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Nonlinear Wave Propagation Through Cold Plasma

S.G. BINDU and V.C. KURIAKOSE

Department of Physics, Cochin University of Science and Technology, Cochin-682022, India

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Abstract

Electromagnetic wave propagation through cold collision free plasma is studied using the nonlinear perturbation method. It is found that the equations can be reduced to the modified Kortweg-de Vries equation.

1 Introduction

An exciting and extremely active area of research investigation during the past years has been the study of solitons and the related issue of the construction of solutions to a wide class of nonlinear equations. The concept of solitons has now become umbiquitous in modern nonlinear science and indeed can be found in various branches of physics. In nonlinear wave propagation through continuous media, steepening of waves arises due to nonlinearities which is balanced by dissipative or dispersive effects. Exciting and important discoveries were made in the nonlinear dynamics of dissipative and conservative systems. There are different methods to study nonlinear systems. The reductive perturbation method for the propagation of a slow modulation of a quasimonochromatic wave was first established by Taniuti and Washimi for the whistler wave in a cold plasma. This method was generalised to a wide class of nonlinear wave systems by Taniuty and Yajima. Kakutani and Ono [1], Kawahara and Taniuti [2] and Taniuti and Wei [3] have investigated the propagation of hydromagnetic waves through a cold collision free plasma using this reductive perturbation method.

In the study of the asymptotic behaviour of nonlinear dispersive waves, Gardner and Morikava [4] were the first to introduce the scale transformation

$$\xi = \varepsilon^{\alpha}(x - vt), \qquad \tau = \varepsilon^{\beta} t.$$

This scale transformation is called the Gardner-Morikawa [4] transformation. They combined this transformation with a perturbation expansion of the dependent variables so as to describe the nonliear asymptotic behaviour and in the process they arrived at the

Kortweg de-Vries [KdV] equation [5] which is a single tractable equation describing the asymptotic behaviour of a wave. This method has established a systematic way for the reduction of a fairly general nonlinear systems to a single tractable nonlinear equation describing the far field behaviour. The reductive perturbation method was first established for the long wave approximation and then for the wave modulation problems.

In the present work we study the propagation of electromagnetic waves through a cold collision free plasma by using a nonlinear reductive perturbation method. It is found that to the lowest order of perturbation the system of equations can be reduced to the modified Kortweg-de Vries equation (mKdV) [6]. In the case of steady state propagation this equation can be integrated to give a solution in terms of hyperbolic functions which exhibit solitary wave nature.

2 Formulation of the problem

When electromagnetic waves pass through a medium , the system gets perturbed. Since electrons are much lighter than ions, electrons respond much more rapidly to the fields and ion motion can be neglected. In the equation of momentum for cold plasma, no pressure term is present. Basic equations relevant to the present problem are the equations of motion of electron and the Maxwell's equations. Here we are interested only in the electronic motion. To obtain a single equation which incorporates weak nonlinear and weak dispersive effects, we employ the expansions of the dependent variables similar to that introduced by Nakata [7].

The equation of motion of an electron in an electromagnetic field is

$$\frac{d\vec{v}}{dt} = -\frac{e}{m}\left[\vec{E} + \vec{v} \times \vec{B}\right]. \tag{1}$$

Taking the leading order terms we get,

$$\frac{\partial\vec{v}}{\partial t} = -\frac{e}{m}\left[\vec{E} + \vec{v} \times \vec{B}\right].$$

For convenience we take the displacement vector field $\vec{S}$, which describes the direction and distance that the plasma has moved from the equilibrium. That is,

$$\vec{v} = \frac{\partial\vec{S}}{\partial t} + (\vec{v}\cdot\nabla)\vec{S}.$$

Therefore Eq.(1) can be written as

$$\frac{\partial^2\vec{S}}{\partial t^2} = -\frac{e}{m}\left[\vec{E} + \left(\frac{\partial\vec{S}}{\partial t} \times \vec{B}\right)\right]. \tag{2}$$

From Maxwell's equation

$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t},$$

and writing

$$E = E_0 \exp i(kx - \omega t)$$

we obtain:

$$kE = \omega B. \tag{3}$$

Substituting Eq.(3) in Eq.(2) we can write:

$$\frac{\partial^2 \vec{S}}{\partial t^2} = \vec{E} + \left[\frac{\partial \vec{S}}{\partial t} \times \vec{E}\right] W, \tag{4}$$

where $W = \frac{k}{\omega}$. Physically each electron is acted upon by an electric field that is parallel to the magnetic field so that there is no perpendicular component of motion that could be affected by the Lorentz force, that is $\vec{v} \times \vec{B} = 0$ [8]. Then Eq.(1) can be written as

$$m\frac{\partial \vec{v}}{\partial t} = -e\vec{E}.$$

Equation (4) then becomes

$$\frac{\partial^2 \vec{S}}{\partial t^2} = \frac{\partial \vec{v}}{\partial t} + \left(\frac{\partial \vec{S}}{\partial t} \times \vec{E}\right) W$$

which can be put as:

$$\frac{\partial \vec{S}}{\partial t} = \vec{v} + (\vec{S} \times \vec{E})W. \tag{5}$$

From Maxwell's equations, we have

$$\vec{B} = \mu_0 \vec{H}, \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{6}$$

$$\nabla \times \vec{H} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{7}$$

Taking the time derivative of Eq.(7) we get,

$$\nabla \times \nabla \times \vec{E} = 1/c^2 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial \vec{v}}{\partial t}. \tag{8}$$

Equation (8) can then be written as

$$c^2 \nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\partial^2 S}{\partial t^2}. \tag{9}$$

Equations (5) and (9) are systems of complicated nonlinear partial differential equations for $\vec{E}$ and $\vec{S}$ describing electromagnetic wave propagation through plasma. Let us seek a solution of these equations in the form of a Fourier expansion in harmonics of the fundamental $E = e^{i(kx-\omega t)}$ as,

$$E = \sum_{n=-\infty}^{+\infty} \vec{E}^n E^n, \tag{10}$$

$$S = \sum_{n=-\infty}^{+\infty} \vec{S}^n E^n. \tag{11}$$

Let us now consider one dimensional plane wave propagating along the x direction in the Cartesian coordinate system (x, y, z). All the physical quantities are assumed to be functions of one space coordinate x and time t. We now introduce the streching variables ξ and τ as,

$$\xi = \varepsilon(x - Vt),$$

$$\tau = \varepsilon^3 t,$$

where the velocity can be determined by the solvability condition of the above equations.

E and S satisfy the following boundary conditions,

$$E_x^i \to 0 \quad \text{except} \quad E_x^0 = E_0 \cos\theta, \tag{12}$$

$$E_y^i \to 0 \quad \text{except} \quad E_y^0 = E_0 \sin\theta, \tag{13}$$

$$E_z^i \to 0, \tag{14}$$

$$\text{as} \qquad \xi \to -\infty, \qquad i = 0, 1, 2, 3, \ldots \tag{15}$$

$$S_x^i \to 0 \quad \text{except} \quad S_x^0 = S_0 \cos\theta, \tag{16}$$

$$S_y^i \to 0 \quad \text{except} \quad S_y^0 = S_0 \sin\theta, \tag{17}$$

$$S_z^i \to 0, \tag{18}$$

$$\text{as} \quad \xi \to -\infty, \qquad i = 0, 1, 2, 3, \ldots \tag{19}$$

The operators in terms of the streching variables can be written as

$$\frac{\partial}{\partial t} = (-\varepsilon v)\frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau},$$

$$\frac{\partial^2}{\partial t^2} = (v^2\varepsilon^2)\frac{\partial^2}{\partial \xi^2} - 2\varepsilon^4 \frac{\partial}{\partial \xi}\frac{\partial}{\partial \tau},$$

$$\frac{\partial^2}{\partial x^2} = (\varepsilon^2)\frac{\partial}{\partial \xi^2}.$$

For an appropriate choice of the coordinate system we can write $E = (E_x, E_y, 0)$ and $S = (S_x, S_y, 0)$. Expressing the Fourier components of E and S in powers of a small parameter ε

$$S^n = \sum_{j=0}^{\infty} \varepsilon^j S_j^n(x,t), \qquad E^n = \sum_{j=0}^{\infty} \varepsilon^j E_j^n(x,t).$$

Before proceeding to the nonlinear problem, it may be instructive to examine the dispersive relation in the linearized limit. Assuming a sinusoidal wave $\exp i(kx - \omega t)$, where k, ω are respectively the wave number and the frequency of the wave. Expanding the above coupled equations,

$$\left[\frac{\partial}{\partial t} - in\omega\right] S^n = \sum_{p+q=n} S^p \times E^q,$$

$$\left[\frac{\partial^2}{\partial t^2} + 2in\omega\frac{\partial}{\partial t} - n^2\omega^2\right][E_s^n + S_s^n] = c^2\left[\frac{\partial^2}{\partial x^2} + 2ink\frac{\partial}{\partial x} - n^2k^2\right] E_s^n(1 - \delta_s, x)$$

Eq.(9) gives the components of $S_1{}_{,s}^{n}$ as funtions of $E_1{}_{,s}^{n}$, $(s = x, y, z)$.

The determinant of this system, $\Delta(n)$ is

$$\Delta(n) = in\omega\left[-n^2\gamma^2\omega^2 + \mu^2 s_x^2 + \gamma\mu(1+\alpha)s_t^2\right],$$

where

$$\mu = (1 + \alpha\gamma)W, \quad \gamma = \left(1 - \frac{k^2}{\omega^2}\right), \quad \alpha = \frac{E_0}{S_0}. \tag{20}$$

For n=1, $\Delta(1)$ is zero if ω satisfies the dispersion relation

$$-\gamma^2\omega^2 + \mu^2 s_x^2 + \gamma\mu(1+\alpha)s_t^2 = 0.$$

From this we obtain $V = \sqrt{\frac{\alpha}{(1+\alpha)}}c$. We assume that $S_0^0 = s$ and $E_0^0 = \alpha s$ are constants and that

$$M_0^n = H_0^n = 0 \qquad \text{for} \quad n \neq 0.$$

The assumed conditions at infinity are, $E_j^n, S_j^n \to 0$ for $j = 0$ for all "n" except for $(j, \mid n \mid) = (1, 1)$, where the limit is assumed to be a finite constant. For $n = 1$, $\Delta(1) = 0$ for $j = 1$. Under this condition the system has a nontrivial solution. But for $n = 2, 3, 4, \ldots$, $\Delta n \neq 0$, we have the trivial solution. That is for $j = 1$ and $n > 1$, we get $E_1^n = S_1^n = 0$.

For $n = 0$, $\Delta(0) = 0$, we can choose $E_1^0 = S_1^0 = 0$. This completes the solution at order $(1, n)$.

For the next order, we can proceed in the same manner. The system will have a solution only if the determinant of the augmented matrix is zero .

Now expanding the dependent variables as,

$$S_x = S_0 + \varepsilon^1 S_x^1 + \varepsilon^2 S_x^2 + \cdots,$$

$$S_y = S_y^0 + \varepsilon^1 S_y^1 + \varepsilon^2 S_y^2 + \cdots,$$

$$S_z = S_z^0 + \varepsilon^1 S_z^1 + \varepsilon^2 S_z^2 + \cdots,$$

$$E_x = E_x^0 + \varepsilon^1 E_x^1 + \varepsilon^2 E_x^2 + \cdots,$$

$$E_y = E_y^0 + \varepsilon^1 E_y^1 + \varepsilon^2 E_y^2 + \cdots,$$

$$E_z = E_z^0 + \varepsilon^1 E_z^1 + \varepsilon^2 E_z^2 + \cdots . \tag{21}$$

Subtituting these expansions in Eqs.(5) and (10), then collecting and solving coefficients of different orders of ϵ^j for $n = 1$ with the boundary conditions given by Eqs.(12) to (17) we get:
at order ε^0

$$S_y^0 E_z^0 - S_z^0 E_y^0 = 0,$$

$$S_z^0 E_x^0 - S_x^0 E_z^0 = 0,$$

$$S_x^0 E_y^0 - S_y^0 E_x^0 = 0,$$

$$\frac{\partial^2 E_x^0}{\partial \xi^2} = 0,$$

$$V^2 \frac{\partial^2}{\partial \xi^2}(\gamma E_y^0 + S_y^0) = 0; \tag{22}$$

at order ε^1

$$(1+\alpha) S_z^0 S_x^1 = -V \frac{\partial S_y^0}{\partial \xi},$$

$$(1+\alpha) S_y^0 S_x^1 = -V \frac{\partial S_z^0}{\partial \xi},$$

$$(E_x^1 + S_x^1) = 0,$$

$$(E_y^1 - \alpha S_y^1) = 0, \tag{23}$$

$$(E_z^1 - \alpha S_z^1) = 0; \tag{24}$$

at order ε^2

$$V \frac{\partial S_x^1}{\partial \xi} = S_y^0 (E_z^2 - \alpha S_z^2) - S_z^0 (E_y^2 - \alpha S_y^2),$$

$$V \frac{\partial S_y^1}{\partial \xi} = S_z^0 (E_x^2 - \alpha S_x^2) - S_x^0 (E_z^2 - \alpha S_z^2), \tag{25}$$

$$V \frac{\partial S_z^1}{\partial \xi} = S_x^0 (E_y^2 - \alpha S_y^2) - S_y^0 (E_x^2 - \alpha S_x^2) + (1+\alpha) S_x^1 S_y^1,$$

$$(E_x^2 + S_x^2) = 0, \tag{26}$$

$$\frac{\partial (E_y^2 - \alpha S_y^2)}{\partial \xi} = -2V \frac{(1+\alpha)^2}{c^2} \frac{\partial S_y^0}{\partial \tau}, \tag{27}$$

$$\frac{\partial(E_z^2-\alpha S_z^2)}{\partial\xi}=-2V\frac{(1+\alpha)^2}{c^2}\frac{\partial S_z^0}{\partial\tau}.$$

Solving for E_y^2, S_y^2 and E_z^2, S_z^2, from Eqs.(23) and (24) we can get

$$(E_y^2-\alpha S_y^2)=\int -2V(1+\alpha)^2/c^2\frac{\partial S_y^0}{\partial\tau}d\xi,\quad (E_z^2-\alpha S_z^2)=\int -2V(1+\alpha)^2/c^2\frac{\partial S_z^0}{\partial\tau}d\xi.(28)$$

$$V\frac{\partial S_x^1}{\partial\xi}=S_y^0(E_z^2-\alpha S_z^2)-S_z^0(E_y^2-\alpha S_y^2).$$

Substituting for

$$(E_y^2-\alpha S_y^2)\qquad\text{and}\qquad(E_z^2-\alpha S_z^2)$$

from Eq.(27) in Eq.(25) we get

$$S_y^0\int\frac{-2V(1+\alpha)^2}{c^2}\frac{\partial}{\partial\tau}S_z^0d\xi-S_Z^0\int\frac{-2V(1+\alpha)^2}{c^2}\frac{\partial}{\partial\tau}S_y^0d\xi=V\frac{\partial S_x^1}{\partial\xi},$$

$$S_y^0\int_{-\infty}^{\xi}\frac{\partial}{\partial\tau}S_Z^0d\xi-S_z^0\int_{-\infty}^{\xi}\frac{\partial}{\partial\tau}S_y^0d\xi=-\frac{c^2}{2(1+\alpha)^2}\frac{\partial S_x^1}{\partial\xi}.$$

Now introducing two new variables A and θ defined by

$$S_y^0=A\cos\theta,\quad S_z^0=A\sin\theta,\quad A=S_0\sin\phi\quad\theta\to 0\quad\text{as}\quad\xi\to\infty.$$

Equation (21) can be written as

$$S_x^1=-\frac{V}{(1+\alpha)}\frac{\partial\theta}{\partial\xi}.$$

Now subtituting the value of S_x^1 and using the new variables Eq.(26) can be written as,

$$\cos\theta\frac{\partial}{\partial\tau}\int_{-\infty}^{\xi}\sin\theta d\xi-\sin\theta\frac{\partial}{\partial\tau}\int_{-\infty}^{\xi}\cos\theta d\xi=-\mu\frac{\partial^2\theta}{\partial\xi^2}.$$

Differentiating Eq.(28) with respect to ξ and simplifying we obtain

$$\frac{\partial}{\partial\xi}\left[\frac{\frac{\partial\theta}{\partial\tau}+\mu\frac{\partial^3\theta}{\partial\xi^3}}{\frac{\partial\theta}{\partial\xi}}\right]=-\mu\frac{\partial^2\theta}{\partial\xi^2}\frac{\partial\theta}{\partial\xi}.$$

This can be integrated with respect to ξ to give,

$$\frac{\partial\theta}{\partial\tau}+\mu\frac{\partial^3\theta}{\partial\xi^3}=-\mu\int_{-\infty}^{\xi}\frac{\partial^2\theta}{\partial\xi^2}\frac{\partial\theta}{\partial\xi}d\xi.$$

Fig. 1. Shows the variation of $f(\xi)$ with respect to ξ.

Multiplying throughout by $\frac{\partial \theta}{\partial \tau}$ we get,

$$\frac{\partial \theta}{\partial \tau} + \mu \frac{\partial^3 \theta}{\partial \xi^3} = -\mu \frac{1}{2} \left(\frac{\partial \theta}{\partial \xi} \right)^2 \frac{\partial \theta}{\partial \xi}.$$

Putting $f = \frac{\partial \theta}{\partial \xi}$, the above equation becomes

$$\frac{\partial f}{\partial \tau} + \frac{3}{2} \mu f^2 \frac{\partial f}{\partial \xi} + \mu \frac{\partial^3 f}{\partial \xi^3} = 0.$$

This equation is the modified Kortweg-de Vries (mKdV) equation. In the case of steady propagation of the wave this equation can be integrated to give a soliton solution

$$f(\zeta) = 2a \operatorname{sech}(a\,\zeta)$$

With $\zeta = \xi - \lambda\tau$, $\lambda =$ constant, $a^2 = \frac{\lambda}{\mu}$ if and only if $\lambda > 0$ $(\mu > 0)$.

Fig. 1 shows the variation of $f(\xi)$ with respect to ξ. Figs. 2 and 3 show the variation of $f(\zeta)$ with respect to ξ and τ for different values of λ. Since $f = \frac{\partial \theta}{\partial \xi}$, θ is defined as

$$\theta = \arccos(1 - 2\operatorname{sech}^2 a\zeta).$$

It is seen that θ increases from 0 to 2π or decreases from 0 to -2π according as $a > 0$ or $a < o$ as ζ goes from $-\infty$ to ∞, since θ is given by $\theta = \int_{-\infty}^{\zeta} f d\zeta$.

3 Conclusion

Starting from the basic equations describing the propagation of electromagnetic waves through the cold plasma we have showed that the system of equations can be reduced to mKdV eqation.

Fig. 2. Show the variation of $f(\zeta)$ with respect to ξ and τ for $\lambda = 21$.

Fig. 3. Show the variation of $f(\zeta)$ with respect to ξ and τ for $\lambda = 30$.

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