dot{f}(t) + \frac{g(t)}{c_4} + F_3.$$

and

$$U = -\frac{c_5}{c_4^2}\int_0^t \left[\int_0^{t'} f(t'')dt''\right] dt' + \frac{c_5}{c_4}\tau_1 t - \frac{c_5}{2c_4^2}\left[t^2 f(t) - 2\int_0^t t' f(t')dt'\right]$$
$$+\frac{c_6}{c_4^2}\left[t f(t) - \int_0^t f(t')dt'\right] - \frac{\tau_2 f(t)}{c_4} + \frac{1}{c_4}\int_0^t g(t')dt' + F_1.$$

The reduced PDE takes the form

$$F_2 F_{1\tau_1\tau_2} + F_{2\tau_1} F_{1\tau_2} + F_{2\tau_2} F_{1\tau_1} + \frac{c_6}{c_4} F_{2\tau_2} = 0,$$
$$F_{2\tau_1\tau_2} - F_2 F_{1\tau_1} F_{1\tau_2} - \frac{c_6}{c_4} F_2 F_{1\tau_2} - \frac{c_5}{c_4}\tau_1 F_2 + F_2 F_3 = 0,$$
$$F_{3\tau_1} - 4F_2 F_{2\tau_2} = 0.$$

Case 3: $c_1, c_2, c_4 = 0$.

$$\tau_1 = x + \frac{1}{c_3}\int_0^t \frac{f(t')dt'}{t'}, \quad \tau_2 = \frac{y}{t} - \frac{c_5}{c_3}\log t - \frac{c_6}{c_3 t}, \quad a = F_2 \sin U, \quad b = F_2 \cos U,$$

$$v = -\frac{\dot{f}(t)}{c_3}\log t^{c_5/c_3} + \frac{f(t)}{c_3 t}\log t^{c_5/c_3} - \frac{c_5}{c_3^2 t}\int_0^t \frac{f(t')dt'}{t'} + \frac{c_5}{c_3^2 t} f(t) - \frac{c_6}{c_3^2 t}\dot{f}(t)$$
$$-\frac{\tau_2}{c_3}\dot{f}(t) + \frac{\tau_2}{c_3 t} f(t) + \frac{g(t)}{c_3 t} + \frac{F_3}{t},$$

and

$$U = -\frac{c_5}{c_3^2}\int_0^t \frac{1}{t'}\left[\int_0^{t'} \frac{f(t'')}{t''}dt''\right] dt' + \frac{c_5\tau_1}{c_3}\log t - \frac{f(t)}{c_3}\log t^{c_5/c_3} + \frac{c_5}{c_3^2}\int_0^t \frac{f(t')dt'}{t'}$$
$$-\frac{c_6 f(t)}{c_3^2 t} - \frac{c_6}{c_3^2}\int_0^t \frac{f(t')}{t'^2}dt' - \frac{\tau_2 f(t)}{c_3} + \int_0^t \frac{g(t')dt'}{c_3 t'} + F_1.$$

The reduced PDE takes the form

$$F_2F_{1\tau_1\tau_2} + F_{1\tau_1}F_{2\tau_2} - \frac{c_5}{c_3}F_{2\tau_2} - \tau_2F_{2\tau_2} = 0,$$
$$F_{2\tau_1\tau_2} - F_2F_{1\tau_1}F_{1\tau_2} + \frac{c_5}{c_3}F_2F_{1\tau_2} + \tau_2F_2F_{1\tau_2} - \frac{c_5}{c_3}\tau_1F_2 + F_2F_3 = 0,$$
$$F_{3\tau_1} - 4F_2F_{2\tau_2} = 0.$$

Case 4: $c_1, c_2, c_6 = 0$.

$$\tau_1 = x + \int_0^t \frac{f(t')}{(c_3t' + c_4)}dt', \quad \tau_2 = \frac{y}{(c_3t + c_4)} - log(c_3t + c_4)^{c_5/c_3^2} - \frac{c_4c_5}{c_3^2(c_3t + c_4)},$$
$$a = F_2 \sin U, \qquad b = F_2 \cos U,$$
$$v = -\frac{1}{(c_3t + c_4)}\Big[\dot{f}(t)(c_3t + c_4)\log(c_3t + c_4)^{c_5/c_3^2} - c_3f(t)\log(c_3t + c_4)^{c_5/c_3^2}$$
$$+c_5\int_0^t \frac{f(t')}{(c_3t' + c_4)}dt' - \frac{c_5}{c_3}f(t) + \frac{c_4c_5}{c_3^2}\dot{f}(t) + \tau_2\dot{f}(t)(c_3t + c_4) - c_3\tau_2f(t)\Big]$$
$$U = -c_5\int_0^t \frac{1}{(c_3t' + c_4)}\left[\int_0^{t'} \frac{f(t'')}{(c_3t'' + c_4)}dt''\right]dt' + \frac{\tau_1c_5}{c_3}\log(c_3t + c_4)$$
$$-f(t)\log(c_3t + c_4)^{c_5/c_3^2} + \frac{c_5}{c_3}\int_0^t \frac{f(t')}{(c_3t' + c_4)}dt' - \frac{c_4c_5f(t)}{c_3^2(c_3t + c_4)}$$
$$-\frac{c_4c_5}{c_3}\int_0^t \frac{f(t')}{(c_3t' + c_4)^2}dt' - \tau_2f(t).$$

The reduced PDE takes the form

$$F_2F_{1\tau_1\tau_2} + F_{1\tau_2}F_{2\tau_1} + F_2\tau_2F_{1\tau_1} - c_3\tau_2F_{2\tau_2} - \frac{c_5}{c_3}F_{2\tau_2} = 0,$$
$$F_{2\tau_1\tau_2} - F_2F_{1\tau_1}F_{1\tau_2} + \frac{c_5}{c_3}F_2F_{1\tau_2} + c_3\tau_2F_2F_{1\tau_2} + c_5\tau_1F_2 + F_2F_3 = 0,$$
$$F_{3\tau_1} - 4F_2F_{2\tau_2} = 0.$$

3.4 Lie symmetries of eq.(3.6)

Applying the Lie algorithm again to the eq.(3.6), one gets the infinitesimals as

$$\xi_1 = -c_7\tau_1 + \frac{c_1^2c_8}{(2c_2c_3 - c_1c_4 - 2c_2^2)}, \qquad \xi_2 = c_7\tau_2,$$
$$\phi_1 = \frac{2c_1c_2c_8\tau_2}{(2c_2c_3 - c_1c_4 - 2c_2^2)} + c_9, \qquad \phi_2 = c_7w_2, \qquad \phi_3 = c_8\tau_2, \tag{3.7}$$

where c_7, c_8 and c_9 are arbitrary constants. Solving the characteristic equation associated with infinitesimal symmetries, (3.7), we get the following similarity variables

$$z = \tau_2\left(c_7\tau_1 - \frac{c_1^2c_8}{(2c_2c_3 - c_1c_4 - 2c_2^2)}\right),$$

$$w_1 = F_1 - \frac{2c_1c_2c_8}{c_7(2c_2c_3 - c_1c_4 - 2c_2^2)}\tau_2 - \frac{c_9}{c_7}\log\tau_2,$$

$$w_2 = \frac{F_2}{\tau_2}, \qquad w_3 = F_3 - \frac{c_8}{c_7}\tau_2.$$

Under this similarity transformation one can reduce the PDE (3.6) into an ODE of the form

$$\begin{aligned} w_2'' &+ \frac{2c_3}{c_1c_7}w_2w_1' - \frac{c_9}{c_7z}w_2w_1' - w_2w_1^2 + \frac{2(c_3-c_2)c_9}{c_1c_7z}w_2 \\ &\qquad - \frac{2c_4}{c_1c_7}w_2 + \frac{2}{z}w_2w_3 + \frac{2}{z}w_3' = 0, \\ w_1''w_2 &+ 2w_1'w_2' + \frac{2}{z}w_1'w_2 - \frac{2c_3}{c_1c_7}w_2' + \frac{c_9}{c_7}w_2' - \frac{2c_3w_2}{c_1c_7z} = 0, \\ w_3' &= \frac{2c_1}{c_7}w_2(w_2 + zw_2'). \end{aligned} \tag{3.8}$$

3.5 Subcases

Eventhough it is very difficult to find the general solution of the eq.(3.8) one can get particular solutions from out of the infinitesimal symmetries by choosing some of the arbitrary constants to be zero. For example, by choosing c_7 as zero ($c_8, c_9 \neq 0$) one gets the similarity variables as

$$z = \tau_2, \quad w_1 = F_1 - \frac{2c_2}{c_1}\tau_1\tau_2 - \frac{c_9k}{c_1^2c_8}\tau_1, \quad w_2 = F_2, \quad w_3 = F_3 + \frac{k}{c_1^2}\tau_1\tau_2,$$

where $k = (2c_2c_3 - c_1c_4 - 2c_2^2)$ and w_1, w_2 and w_3 are arbitrary functions of τ_1 and τ_2. Under this similarity transformation the PDE (3.6) gets reduced to an ODE of the following form,

$$\begin{aligned} &w_2'\left[\frac{4c_2 - 2c_3}{c_1}z + \frac{c_9k}{c_1^2c_8}\right] = 0, \qquad w_2w_2' + \frac{kz}{c_1^3} = 0, \\ &\frac{2(c_3-c_2)}{c_1}zw_2w_1' - w_2w_1'\left[\frac{2c_2}{c_1}z + \frac{c_9k}{c_1^2c_8}\right] + 2w_2w_3 = 0. \end{aligned} \tag{3.9}$$

A simple solution can be obtained from (3.9) by restricting $k = 0$, as

$$w_1 = \frac{c_1}{2c_2 - c_3}\int_0 \frac{w_3}{z}dz + I_2, \qquad w_2 = I_1,$$

where w_3 is arbitrary.

Similarly for the case $c_7 = 0$ one ends up with the following ODE:

$$\begin{aligned} &w_2'' + \frac{2c_3}{c_1}w_2w_1' + \frac{c_9}{c_7z}w_1'w_2 - w_2w_1'^2 - \frac{2c_4}{c_1}w_2 + \frac{2}{z}w_2w_3 = 0, \\ &w_2w_1'' + 2w_1'w_2' - \frac{c_9}{c_1z}w_2' - \frac{2c_3}{c_1}w_2' = 0, \qquad w_3' - \frac{2c_1}{z}w_2w_2' = 0. \end{aligned}$$

One can extend the same analysis for all sub-cases mentioned in Sec. 3.3 and bring out particular solutions.

4 Conclusions

In this paper we have carried out a detailed invariance analysis of two different nonlinear evolution equations in (2+1)-dimensions, namely, (i) breaking soliton equation and (ii) (2+1) NLS equation introduced by Zakharov, which attracted considerable attention in the recent literature and pointed out that the above two equations do not admit Virasoro type algebras even though they are integrable. We have also briefly reviewed the existence of Kac-Moody-Virasoro algebras in other integrable systems. The fuller implication of the absence of the Kac-Moody-Virasoro type subalgebras in both the systems and their connection with integrability deserves much further study. As far as our knowledge goes no one has pointed out in the literature that any nonintegrable system admits Kac-Moody-Virasoro type algebras. Thus from our studies we have also concluded that one can not distinguish the integrable systems with the existence of Kac-Moody-Virasoro algebras. Currently we are investigating the possible new similarity reductions through non-classical and direct methods of Clarkson and Kruskal.

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5 Appendix

In the following we briefly summarize the existence of Virasoro type algebras in other important integrable (2+1)-dimensional nonlinear systems.

A Nizhnik-Novikov-Veselov (NNV) equation

A symmetric generalization of the KdV equation in (2+1)-dimensions is the NNV equation [26]

$$\begin{aligned} &u_t + u_{xxx} + u_{yyy} + u_x + u_y = 3(uv)_x + 3(uq)_y, \\ &u_x = v_y, \qquad u_y = q_x. \end{aligned} \tag{A.1}$$

It has been shown that eq.(A.1) admits weak Lax pair [27], Painlevé property and dromion solutions [28]. Further, eq.(A.1) admits the following infinite dimensional Lie vector fields of the form [17]

$$V = V_1(f) + V_2(g) + V_3(h),$$

where

$$\begin{aligned} V_1(f) &= \frac{x}{3}\dot{f}(t)\frac{\partial}{\partial x} + \frac{y}{3}\dot{f}(t)\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} - \frac{2}{3}u\dot{f}(t)\frac{\partial}{\partial u} \\ &\quad + \left(\frac{2}{9}\dot{f}(t) - \frac{2}{3}v\dot{f}(t) - \frac{1}{9}x\ddot{f}(t)\right)\frac{\partial}{\partial v} + \left(\frac{2}{9}\dot{f}(t) - \frac{2}{3}q\dot{f}(t) - \frac{1}{9}y\ddot{f}(t)\right)\frac{\partial}{\partial q}, \\ V_2(g) &= g(t)\frac{\partial}{\partial x} - \frac{1}{3}\dot{g}(t)\frac{\partial}{\partial v}, \qquad V_3(g) = h(t)\frac{\partial}{\partial y} - \frac{1}{3}\dot{h}(t)\frac{\partial}{\partial q}. \end{aligned}$$

where $f(t)$, $g(t)$ and $h(t)$ are arbitrary functions of t and dot denotes differentiation with respect to t.

The associated Lie algebra between these vector fields become

$$
\begin{aligned}
&[V_1(f_1), V_1(f_2)] = V_1(f_1\dot{f}_2 - f_2\dot{f}_1), \qquad [V_2(g_1), V_2(g_2)] = 0,\\
&[V_3(h_1), V_3(h_2)] = 0, \qquad [V_1(f), V_2(g)] = V_2\left(f\dot{g} - \frac{1}{3}g\dot{f}\right),\\
&[V_1(f), V_3(h)] = V_3\left(f\dot{h} - \frac{1}{3}h\dot{f}\right), \qquad [V_2(g), V_3(h)] = 0,
\end{aligned}
$$

which is obviously an infinite dimensional Lie algebra of symmetries. A Virasoro-Kac-Moody type subalgebra is immediately obtained by restricting the arbitrary functions f, g and h to Laurent polynomials so that we have the commutators

$$
\begin{aligned}
&[V_1(t^n), V_1(t^m)] = (m-n)V_1(t^{n+m-1}), \quad [V_1(t^n), V_2(t^m)] = \left(m - \frac{1}{3}n\right)V_2(t^{n+m-1}),\\
&[V_1(t^n), V_3(t^m)] = \left(m - \frac{1}{3}n\right)V_3(t^{n+m-1}), \quad [V_2(t^n), V_2(t^m)] = 0,\\
&[V_3(t^n), V_3(t^m)] = 0, \quad [V_2(t^n), V_3(t^m)] = 0.
\end{aligned}
$$

B Generalized nonlinear Schrödinger equation introduced by Fokas

Recently Fokas has introduced a (2+1)-dimensional generalized nonlinear Schrödinger equation of the form [29]

$$
\begin{aligned}
&iq_t - (\alpha-\beta)q_{xx} + (\alpha+\beta)q_{yy} - 2\lambda q\left[(\alpha+\beta)v - (\alpha-\beta)u\right] = 0,\\
&v_x = |q|^2_y, \qquad u_y = |q|^2_x.
\end{aligned}
\tag{B.1}
$$

Eq.(B.1) is a symmetric generalization of a (1+1)-dimensional NLS equation. Interestingly it includes the following three important systems:

(i) $\alpha = \beta = 1/2$: Simplest complex scalar equation in (2+1)-dimensions;

(ii) $\alpha = 0$, $\beta = 1$: Davey-Stewartson equation I (DSI);

(iii) $\alpha = 1$, $\beta = 0$: Davey-Stewartson equation III (DSIII).

By introducing the transformation $q = a + ib$ Eq.(B.1) can be rewritten as

$$
\begin{aligned}
&a_t - (\alpha-\beta)b_{xx} + (\alpha+\beta)b_{yy} - 2\lambda(\alpha+\beta)bv + 2\lambda(\alpha-\beta)bu = 0,\\
&b_t + (\alpha-\beta)a_{xx} - (\alpha+\beta)a_{yy} + 2\lambda(\alpha+\beta)av - 2\lambda(\alpha-\beta)au = 0,\\
&v_x - 2aa_y - 2bb_y = 0, \qquad u_y - 2aa_x - 2bb_x = 0.
\end{aligned}
\tag{B.2}
$$

Recently Radha and Lakshmanan [30] have investigated the system (B.2) and shown that it admits P-property and constructed multidromion and localized breather solutions. Eq.(B.2) admits Lie vector fields of the form [17]

$$
V = V_1(f) + V_2(g) + V_3(h) + V_4(l) + V_5(m),
$$

where

$$
V_1(f) = \frac{x}{2}\dot{f}(t)\frac{\partial}{\partial x} + \frac{y}{2}\dot{f}(t)\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} - \left(\frac{a}{2}\dot{f}(t) - \frac{b}{8A}x^2\ddot{f}(t) + \frac{b}{8B}y^2\ddot{f}(t)\right)\frac{\partial}{\partial a}
$$

$$-\left(\frac{b}{2}\dot{f}(t)+\frac{a}{8A}x^2\ddot{f}(t)-\frac{a}{8B}y^2\ddot{f}(t)\right)\frac{\partial}{\partial b}-\left(v\dot{f}(t)-\frac{1}{16B^2C}y^2\ddot{f}(t)\right)\frac{\partial}{\partial v}$$

$$-\left(u\dot{f}(t)+\frac{1}{16A^2C}x^2\ddot{f}(t)\right)\frac{\partial}{\partial u},$$

$$V_2(g)=g(t)\frac{\partial}{\partial x}+\frac{1}{2A}bx\dot{g}(t)\frac{\partial}{\partial a}-\frac{1}{2A}ax\dot{g}(t)\frac{\partial}{\partial b}-\frac{1}{4A^2C}x\ddot{g}(t)\frac{\partial}{\partial u},$$

$$V_3(h)=h(t)\frac{\partial}{\partial y}-\frac{1}{2B}by\dot{h}(t)\frac{\partial}{\partial a}+\frac{1}{2B}ay\dot{h}(t)\frac{\partial}{\partial b}-\frac{1}{4B^2C}y\ddot{h}(t)\frac{\partial}{\partial v},$$

$$V_4(l)=-bl(t)\frac{\partial}{\partial a}+al(t)\frac{\partial}{\partial b}+\frac{1}{2AC}\dot{l}(t)\frac{\partial}{\partial u},\qquad V_5(m)=m(t)\frac{\partial}{\partial u}+\frac{B}{A}m(t)\frac{\partial}{\partial v},$$

where f, g, h, l, m are arbitrary functions of t and $A=(\alpha-\beta)$, $B=(\alpha+\beta)$ and $c=\lambda$. The nonzero commutation relations between the Lie vector fields are

$$[V_1(f_1),V_1(f_2)]=V_1(f_1\dot{f}_2-f_2\dot{f}_1),\qquad [V_2(g_1),V_2(g_2)]=-\frac{1}{2A}V_4(g_1\dot{g}_2-g_2\dot{g}_1),$$
$$[V_3(h_1),V_3(h_2)]=-\frac{1}{2B}V_4(h_1\dot{h}_2-h_2\dot{h}_1),\qquad [V_1(f),V_2(g)]=V_2\left(f\dot{g}-\frac{g\dot{f}}{2}\right),$$
$$[V_1(f),V_3(h)]=V_3\left(f\dot{h}-\frac{h\dot{g}}{2}\right),\qquad [V_1(f),V_4(h)]=V_4(f\dot{l}),$$
$$[V_1(f),V_5(m)]=V(m\dot{f}+f\dot{m}).$$

By restricting the arbitrary functions f, g, h, l and m to be polynomials in t one can get Kac-Moody-Virasoro type subalgebras of the form

$$[V_1(t^n),V_1(t^m)]=(m-n)V_1(t^{n+m-1}),\quad [V_2(t^n),V_2(t^m)]=\frac{-(m-n)}{2A}V_4(t^{n+m-1}),$$
$$[V_3(t^n),V_3(t^m)]=\frac{-(m-n)}{2B}V_4(t^{n+m-1}),\quad [V_1(t^n),V_2(t^m)]=(m-\frac{n}{2})V_2(t^{n+m-1}),$$
$$[V_1(t^n),V_3(t^m)]=(m-\frac{n}{2})V_3(t^{n+m-1}),\quad [V_1(t^n),V_5(t^m)]=(m+n)V_5(t^{n+m-1}).$$

C (2+1)-dimensional sine-Gordon equation

The (2+1)-dimensional integrable sine-Gordon equation introduced by Konopelchenko and Rogers [31] in appropriate variables has the form

$$\theta_{xyt}+\frac{1}{2}\theta_y\rho_x+\frac{1}{2}\theta_x\rho_y=0,\qquad \rho_{xy}-\frac{1}{2}(\theta_x\theta_y)_t=0. \tag{C.1}$$

Recently Radha and Lakshmanan have studied singularity structure and localized solutions of the eq.(C.1) and shown that eq.(C.1) admits P-property [32]. Eq.(C.1) admits the following Lie vector fields [17]

$$V=V_1(f)+V_2(g)+V_3(h)+V_4(l)+V_5(N)$$

where

$$V_1 = f(x)\frac{\partial}{\partial x}, \qquad V_2 = g(y)\frac{\partial}{\partial y}, \qquad V_3 = h(t)\frac{\partial}{\partial t} - \rho\dot{h}(t)\frac{\partial}{\partial \rho},$$
$$V_4 = l(t)\frac{\partial}{\partial \rho}, \qquad V_5 = N(t)\frac{\partial}{\partial \theta},$$

where f, g, h and l, N are arbitrary functions of x, y and t respectively. The nonzero commutation relations between the vector fields are

$$[V_1(f_1), V_1(f_2)] = V_1(f_1 f_2' - f_2 f_1'), \quad [V_2(g_1), V_2(g_2)] = V_2(g_1 g_2' - g_2 g_1'),$$
$$[V_3(h_1), V_3(h_2)] = V_3(h_1\dot{h}_2 - h_2\dot{h}_1), \quad [V_3(h), V_4(l)] = V_4(l\dot{h} + h\dot{l}),$$
$$[V_3(h), V_5(m)] = V_5(h\dot{m}).$$

All other commutators vanish.

By restricting the arbitrary functions $f(x)$, $g(y)$, $h(t)$, $l(t)$ and $N(t)$ to be polynomials in the variables x, y and t one can get immediately Virasoro type subalgebras of the form

$$[V_1(x^n), V_1(x^m)] = (m-n)V_1(x^{n+m-1}), \quad [V_2(y^n), V_2(y^m)] = \frac{-(m-n)}{2A}V_4(y^{n+m-1}),$$
$$[V_3(t^n), V_3(t^m)] = -(m-n)2BV_4(t^{n+m-1}), \quad [V_3(t^n), V_4(t^m)] = (m+n)V_4(t^{n+m-1}),$$
$$[V_3(t^n), V_5(t^m)] = (m+n)V_5(t^{n+m-1}).$$

D (2+1)-dimensional long dispersive wave equation

Recently Chakravarthy, Kent and Newman [33] have introduced a (2+1)-dimensional long dispersive wave equation of the form

$$\begin{aligned} &\lambda q_t + q_{xx} - 2q\int (qr)_x d\eta = 0, \\ &\lambda r_t - r_{xx} + 2r\int (qr)_x d\eta = 0. \end{aligned} \tag{D.1}$$

Eq.(D.1) is the (2+1)-dimensional generalization of the one dimensional long dispersive wave equation [34]. By introducing the transformation $(qr)_x = v_\eta$ and rewriting the above equation we get

$$\begin{aligned} &q_t + q_{xx} - 2qv = 0, \\ &r_t - r_{xx} + 2rv = 0, \\ &v_y - rq_x - qr_x = 0. \end{aligned} \tag{D.2}$$

Eq.(D.2) admits P-property and line solitons and dromions [35]. Eq.(D.2) admits the following vector fields [18]

$$V = V_1(f) + V_2(g) + V_3(m) + V(N),$$

where

$$V_1(f) = \frac{x}{2}\dot{f}(t)\frac{\partial}{\partial x} + f(t)\frac{\partial}{\partial t} + \left(\frac{1}{8}\ddot{f}(t)qx^2 - \frac{1}{2}q\dot{f}(t)\right)\frac{\partial}{\partial q}$$
$$-\frac{1}{8}\ddot{f}(t)x^2 r\frac{\partial}{\partial r} + \left(\frac{1}{16}\frac{d^3 f}{dt^3}x^2 - v\dot{f}(t)\right)\frac{\partial}{\partial v},$$

$$V_2(g) = g(t)\frac{\partial}{\partial x} + \frac{1}{2}\dot{g}(t)xq\frac{\partial}{\partial q} - \frac{1}{2}\dot{g}(t)xr\frac{\partial}{\partial r} + \frac{1}{4}\ddot{g}(t)x\frac{\partial}{\partial v},$$

$$V_3(m) = m(y)\frac{\partial}{\partial y} - m'(y)q\frac{\partial}{\partial q}, \qquad V_4(N) = -qN(y,t)\frac{\partial}{\partial q} + rN(y,t)\frac{\partial}{\partial r}.$$

The associated Lie algebra between these vector fields become

$$[V_1(f_1), V_1(f_2)] = V_1(f_1\dot{f}_2 - f_2\dot{f}_1),$$

$$[V_2(g_1), V_2(g_2)] = \frac{g_1\dot{g}_2 - g_2\dot{g}_1}{2}\left(q\frac{\partial}{\partial q} - r\frac{\partial}{\partial r}\right) + \frac{g_1\ddot{g}_2 - g_2\ddot{g}_1}{4}\frac{\partial}{\partial v},$$

$$[V_3(m_1), V_3(m_2)] = V_3(m_1m_2' - m_2m_1'), \quad [V_1(f), V_2(g)] = V_2\left(f\dot{g} - \frac{1}{2}g\dot{f}\right),$$

$$[V_1(f), V_4(N)] = V_4(f\dot{N}), \qquad [V_3(m), V_4(N)] = V_4(mN'),$$

which is obviously an infinite dimensional Lie algebra of symmetries. A Virasoro-Kac-Moody type subalgebra is immediately obtained by restricting the arbitrary functions f and m to Laurent polynomials so that we have the commutators

$$[V_1(t^n), V_1(t^m)] = (m-n)V_1(t^{n+m-1}), \qquad [V_3(y^n), V_3(y^m)] = (m-n)V_3(y^{n+m-1}).$$

References

[1] Ablowitz M.J. and Clarkson P.A., Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1990.

[2] Konopelchenko B.G., Solitons in Multidimensions, World Scientific, Singapore, 1993.

[3] Olver P.J., Applications of Lie Groups to Differential Equations, Springer, New York, 1986.

[4] Bluman G.W. and Kumei S., Symmetries and Differential Equations, Springer, New York, 1989.

[5] Stephani H., Differential Equations: Their Solutions Using Symmetries, Cambridge University Press, Cambridge, 1990.

[6] Ibragimov N.H., CRC Handbook of Lie Group Analysis of Differential Equations, CRC Press, Boca Raton, 1996.

[7] Hill J.M., Solutions of Differential Equations by Means of One-parameter Groups, Bitman, Boston, 1982.

[8] Sesahdri R. and Na T.Y., Group Invariance in Engineering Boundary Value Problems, Springer, New York, 1985.

[9] Lakshmanan M. and Kaliappan P., *J. Math. Phys.*, 1983, V.24, 795.

[10] Clarkson P.A., *Chaos, Solitons* & *Fractals*, 1995, V.5, 2261.

[11] Ames W.F. and Rogers C. (Eds.), Nonlinear Equations in the Applied Sciences, Academic Press, Boston, 1992, Chapter II.

[12] David D., Kamran N., Levi D. and Winternitz P., *J. Math. Phys.*, 1986, V.27, 1225.

[13] Champagne B. and Winternitz P., *J. Math. Phys.*, 1988, V.29, 1.

[14] Martina L. and Winternitz P., *Ann. Phys. (N.Y.)*, 1989, V.196, 231.

[15] Levi D. and Winternitz P., *Phys. Lett. A*, 1988, V.129, 165.

[16] Levi D., Menyuk C.R. and Winternitz P., *Phys. Rev. A*, 1994, V.49, 2844.

[17] Lakshmanan M. and Senthil Velan M., *J. Nonlin. Math. Phys*, 1995, V.3, 24.

[18] Senthil Velan M. and Lakshmanan M., *J. Nonlin. Math. Phys.*, 1997, V.4, 251.

[19] Faucher M. and Winternitz P., *Phys. Rev. E*, 1993, V.48, 3066.

[20] Paquin G. and Winternitz P., *Physica D*, 1990, V.46, 122.

[21] Bogoyavlenskii O.I., *Russian Math. Surveys*, 1990, V.45, 1.

[22] Strachan I.A.B., *Inv. Prob.*, 1992, V.8, L21.

[23] Radha R. and Lakshmanan M., *Inv. Prob.*, 1994, V.10, L29.

[24] Radha R. and Lakshmanan M., *Phys. Lett. A*, 1995, V.197, 7; *J. Phys. A: Math. Gen.*, 1997, V.30, 3229.

[25] Head A., *Comput. Phys. Comm.*, 1993, V.77, 241.

[26] Novikov S.P. and Veselov A.P., *Physica D*, 1986, V.18, 267.

[27] Boiti M., Martina L., Manna M. and Pempinelli F., *Inv. Prob.*, 1986, V.2, 271.

[28] Radha R. and Lakshmanan M., *J. Math. Phys.*, 1994, V.35, 4746.

[29] Fokas A.S., *Inv. Prob.*, 1994, V.10, L19.

[30] Radha R. and Lakshmanan M., *Chaos, solitons & Fractals*, 1997, V.8, 17.

[31] Konopelchenko B.G. and Rogers C., *J. Math. Phys.*, 1993, V.34, 214.

[32] Radha R. and Lakshmanan M., *J. Phys. A*, 1996, V.29, 1551.

[33] Chakravarthy S., Kent S.L. and Newman E.I., *J. Math. Phys.*, 1995 V.36, 763.

[34] Boiti M., Leon J.J.P. and Pempinelli F., *Inv. Prob.*, 1987 V.3, 371.

[35] Radha R. and Lakshmanan M., *J. Math. Phys.*, 1997, V.38, 292.