Journal of Nonlinear Mathematical Physics 1998, V.5, N 2, 162–180. Article

Hamiltonian Formalism in Quantum Mechanics

Boris A. KUPERSHMIDT

The University of Tennessee Space Institute, Tullahoma, TN 37388 USA
E-mail: bkupersh@utsi.edu

Received January 29, 1998

To Vladimir Igorevich Arnol'd with admiration, on occasion of his 60th birthday.

Abstract

Heisenberg motion equations in Quantum mechanics can be put into the Hamilton form. The difference between the commutator and its principal part, the Poisson bracket, can be accounted for exactly. Canonical transformations in Quantum mechanics are not, or at least not what they appear to be; their properties are formulated in a series of Conjectures.

1 Introduction

The motion equations of Classical mechanics, in the Hamilton form, are:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{1.1a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{1.1b}$$

Here $i = 1, \ldots, N$, and H, the Hamiltonian, is a function of the p_i's and q_i's, most often polynomial in the momenta p_i's. The overdot, as usual, denotes the time derivative.

The motion equations of Quantum mechanics, in the Heisenberg form, are:

$$\dot{q}_i = h^{-1}[H, q_i], \tag{1.2a}$$

$$\dot{p}_i = h^{-1}[H, p_i]. \tag{1.2b}$$

Here H again is a "function" of the p_i's and q_i's; the latter, however, *no longer commute* between themselves but are, instead, subject to the commutation relations

$$[p_k, q_\ell] = h\delta_{k\ell}, \qquad [p_k, p_\ell] = [q_k, q_\ell] = 0, \tag{1.3}$$

the complex number $\sqrt{-1}$ having been absorbed into h for future convenience. The straight bracket notation stands for the commutator:

$$[u, v] = uv - vu. \tag{1.4}$$

These two types of motion equations are known as not entirely unrelated. For example, if the p_i's and the q_i's are treated as operators, then the Classical equations (1.1) describe the motion of the mean values of these operators provided the Hamiltonian is quadratic in its arguments. (This is a Corollary of Ehrenfest's Theorem. These and other mysteries are revealed in Messiah's classic text on Quantum Mechanics [8].)

The first main result of this paper is an observation that the Quantum motion equations (1.2) can be recast into the Classical form (1.1) *provided one properly defines the notion of partial derivatives* entering into the RHS of the equations (1.1). This is done in the next Section. The main idea is to treat Quantum notions as special instances of noncommutative objects and then utilize noncommutative algebra concepts.

If the motion equations (1.1) and (1.2) are rewritten in the equivalent form as, respectively,

$$\dot{F} = \{H, F\}, \tag{1.5}$$

$$\dot{F} = h^{-1}[H, F], \tag{1.6}$$

where F is an arbitrary function of the p_i's and q_i's and $\{\cdot, \cdot\}$ denotes the Poisson bracket:

$$\{H, F\} = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \right), \tag{1.7}$$

one can ask whether these two forms are related in some precise manner. Certainly, one knows that the Poisson bracket is the "main part" of the commutator, in the sense that

$$\lim_{h \to 0} h^{-1}[H, F] = \{H, F\}, \tag{1.8}$$

as a physicist would say, or

$$\{H, F\} = h^{-1}[H, F] \pmod{h} \tag{1.9}$$

as is preferred by mathematicans. In Section 3 we shall verify that, when the number N of degrees of freedom equals 1,

$$h^{-1}[H, F] = \sum_{s \geq 1} \frac{(-h)^{s-1}}{s!} \left(\frac{\partial^s H}{\partial p^s} \frac{\partial^s F}{\partial q^s} - \frac{\partial^s F}{\partial p^s} \frac{\partial^s H}{\partial q^s} \right), \tag{1.10}$$

where the partial derivatives in the RHS are understood in the same sense, to be defined in Section 2, as those entering formulae (1.1) when considered noncommutatively. (The general case $N \geq 1$ is covered by formula (3.22).) We shall see that formula (1.10) is related to the definition of multiplication on the space of normally quantised Hamiltonians.

In Section 4 we consider the question of canonical transformations in Quantum mechanics, reformulate the Classical Jacobian conjecture into a symplectic object, quantize it, and state various generalizations of it.

2 Heisenberg as Hamilton in disguise

Let us first fix notations and conventions. Our basic number field $\mathcal{F}$ (such as $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, etc.) will be of characteristic zero; this is not essential for results, but allows shortcuts in proofs. Instead of a field $\mathcal{F}$ one can take any associative ring (or $\mathbf{Q}$-algebra) commuting with the function-ring generators, but we shan't travel this route either, to avoid interruptions by remarks. Our function rings will always be *polynomial*, again to bypass necessary pedantic comments; nothing much will change if we allow unspecified *functions* of the q_i's (rational, algebraic, etc.) as is the case in practical mechanics, because all our formulae will describe identities between differential operators, and the said identities remain true no matter what objects these differential operators are allowed to act upon.

We start with the associative ring

$$C = C_u = \mathcal{F}\langle u_1, \ldots, u_m \rangle, \tag{2.1}$$

consisting of polynomials in *noncommuting variables* $u_1, \ldots, u_m$; all with coefficients in $\mathcal{F}$. (The coefficients are always assumed to commute with the field variables u_i's.) If $x \in C$ then $\hat{L}_x$ and $\hat{R}_x$ denote the operators of left and right multiplication by x in C:

$$\hat{L}_x(y) = xy, \qquad \hat{R}_x(y) = yx, \qquad \forall\ x, y \in C. \tag{2.2}$$

The associative ring generated by the operators $\hat{L}_x$ and $\hat{R}_x$, for all x in C, is denoted

$$Op_0(C). \tag{2.3}$$

We shall utilize the following useful elements in this operator ring ([7]): For any $H \in C$,

$$\frac{\partial^{\sim} H}{\partial u_k} \in Op_0(C) \tag{2.4}$$

is the following operator:

$$\frac{\partial^{\sim} H}{\partial u_k}(x) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left(H\Big|_{u_k \to u_k + \epsilon x} \right), \quad x \in C. \tag{2.5}$$

Alternatively, we can describe the operation $\dfrac{\partial^{\sim}}{\partial u_k}$ *itself* as a *derivation* (over $\mathcal{F}$) of C into $Op_0(C)$:

$$\frac{\partial^{\sim}}{\partial u_k} :\ C \to Op_0(C), \tag{2.6a}$$

which acts on the generators of C by the rule

$$\frac{\partial^{\sim}}{\partial u_k}(u_s) = \delta_{ks}. \tag{2.6b}$$

If $X \in \mathrm{Der}(C)$ is a derivation of C (over $\mathcal{F}$) then, obviously,

$$X(H) = \sum_k \frac{\partial^{\sim} H}{\partial u_k}(X(u_k)) =: \frac{\partial^{\sim} H}{\partial \boldsymbol{u}}(\boldsymbol{X}), \qquad \forall\ H \in C. \tag{2.7}$$

The same equality can be described in a more familiar form. First, let us write suggestively, but imprecisely,

$$X = \sum_k X(u_k) \frac{\partial}{\partial u_k}, \qquad \forall\, X \in \mathrm{Der}(C), \tag{2.8}$$

to mean nothing more than $X \in \mathrm{Der}(C)$ is uniquely determined by the action of X on the u_k's. Second, let

$$\Omega^1(C) = \left\{ \sum_{ks} \varphi_{ks} du_k \psi_{ks} \mid \varphi_{ks}, \psi_{ks} \in C \right\} \tag{2.9}$$

be the C-bimodule of 1-forms over C, with the universal derivation $d: \ C \to \Omega^1(C)$ acting naturally on the generators of C by the rule

$$d(u_k) = du_k, \qquad k = 1, \ldots, m. \tag{2.10}$$

Then

$$d(H) = \sum_k \frac{\partial^\sim H}{\partial u_k}(du_k), \tag{2.11}$$

where $\dfrac{\partial^\sim H}{\partial u_k}$, as an element of $Op_0(C)$, is extended naturally to act on any C-bimodule, in this case $\Omega^1(C)$. If we now define the familiar pairing

$$\Omega^1(C) \times \mathrm{Der}(C) \to C \tag{2.12}$$

by the rule

$$\left\langle \sum \varphi_{ks} du_k \psi_{ks}, X \right\rangle = \left(\sum \varphi_{ks} du_k \psi_{ks} \right)(X) = \sum \varphi_{ks} X(u_k) \psi_{ks}, \tag{2.13}$$

then formula (2.7) can be rewritten in the familiar form

$$X(H) = \langle dH, X \rangle = dH(X). \tag{2.14}$$

So far we haven't met any p's or q's. We shall get to them at the very end of this Section, for more general formulae we work with now are more transparent and easier to handle.

For lack of better notation, we shall denote by $\dfrac{\partial H}{\partial u_k}$ the following element of the ring C, *not* of the ring $Op_0(C)$:

$$\frac{\partial H}{\partial u_k} = \frac{\partial^\sim H}{\partial u_k}(1) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left(H \bigg|_{u_k \mapsto u_k + \epsilon} \right). \tag{2.15}$$

If H is a homogeneous polynomial of degree $\ell = \deg(H)$ and $X^{rad} \in \mathrm{Der}(C)$ is the radial derivation of C:

$$X^{rad}(u_k) = u_k, \qquad k = 1, \ldots, m, \tag{2.16}$$

then we have the following noncommutative analog of the Euler Theorem on homogeneous functions:

$$X^{rad}(H) = \left(\sum u_k \frac{\partial}{\partial u_k}\right)(H) = \ell H = \deg(H)H. \tag{2.17}$$

Suppose now that we impose some *commutation relations* on the u_i's. This means that we are given a finite or infinite system of polynomials (or, in more general circumstances, power series, etc.)

$$R_r = \sum_\sigma c_{r\sigma} u^\sigma, \quad c_{r\sigma} \in \mathcal{F}, \tag{2.18a}$$

$$u^\sigma := u_{\sigma_1} \dots u_{\sigma_s} \text{ for } \sigma = (\sigma_1, \dots, \sigma_s), \quad \sigma_c = 1, \dots, m, \tag{2.18b}$$

and we form the factor-ring

$$C_u^{new} = C_u / I_{\mathcal{R}}, \tag{2.19}$$

where $I_{\mathcal{R}}$ is the two-sided ideal in C_u generated by the polynomials R_r's. If we want now to consider some "motion equations" in the ring C_u^{new}, i.e., elements of the Lie algebra $\mathrm{Der}(C_u^{new})$, we have to look at only those derivations $X \in \mathrm{Der}(C_u)$ which *preserve the ideal* $I_{\mathcal{R}}$. There exists quite a number of such special derivations, namely the elements

$$\{\mathrm{ad}_F := \hat{L}_F - \hat{R}_F \mid F \in C_u\}. \tag{2.20}$$

Indeed, any element of the ideal $I_{\mathcal{R}}$ is a finite sum of the terms

$$\{\varphi P_r \psi \mid \varphi, \psi \in C\}. \tag{2.21}$$

But then

$$\mathrm{ad}_F(\varphi P_r \psi) = F\varphi P_r \psi - \varphi P_r \psi F \tag{2.22}$$

is again an element of I_R. In the physical language, if

$$\dot{u}_i = [F, u_i], \qquad i = 1, \dots, m, \tag{2.23}$$

so that

$$u_i(t + \Delta t) = u_i(t) + \Delta t[F, u_i(t)] + O(\Delta t)^2, \tag{2.24}$$

then

$$\sum c_{r\sigma} u^\sigma(t + \Delta t) = \sum c_{r\sigma} u^\sigma(t) + \Delta t \left[F, \sum c_{r\sigma} u^\sigma(t)\right] + O(\Delta t)^2, \tag{2.25}$$

so that the commutation relations on the u_i's are preserved in time.

There may exist also some other derivations of the ring C_u which preserve a particular ideal $I_{\mathcal{R}}$. This is the case we are interested in, with the derivations in question being the "partial derivatives" $\dfrac{\partial}{\partial u_k}$ (2.15).

Lemma 2.26. *Suppose we are given the relations*

$$P_{ij} = u_i u_j - u_j u_i - c_{ij}, \qquad c_{ij} = -c_{ij} \in \mathcal{F}. \tag{2.27}$$

Then the derivations $\dfrac{\partial}{\partial u_k}$ *preserve the two-sided ideal generated by these relations.*

Proof. We have

$$\frac{\partial}{\partial u_k}(P_{ij}) = \delta_{ik} u_j + \delta_{jk} u_i - \delta_{jk} u_i - \delta_{ik} u_j = 0, \tag{2.28}$$

and hence

$$\frac{\partial}{\partial u_k}(\varphi P_{ij} \psi) = \frac{\partial \varphi}{\partial u_k} P_{ij} \psi + \varphi P_{ij} \frac{\partial \psi}{\partial u_k} \ \in I_{\mathcal{R}}. \qquad \blacksquare \tag{2.29}$$

Corollary 2.30. *In the ring* C_u^{new}*:*

$$C_u^{new} = \mathcal{F}\langle u_1, \ldots, u_m \rangle / \left([u_i, u_j] = c_{ij} \right) \tag{2.31}$$

the objects

$$\left\{ \frac{\partial H}{\partial u_k} \ \middle| \ H \in C_u^{new}, \quad k = 1, \ldots, m \right\} \tag{2.32}$$

are well-defined and satisfy formulae

$$\mathrm{ad}_{u_i}(H) = \sum_k \frac{\partial H}{\partial u_k} c_{ik}. \tag{2.33}$$

Proof. By formula (2.7),

$$\mathrm{ad}_{u_i}(H) = \sum_k \frac{\partial^{\sim} H}{\partial u_k} (\mathrm{ad}_{u_i}(u_k)) \quad \text{in } C_u. \tag{2.34}$$

By formula (2.27),

$$[u_i, u_j] = c_{ij} \quad \text{in } C_u^{new}. \tag{2.35}$$

Hence, now in C_u^{new},

$$\mathrm{ad}_{u_i}(H) = \sum_k \frac{\partial^{\sim} H}{\partial u_k}(c_{ik}) = \sum c_{ik} \frac{\partial^{\sim} H}{\partial u_k}(1) = \sum c_{ik} \frac{\partial H}{\partial u_k}. \qquad \blacksquare \tag{2.36}$$

Corollary 2.37. *Consider the case where* $\mathcal{F}$ *is replaced by*

$$\mathcal{F}_h = \mathcal{F}[\,[h]\,], \tag{2.38}$$

the ring of formal power series in h*, and the* u_i*'s are taken to be the* p_i*'s and the* q_i*'s, with the commutation relations*

$$[p_i, q_i] = h\delta_{ij}, \qquad [p_i, p_j] = [q_i, q_j] = 0. \tag{2.39}$$

Then the Heisenberg motion equations (1.2) take the Hamiltonian form (1.1).

Proof. We can transform formulae (1.2) as follows:

$$\dot{q}_i = h^{-1}[H, q_i] = -h^{-1}\mathrm{ad}_{q_i}(H) \overset{[\text{by } (2.33,39)]}{=} -h^{-1}\frac{\partial H}{\partial p_i}(-h) = \frac{\partial H}{\partial p_i},$$

$$\dot{p}_i = h^{-1}[H, p_i] = -h^{-1}\mathrm{ad}_{p_i}(H) = -h^{-1}\frac{\partial H}{\partial q_i}h = -\frac{\partial H}{\partial q_i}. \qquad \blacksquare$$

Remark 2.40. Like in the commutative algebra and analysis, partial derivatives commute between themselves both in C_u and C_u^{new}:

$$\frac{\partial^2 H}{\partial u_i \partial u_j} = \frac{\partial^2 H}{\partial u_j \partial u_i}. \tag{2.41}$$

This is clear from the definition (2.15).

Remark 2.42. The *operator-valued* partial derivatives $\dfrac{\partial^\sim H}{\partial u_k}$ satisfy the chain rule: If the u_k's are functions of the φ_α's then

$$\frac{\partial^\sim H}{\partial \varphi_\alpha} = \sum_k \frac{\partial^\sim H}{\partial u_k}\frac{\partial^\sim u_k}{\partial \varphi_\alpha}. \tag{2.43}$$

Indeed,

$$u_k(\varphi_1, \dots, \varphi_\alpha + \epsilon x, \dots) = u_k(\varphi) + \epsilon\frac{\partial^\sim u_k}{\partial \varphi_\alpha}(x) + O\left(\epsilon^2\right). \tag{2.44}$$

Therefore,

$$\begin{aligned} H(u_1(\varphi_\alpha + \epsilon x), \dots, u_m(\varphi_\alpha + \epsilon x)) &= H\left(u_1(\varphi) + \epsilon\frac{\partial^\sim u_1}{\partial \varphi_\alpha}(x) + O\left(\epsilon^2\right), \dots\right) \\ &= H(u(\varphi)) + \epsilon\sum_k \frac{\partial^\sim H}{\partial u_k}\left(\frac{\partial^\sim u_k}{\partial \varphi_\alpha}(x)\right), \end{aligned} \tag{2.45}$$

so that

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} H(\varphi_1, \dots, \varphi_\alpha + \epsilon x, \dots) = \frac{\partial^\sim H}{\partial \varphi_\alpha}(x) = \sum \frac{\partial^\sim H}{\partial u_k}\frac{\partial^\sim u_k}{\partial \varphi_\alpha}(x), \qquad \forall\ x, \tag{2.46}$$

and formula (2.43) follows.

3 Commutator vs Poisson bracket

On the way to verify formula (1.10), we shall prove first a more general statement. Suppose we impose the relations

$$[u_i, u_j] = hc_{ij}, \qquad 1 \leq i,\ j \leq m, \quad c_{ij} = -c_{ij} \in \mathcal{F}, \tag{3.1}$$

on the ring $\mathcal{F}_h\langle u_1, \dots, u_m\rangle$. We can think of these relations as the rules allowing us to reduce every polynomial in the u_i's to a specific lexicographic form by choosing an ordering among the generators u_i's. The original relations (3.1), in the form

$$u_i u_j = u_j u_i + hc_{ij}, \tag{3.2}$$

imply, and are equivalent to, the series of relations

$$\frac{u_i^n}{n!}\frac{u_j^m}{m!} = \sum_{s\geq 0}(hc_{ij})^s \frac{u_j^{m-s}}{(m-s)!}\frac{u_i^{n-s}}{(n-s)!}, \qquad n, m \in \mathbf{N}. \tag{3.3}$$

This series of relations, in turn, is equivalent to the *single* formal relation

$$E^{\boldsymbol{\lambda}\cdot\boldsymbol{u}} E^{\boldsymbol{\mu}\cdot\boldsymbol{u}} = E^{\boldsymbol{\mu}\cdot\boldsymbol{u}} E^{\boldsymbol{\lambda}\cdot\boldsymbol{u}} e^{h\langle\boldsymbol{\lambda},\boldsymbol{\mu}\rangle} \tag{3.4}$$

in $\mathcal{F}_h\langle u_1, \ldots, u_m\rangle[[\boldsymbol{\lambda}, \boldsymbol{\mu}]]$, where

$$E^{\boldsymbol{\lambda}\cdot\boldsymbol{u}} = e^{\lambda_1 u_1} \ldots e^{\lambda_m u_m}, \tag{3.5}$$

$$\langle\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle = \sum c_{ij}\lambda_i\mu_j = -\langle\boldsymbol{\mu}, \boldsymbol{\lambda}\rangle. \tag{3.6}$$

Lemma 3.7. *Define the coefficients* $\{\theta_{\sigma\sigma'}\}$ *in* $\mathcal{F}$ *by the identity:*

$$\sum_{s\geq 1}\frac{(-h)^{s-1}}{s!}\langle\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle^s = \sum_{\sigma\sigma'}\theta_{\sigma\sigma'}\lambda^\sigma\mu^{\sigma'}\frac{(-h)^{-1+(|\sigma|+|\sigma'|)/2}}{((|\sigma|+|\sigma'|)/2)!}, \tag{3.8}$$

where

$$\lambda^\sigma = \lambda_1^{\sigma_1}\ldots\lambda_m^{\sigma_m} \quad \textit{for} \quad \sigma = (\sigma_1, \ldots, \sigma_m), \quad \sigma_i \in \mathbf{Z}_+, \tag{3.9a}$$

$$|\sigma| = \sigma_1 + \cdots + \sigma_m. \tag{3.9b}$$

Then

$$h^{-1}[H, F] = \sum_{s\geq 1}\frac{(-h)^{s-1}}{s!}\sum_{|\sigma|=|\sigma'|=s}\theta_{\sigma\sigma'}\frac{\partial^{|\sigma|}H}{\partial u^\sigma}\frac{\partial^{|\sigma'|}F}{\partial u^{\sigma'}}, \tag{3.10}$$

where

$$\frac{\partial^{|\sigma|}H}{\partial u^\sigma} = \frac{\partial^{\sigma_1}}{\partial u_1^{\sigma_1}}\cdots\frac{\partial^{\sigma_m}}{\partial u_m^{\sigma_m}}(H) \qquad \textit{for} \quad \sigma = (\sigma_1, \ldots, \sigma_m). \tag{3.11}$$

Proof. It's enough to check formula (3.10) for the case

$$H = E^{\boldsymbol{\lambda}\cdot\boldsymbol{u}}, \qquad F = E^{\boldsymbol{\mu}\cdot\boldsymbol{u}}. \tag{3.12}$$

Then

$$\begin{aligned}
h^{-1}[H, F] &= h^{-1}\left(\boldsymbol{E}^{\boldsymbol{\lambda}\cdot\boldsymbol{u}}\boldsymbol{E}^{\boldsymbol{\mu}\cdot\boldsymbol{u}} - \boldsymbol{E}^{\boldsymbol{\mu}\cdot\boldsymbol{u}}\boldsymbol{E}^{\boldsymbol{\lambda}\cdot\boldsymbol{u}}\right) \\
&\overset{[\text{by }(3.4)]}{=} HF\frac{1-e^{h\langle\boldsymbol{\mu},\boldsymbol{\lambda}\rangle}}{h} = HF\frac{1-e^{-h\langle\boldsymbol{\lambda},\boldsymbol{\mu}\rangle}}{h} = \frac{e^{-h\langle\boldsymbol{\lambda},\boldsymbol{\mu}\rangle}-1}{-h}HF \\
&= \sum_{s\geq 1}\frac{(-h)^{s-1}}{s!}\langle\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle^s HF \overset{[\text{by }(3.8)]}{=} \sum\theta_{\sigma\sigma'}\lambda^\sigma\mu^{\sigma'}\frac{(-h)^{-1+\cdots}}{(\cdots)!} \\
&= \sum_{s\geq 1}\frac{(-h)^{s-1}}{s!}\sum_{|\sigma|=|\sigma'|=s}\theta_{\sigma\sigma'}\frac{\partial^{|\sigma|}H}{\partial u^\sigma}\frac{\partial^{|\sigma'|}F}{\partial u^{\sigma'}}. \qquad \blacksquare
\end{aligned}$$

The terms with $s = 1$ in the RHS of formula (3.10) comprise the Poisson bracket part, for formula (3.8) implies that

$$\theta_{ij} = c_{ij}. \tag{3.13}$$

If we now specialize to the Quantum case when $[p_i, q_j] = h\delta_{ij}$, we *will not* get formula (1.10), for in the RHS of formula (3.10) H stands always to the *left* of F and the (H, F) – skewsymmetry is thus hidden. But we can emulate the *proof* of Lemma 3.7. First, we convert the relations

$$[p_i, q_j] = h\delta_{ij}, \qquad [p_i, p_j] = [q_i, q_j] = 0, \tag{3.14}$$

into the singl formal relation

$$e^{\boldsymbol{\lambda}\cdot\boldsymbol{p}} e^{\boldsymbol{\alpha}\cdot\boldsymbol{q}} = e^{h\boldsymbol{\lambda}\cdot\boldsymbol{\alpha}} e^{\boldsymbol{\alpha}\cdot\boldsymbol{q}} e^{\boldsymbol{\lambda}\cdot\boldsymbol{p}}. \tag{3.15}$$

Next, we take

$$H = e^{\boldsymbol{\alpha}\cdot\boldsymbol{q}} e^{\boldsymbol{\lambda}\cdot\boldsymbol{p}}, \qquad F = e^{\boldsymbol{\beta}\cdot\boldsymbol{q}} e^{\boldsymbol{\mu}\cdot\boldsymbol{p}}. \tag{3.16}$$

Now, consider the operators

$$\mathcal{O}_{HF} = \frac{\partial^H}{\partial \boldsymbol{p}} \cdot \frac{\partial^F}{\partial \boldsymbol{q}} : \ HF \mapsto \sum_i \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i}, \tag{3.17a}$$

$$\mathcal{O}_{FH} = \frac{\partial^F}{\partial \boldsymbol{p}} \cdot \frac{\partial^H}{\partial \boldsymbol{q}} : \ FH \mapsto \sum_i \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i}. \tag{3.17b}$$

Let us verify that

$$e^{-h\mathcal{O}_{HF}}(HF) = (\mathrm{smbl}(H)\mathrm{smbl}(F))_{\mathrm{normal}}, \tag{3.18}$$

where, for $H \in C_u^{new}$, $\mathrm{smbl}(H) \in \mathcal{F}[u_1, \ldots, u_m]$ is the symbol of H which results by letting h vanish (in $C_u^{new}/(hC_u^{new})$), and the subscript "normal" denotes the normal quantization, with the q_i's standing to the left of the p_i's. Indeed, for H and F given by formula (3.16),

$$e^{-h\mathcal{O}_{HF}}(HF) = \sum_{s \geq 0} \frac{(-h\boldsymbol{\lambda}\cdot\boldsymbol{\beta})^s}{s!} HF = e^{(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{q}} e^{(\boldsymbol{\lambda}+\boldsymbol{\mu})\cdot\boldsymbol{p}}. \tag{3.19}$$

Since

$$\mathrm{smbl}(H)\mathrm{smbl}(F) = \mathrm{smbl}(F)\mathrm{smbl}(H), \tag{3.20}$$

formula (3.18) implies:

$$e^{-h\mathcal{O}_{HF}}(HF) = e^{-h\mathcal{O}_{FH}}(FH), \tag{3.21}$$

so that

$$\begin{aligned} h^{-1}[H, F] = h^{-1}(HF - FH) &= \frac{e^{-h\mathcal{O}_{HF}} - 1}{-h}(HF) - \frac{e^{-h\mathcal{O}_{FH}} - 1}{-h}(FH) \\ &= \sum_{s \geq 1} \frac{(-h)^{s-1}}{s!} \left((\mathcal{O}_{HF})^s (HF) - (\mathcal{O}_{FH})^s (FH) \right). \end{aligned} \tag{3.22}$$

For the case when the number of degrees of freedom $N = 1$,

$$\mathcal{O}_{HF}(HF) = \frac{\partial H}{\partial p}\frac{\partial F}{\partial q}, \tag{3.23}$$

and formula (3.22) yields formula (1.10).

Remark 3.24. Formula (3.21) implies that we have a symmetric bilinear form in the *noncommutative* ring $C_{p,q}^{new}$:

$$(H, F) = e^{-h\mathcal{O}_{HF}}(HF). \tag{3.25}$$

There exists another attractive bilinear form on this ring, this time with values in the *commutative* ring $\mathcal{F}[q_i][[h]]$:

$$(H, F) = \mathrm{Res}(HF^\dagger), \tag{3.26}$$

where

$$\mathrm{Res}\left(\sum c_{\sigma\sigma'}q^\sigma p^{\sigma'}\right) := \sum c_{\sigma 0}q^\sigma, \tag{3.27}$$

and $\dagger$ is an antiinvolution (over $\mathcal{F}_h$):

$$(HF)^\dagger = F^\dagger H^\dagger, \tag{3.28}$$

defined on the generators q_i's and p_i's by the rule

$$q_i^\dagger = q_i, \qquad p_i^\dagger = -p_i, \qquad i = 1, \ldots, N. \tag{3.29}$$

The bilinear form (H, F) (3.26) is not symmetric in the *linear algebra* sense, but it is symmetric in the *differential algebra* sense:

$$(H, F) \sim (F, H), \tag{3.30}$$

where, for elements $a, b \in \mathcal{F}[q_i][\,[h]\,]$, we write

$$a \sim b \quad \text{to mean} \quad (a - b) \in \sum_i Im\frac{\partial}{\partial q_i}. \tag{3.31}$$

To prove formula (3.30), we take H and F given by formula (3.16). Then

$$\begin{aligned}(H, F) &= \mathrm{Res}(HF^\dagger) = \mathrm{Res}\left(e^{\boldsymbol{\alpha}\cdot\boldsymbol{q}}e^{\boldsymbol{\lambda}\cdot\boldsymbol{p}}e^{-\boldsymbol{\mu}\cdot\boldsymbol{p}}e^{\boldsymbol{\beta}\cdot\boldsymbol{q}}\right)\\ &= \mathrm{Res}\left(e^{(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{q}}e^{(\boldsymbol{\lambda}-\boldsymbol{\mu})\cdot\boldsymbol{p}}e^{(\boldsymbol{\lambda}-\boldsymbol{\mu})\cdot\boldsymbol{\beta}h}\right) = e^{(\boldsymbol{\lambda}-\boldsymbol{\mu})\cdot\boldsymbol{\beta}h}e^{(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{q}}.\end{aligned} \tag{3.32}$$

Hence,

$$\begin{aligned}(F, H) &= e^{(\boldsymbol{\mu}-\boldsymbol{\lambda})\cdot\boldsymbol{\alpha}h}e^{(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{q}} = e^{(\boldsymbol{\mu}-\boldsymbol{\lambda})\cdot(\boldsymbol{\alpha}+\boldsymbol{\beta})h}(H, F)\\ &= e^{h(\boldsymbol{\mu}-\boldsymbol{\lambda})\cdot\partial/\partial\boldsymbol{q}}(H, F) \sim (H, F).\end{aligned} \tag{3.33}$$

4 Canonical transformations, special and general

If M is a smooth manifold and T^*M is the contangent bundle (= the phase space) of M, then any transformation

$$\varphi: \ M \to M \tag{4.1}$$

is uniquely lifted to a transformation

$$\bar{\varphi}: T^*M \to T^*M \tag{4.2}$$

covering φ, by the requirement that the canonical 1-form

$$\rho = \boldsymbol{p} d\boldsymbol{q} \tag{4.3}$$

on T^*M be preserved:

$$\bar{\varphi}^*(\rho) = \rho. \tag{4.4}$$

Re-expressing this picture analytically/algebraically, we start with an automorphism Φ of the ring C_q

$$\Phi: C_q \to C_q, \qquad C_q = \mathcal{F}[q_1, \dots q_N], \ C^\infty(q_1, \dots, q_N), \ \dots \tag{4.5}$$

$$\Phi(q_i) = Q_i = Q_i(q_1, \dots, q_N), \qquad i = 1, \dots, N, \tag{4.6}$$

and then determine the elements

$$\bar{\Phi}(p_i) = P_i = P_i(q, p) \tag{4.7}$$

from the requirement that

$$\boldsymbol{pdq} = \boldsymbol{PdQ}: \tag{4.8}$$

$$\sum_j p_j dq_j = \sum_i P_i dQ_i = \sum P_i Q_{i,j} dq_j. \tag{4.9}$$

Thus,

$$p_j = \sum_i P_i Q_{i,j}, \qquad j = 1, \dots, N. \tag{4.10}$$

Denote by

$$J = J_{Q|q} = \left(\frac{\partial Q_i}{\partial q_j} \right) \tag{4.11}$$

the Jacobian of the map Φ. The transformation formulae (4.10) can be rewritten in one of the equivalent forms:

$$\boldsymbol{p}^t = \boldsymbol{P}^t J, \tag{4.12a}$$

$$\boldsymbol{P}^t = \boldsymbol{p}^t J^{-1}, \tag{4. 12b}$$

$$\boldsymbol{P} = (J^{-1})^t \boldsymbol{p}, \tag{4.12c}$$

where $\boldsymbol{p}$, $\boldsymbol{P}$, $\boldsymbol{q}$, $\boldsymbol{Q}$ are thought of as column-vectors. Since the canonical 1-form $\boldsymbol{p}d\boldsymbol{q}$ is preserved, the symplectic 2-form $d\boldsymbol{p} \wedge d\boldsymbol{q}$ is preserved as well. Therefore, the basic Poisson brackets are also preserved:

$$\{P_i, P_j\} = \{Q_i, Q_j\} = 0, \qquad \{P_i, Q_j\} = \delta_{ij}. \tag{4.13}$$

Remark 4.14. If one concentrates on the preservation of the Poisson brackets *only*, that is, of the 2-form $d\boldsymbol{q} \wedge d\boldsymbol{q}$, rather than the canonical 1-form $\boldsymbol{p}d\boldsymbol{q}$, the uniqueness of the lifting of φ into $\overline{\varphi}$ no longer holds. For example, we can replace formula (4.4) by the relation

$$\overline{\varphi}^{\,*}(\rho) = \rho + \omega, \tag{4.15}$$

where ω is a closed 1-form on M lifted into T^*M. Taking

$$\omega = d(f), \qquad f \in C_q, \tag{4.16}$$

we find, instead of formula (4.10), the relations

$$p_j = \sum P_i Q_{i,j} + f_{,j}\,, \tag{4.17}$$

$$\boldsymbol{P} = (J^{-1})^t \left(\boldsymbol{p} - \boldsymbol{\nabla}(f)\right). \tag{4.18}$$

We shall see below that such nonuniqueness is *unavoidable* in Quantum mechanics.

Lemma 4.19. *Formulae (4.6,12c) preserve the Quantum commutation relations*

$$[q_i, q_j] = [p_i, p_j] = 0, \qquad [p_i, q_j] = h\delta_{ij}, \quad 1 \le i, j \le N. \tag{4.20}$$

Proof. Obviously,

$$[Q_i, Q_j] = 0. \tag{4.21}$$

Next,

$$[P_i, Q_j] = \left[\sum_\alpha \left(J^{-1}\right)^t_{i\alpha} p_\alpha,\ Q_j\right] = \sum \left(J^{-1}\right)_{\alpha i} hQ_{j,\alpha} = h \sum_\alpha \left(J^{-1}\right)_{\alpha i} J_{j\alpha} = h\delta_{ij}. \tag{4.22}$$

Finally,

$$P_i P_j = \sum \left(J^{-1}\right)_{\alpha i} p_\alpha \left(J^{-1}\right)_{\beta j} p_\beta = \sum \left(J^{-1}\right)_{\alpha i} \left\{\left(J^{-1}\right)_{\beta j} p_\alpha + h\left(J^{-1}\right)_{\beta j,\alpha}\right\} p_\beta, \tag{4.23}$$

whence

$$h^{-1}[P_i, P_j] = \sum_\beta \left\langle \sum_\alpha \left(J^{-1}\right)_{\alpha i} \left(J^{-1}\right)_{\beta j,\alpha} - \sum_\nu \left(J^{-1}\right)_{\nu j} \left(J^{-1}\right)_{\beta i,\nu} \right\rangle p_\beta. \tag{4.24}$$

Now,

$$\left(J^{-1}\right)_{\beta j,\alpha} = -\left(J^{-1} J_{,\alpha} J^{-1}\right)_{\beta j} = -\sum_{\mu\nu} \left(J^{-1}\right)_{\beta\mu} J_{\mu\nu,\alpha} \left(J^{-1}\right)_{\nu j}\,, \tag{4.25a}$$

and thus

$$\left(J^{-1}\right)_{\beta i,\nu} = -\sum_{\mu\alpha} \left(J^{-1}\right)_{\beta\mu} J_{\mu\alpha,\nu} \left(J^{-1}\right)_{\alpha i}. \tag{4.25b}$$

Substituting formulae (4.25) into formula (4.24) and noticing that

$$J_{\mu\nu,\alpha} = \frac{\partial^2 Q_\mu}{\partial q_\nu \partial q_\alpha} = J_{\mu\alpha,\nu}, \tag{4.26}$$

we find that

$$[P_i, P_j] = 0. \quad \blacksquare \tag{4.27}$$

The nonuniqueness of quantum formulae (4.12c) can be demonstrated in two ways.

Lemma 4.28. *The transformation*

$$Q_i = q_i, \qquad P_i = p_i + g_{,i}\,, \qquad i = 1, \dots, m, \quad g \in C_q, \tag{4.29}$$

preserves the quantum commutation relations (4.20).

Proof. We have

$$[P_i, P_j] = [p_i + g_{,i}\,,\ p_j + g_{,j}\,] = hg_{,ij} - hg_{,ji} = 0, \tag{4.30}$$

and the rest of the relations are obviously satisfied. $\blacksquare$

Lemma 4.31. *The transformation*

$$Q_i = Q_i(\boldsymbol{q}), \tag{4.32a}$$

$$P_i = \sum_\alpha p_\alpha \left(J^{-1}\right)_{\alpha_i}, \qquad i = 1, \dots, N, \tag{4.32b}$$

is also a quantum canonical transformation.

Proof. (A) The new formulae (4.32) are just the mirror image of the old ones, (4.6,12c), and † is an (anti)isomorphism. (B) Alternatively, we can straightforwardly calculate like in the proof of Lemma 4.19, and keep all the p_α's to the *left* of the q_β's. $\blacksquare$

Thus, given a transformation $\Phi : \ C_q \to C_q$, we have two *different* lifts of it into quantum canonical maps, Φ_r (4.6,12c), and Φ_ℓ (4.32):

$$\Phi_r(q_i) = Q_i(\boldsymbol{q}), \qquad \Phi_r(p_i) = \sum_\alpha \left(J^{-1}\right)_{\alpha i} p_\alpha, \tag{4.33}$$

$$\Phi_\ell(q_i) = Q_i(\boldsymbol{q}), \qquad \Phi_\ell(p_i) = \sum_\alpha p_\alpha \left(J^{-1}\right)_{\alpha i}. \tag{4.34}$$

How are these two maps related? Let us consider the composition $\Psi = \Phi_r \Phi_\ell^{-1}$:

$$\Psi(q_i) = q_i, \qquad \Psi(p_i) = \sum_{\alpha\beta} \left(J^{-1}\right)_{\alpha\beta} p_\alpha J_{\beta i}, \qquad 1 \le i \le N. \tag{4.35}$$

Lemma 4.36.

$$\Psi(\boldsymbol{p}) = \boldsymbol{p} + h\boldsymbol{\nabla}(g), \tag{4.37a}$$

$$g = \ln\det(J). \tag{4.37b}$$

Proof. From formulae (4.35) we find:

$$\Psi(p_i) = \sum \left(J^{-1}\right)_{\alpha\beta} \{J_{\beta i} p_\alpha + hJ_{\beta i,\alpha}\} = p_i + h \sum \left(J^{-1}\right)_{\alpha\beta} J_{\beta i,\alpha}. \tag{4.38}$$

But

$$\begin{aligned} \sum \left(J^{-1}\right)_{\alpha\beta} J_{\beta i,\alpha} &= \sum \left(J^{-1}\right)_{\alpha\beta} J_{\beta\alpha,i} = \operatorname{Tr}\left(J^{-1}J_{,i}\right) \\ &\overset{\text{[by formula (4.42) below]}}{=} (\ln\det(J))_{,i}\,. \quad \blacksquare \end{aligned} \tag{4.39}$$

Remark 4.40. Recall that if $A \in \mathrm{Mat}_n(C_q)$ then

$$d(\ln\det(A)) = \operatorname{Tr}(A^{-1}dA), \tag{4.41}$$

in the sense that

$$(\ln\det(A))_{,i} = \operatorname{Tr}(A^{-1}A_{,i}) = \sum (A^{-1})_{\alpha\beta} A_{\beta\alpha,i}. \tag{4.42}$$

Indeed, Let $B = \ln(A)$, so that $A = e^B$. Then

$$\begin{aligned} \operatorname{Tr}(A^{-1}dA) &= \operatorname{Tr}\left(e^{-B}d(e^B)\right) = \operatorname{Tr}\left(e^{-B}\sum \frac{B^s d(B) B^r}{(r+s+1)!}\right) \\ &= \operatorname{Tr}\left(e^{-B}\sum \frac{B^r B^s d(B)}{(r+s+1)!}\right) = \operatorname{Tr}\left(e^{-B}\sum \frac{B^\ell d(B)}{\ell!}\right) = \operatorname{Tr}(dB)) \\ &= d\operatorname{Tr}(B) = d(\ln\det(e^B)) = d(\ln\det(A)). \end{aligned}$$

Remark 4.43. Formula (4.41) is rational in A. An equivalent formulation, regular in A, is

$$d(\det(A)) = \operatorname{Tr}(\operatorname{adj}(A)dA), \tag{4.44}$$

where $\operatorname{adj}(A)$ is the adjugate matrix of A:

$$\operatorname{adj}(A)A = A\operatorname{adj}(A) = \det(A)\mathbf{1}. \tag{4.45}$$

Remark 4.46. Which one of the maps Φ_r or Φ_ℓ is *right* in practice? Unfortunately, this is the sort of question akin to the problem of "right" quantization, that is to say, a wrong and misleading one. The "right" answer depends on the problem at hand, i.e., the Hamiltonian, and it may be nonunique nonetheless. I shall leave an elaboration of this point to the future. Let us consider instead an instructive case of the *mechanical* Hamiltonians, those of the form

$$H = \sum a^{ij}(Q)P_iP_j + V(Q). \tag{4.47}$$

It is well-known in Quantum mechanics that if P_k's are treated as $h\dfrac{\partial}{\partial Q_k}$'s (recall that $\sqrt{-1}$ has been absorbed into h) then the selfadjoint form of H is

$$H = \sum P_i a^{ij}(Q) P_j + V(Q). \tag{4.48}$$

In other words,

$$H^\dagger = H. \tag{4.49}$$

How does one transform such an H under a change of variables $q_i \mapsto \Phi(q_i) = Q_i(\boldsymbol{q})$ (4.6), and still preserve the selfadjointness of H? Let us look at the simple example of a free particle in polar coordinates:

$$x = r\cos\theta,\ y = r\sin\theta, \tag{4.50a}$$

$$J = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \Rightarrow J^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.50b}$$

Thus, for the left form (4.34) we get

$$\begin{aligned}(p_x, p_y) &= (p_r, p_\varphi) \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= (p_r\cos\theta - p_\theta r^{-1}\sin\theta,\ p_r\sin\theta + p_\theta r^{-1}\cos\theta).\end{aligned} \tag{4.51}$$

Hence, for the right form (4.33) we obtain

$$(p_x, p_y) = \left(\cos\theta p_r - r^{-1}\sin\theta p_\theta, \sin\theta p_r + r^{-1}\cos\theta p_\theta\right). \tag{4.52}$$

Now, the Hamiltonian $p_x^2 + p_y^2$ becomes:

$$p_x^2 + p_y^2 = p_r^2 + r^{-2}p_\theta^2 - hp_r r^{-1} \qquad \text{(left form)}, \tag{4.53ℓ}$$

$$p_x^2 + p_y^2 = p_r^2 + r^{-2}p_\theta^2 + hr^{-1}p_r \qquad \text{(right form)}, \tag{4.53r}$$

and neither of these is physically palatable by virtue of not being selfadjoint. This observation seems to suggest that a substantial fraction of literature on Quantum mechanics is beside the point. What the point or points is or are I'll again leave for the future can-of-worms operations. Let us return to the mechanical Hamiltonian H (4.48): how *should* it transform in order to preserve its selfadjointness? We have seen above that neither the left nor the right transformation is satisfactory.

Lemma 4.54. *Denote the left and right transformations as*

$$P_i^\ell = \sum p_\alpha \left(J^{-1}\right)_{\alpha i}, \qquad P_i^r = \sum \left(J^{-1}\right)_{\alpha i} p_\alpha. \tag{4.55}$$

Set

$$H^{\ell r} = \sum P_i^\ell a^{ij}(Q) P_j^r, \qquad H^{r\ell} = \sum P_i^r a^{ij}(Q) P_j^\ell. \tag{4.56}$$

Then

$$(H^{\ell r})^\dagger = H^{\ell r}, \qquad (H^{r\ell})^\dagger = H^{r\ell}. \tag{4.57}$$

Proof.

$$(P_i^\ell)^\dagger = -P_i^r, \qquad (P_i^r)^\dagger = -P_i^\ell. \qquad \blacksquare \tag{4.58}$$

In coordinates,

$$H^{\ell r} = \sum p_\alpha \left(J^{-1}\right)_{\alpha i} a^{ij}(Q) \left(J^{-1}\right)_{\beta j} p_\beta, \tag{4.59}$$

$$H^{r\ell} = \sum \left(J^{-1}\right)_{\alpha i} p_\alpha a^{ij}(Q) p_\beta \left(J^{-1}\right)_{\beta j}. \tag{4.60}$$

One can try other remedies, e.g.

$$P_i = (P_i^\ell + P_j^r)/2, \qquad P_i = \sqrt{P_i^\ell P_i^r}, \qquad P_i = \sqrt{P_i^r P_i^\ell}, \tag{4.61}$$

etc., but they appear too artificial. It does seem unavoidable to work with *two different* type of momenta in Quantum mechanics, left and right, and transform each one accordingly. Formula (4.59) appears to offer slight advantages in this regard. In particular, for the free particle in polar coordinates, we find

$$H^{\ell r} = p_x^\ell p_x^r + p_y^\ell p_y^r = p_r^2 + r^{-2} p_\theta^2, \tag{4.62a}$$

$$H^{r\ell} = p_x^r p_x^\ell + p_y^r p_y^\ell = p_r^2 + r^{-2} p_\theta^2, \tag{4.62b}$$

and each one of these formulae is satisfactory. In general,

$$\begin{aligned}
h^{-2}(H^{\ell r} - H^{r\ell}) &= h^{-2} \sum \left(\left(J^{-1}\right)_{\alpha i} p_\alpha + h \left(J^{-1}\right)_{\alpha i,\alpha} \right) a^{ij} \left(J^{-1}\right)_{\beta j} p_\beta \\
&\quad - h^{-2} \sum \left(J^{-1}\right)_{\alpha i} p_\alpha a^{ij} \left(\left(J^{-1}\right)_{\beta j} p_\beta + h \left(J^{-1}\right)_{\beta j,\beta} \right) \\
&= h^{-1} \sum \left(J^{-1}\right)_{\mu j,\mu} a^{ij} \left(J^{-1}\right)_{\beta i} p_\beta - h^{-1} \sum \left(J^{-1}\right)_{\beta i} p_\beta a^{ij} \left(J^{-1}\right)_{\mu j,\mu} \\
&= h^{-1} \sum \left[\left(J^{-1}\right)_{\mu j,\mu} a^{ij}, \left(J^{-1}\right)_{\beta i} p_\beta \right] \\
&= -h^{-1} \sum \left(J^{-1}\right)_{\beta i} h \left(\left(J^{-1}\right)_{\mu j,\mu} a^{ij} \right)_{,\beta} \\
&\overset{\text{[by 4.64)]}}{=} \sum \left(J^{-1}\right)_{\beta i} \left((\ln\det(J))_{,\psi} \left(J^{-1}\right)_{\psi j} a^{ij} \right)_{,\beta},
\end{aligned} \tag{4.63}$$

where we used the formula

$$\begin{aligned}
-\sum_\mu \left(J^{-1}\right)_{\mu j,\mu} &= \sum \left(J^{-1} J_{,\mu} J^{-1}\right)_{\mu j} = \sum \left(J^{-1}\right)_{\mu\varphi} J_{\varphi\psi,\mu} \left(J^{-1}\right)_{\psi j} \\
&\overset{\text{[by (4.42)]}}{=} \sum_\psi (\ln\det(J))_{,\psi} \left(J^{-1}\right)_{\psi j}.
\end{aligned} \tag{4.64}$$

Let us return now to the formula (4.37). It can be looked at from a slightly different perspective, if we notice that $\Psi(\boldsymbol{p}) = \boldsymbol{p}$ whenever

$$\det(J) = \text{const} \neq 0. \tag{4.65}$$

Namely, from formulae (4.33,34) we find that

$$\begin{aligned}\Phi_r(p_i)-\Phi_\ell(p_i) &= \sum \left(J^{-1}\right)_{\alpha i} p_\alpha - p_\alpha \left(J^{-1}\right)_{\alpha i}) = \sum \left[\left(J^{-1}\right)_{\alpha i}, p_\alpha\right] \\ &= -h\sum \left(J^{-1}\right)_{\alpha i,\alpha} \overset{[\text{by (4.64)}]}{=} = h\sum \left(J^{-1}\right)_{\alpha i} (\ln\det(J))_{,\alpha}\,.\end{aligned} \tag{4.66}$$

Thus, the condition of constant $\det(J)$ (4.65) is necessary and sufficient to have the left and right formulae coincide, and thus provide a *unique* lift from an automorphism Φ of C_q into a Quantum automorphism $\overline{\Phi}$ of $C_{p,q}$. That a polynomial map Φ with a constant non-zero $\det(J)$ does indeed define an automorphism of C_q, has been conjectured originally by Keller in [5]; this conjecture is known as the Jacobian Conjecture, and it is related to many other open problems in algebra; see , e.g., reviews in [2]. Let us discuss this Conjecture, thereafter called Conjecture K, from the physical point of view. First, since an automorphism of $\mathcal{F}[q]$ extends, via formulae (4.12), to a Poisson automorphism of $\mathcal{F}[q,p]$, Conjecture K is implied by the more general symplectic

Conjecture *S*. A polynomial Poisson endomorphism of $\mathcal{F}[p,q]$ is an automorphism. (In other words, if $P_i, Q_i \in \mathcal{F}[p,q]$ are such that

$$\{P_i,P_j\}=\{Q_i,Q_j\}=0, \qquad \{P_i,Q_j\}=\delta_{ij}, \qquad 1\le i,j\le N, \tag{4.67}$$

then the p_i's and the q_i's can be re-expressed as polynomials in the P's and the Q's. In other words still, $\mathcal{F}[P,Q]=\mathcal{F}[p,q]$.)

Vice versa, the symplectic Conjecture S is implied by the Conjecture K. Indeed, if the form $d\boldsymbol{p}\wedge d\boldsymbol{q}$ is preserved then so is the volume form $(d\boldsymbol{p}\wedge \boldsymbol{q})^{\wedge N}$; thus, the $\det(J)$ in this case equals to 1.

The symplectic Conjecture S is a quasiclassical limit of the Quantum

Conjecture *Q*. Let $W_N=W_N(k;h)$ be the h-scaled Weyl algebra over a commutative ring k, (see , e.g., [1]) with the generators $q_1,\dots,q_N$, $p_1,\dots,p_N$ and the relations

$$[q_i,q_j]=[p_i,p_j]=0, \qquad [p_i,q_j]=h\delta_{ij}, \qquad 1\le i,j\le N. \tag{4.68}$$

If $Q_1,\dots,Q_N$, $P_1,\dots,P_N\in W_N$ are such that

$$[Q_i,Q_j]=[P_i,P_j]=0, \qquad [P_i,Q_j]=h\delta_{ij}, \qquad 1\le i,j\le N, \tag{4.69}$$

then the q_i's and p_i's can be re-expressed as polynomials in the P's and Q's (with coefficients in $k[h]$ or $k_h=k[[h]]$ depending upon the version of W_N).

In-between Conjectures S and Q is located

Conjecture *C–Q*. (i) Every Poisson endomorphism (resp. automorphism) of $\mathcal{F}[p,q]$ can be quantized; (ii) Such quantization is unique over $k[h]$.

Quantization is certainly nonunique over $k[[h]]$. For example,

$$P_i=\psi(h)p_i, \qquad Q_i=q_i/\psi(h), \qquad i=1,\dots,N, \tag{4.70}$$

is a quantum automorphism for any $\psi(h)\in 1+hk[[h]]$, and it reduces to an identical Poisson map for $h=0$ no matter what $\psi(h)$ is; such nonuniqueness, therefore, attaches to *every* Quantum endomorphism. On the other hand, results of Wollenberg [9, 3] show that the (i) part of the Conjecture C–Q fails for *infinitesimal* endomorphisms, i.e., derivations. (See also the last Remark at the end of this Section.)

Conjectures K and S have noncommutative analogs.

Conjecture $\mathcal{K}$. Let R be an associative ring and $R\langle x\rangle = R\langle x_1, \dots, x_m\rangle$ a ring of polynomials in noncommuting variables $x_1, \dots, x_m$ with coefficients in R which *do not* necessarily commute with the x_i's. Let $F_1, \dots, F_m \in R\langle x\rangle$ be such that the Jacobian matrix $J(F)$:

$$J(F)_{ij} = \frac{\partial^{\sim} F_i}{\partial x_j} \ \in \ Op_0(R\langle x\rangle) \tag{4.71}$$

is invertible, so that there exists a matrix $\mathcal{M} \in \mathrm{Mat}_m(Op_0(R\langle x\rangle))$ such that

$$\mathcal{M}J(F) = \mathbf{1}. \tag{4.72}$$

Then there exist polymonials $G_1, \dots, G_m \in R\langle x\rangle$ such that

$$G_i(F_1, \dots, F_m) = x_i, \qquad 1 \leq i \leq m. \tag{4.73}$$

Conjecture $\mathcal{S}$. In $R\langle p, q\rangle = R\langle p_1, \dots, p_N, q_1, \dots, q_N\rangle$, let the noncommutative Hamiltonian structure [7] be given by the Hamiltonian matrix

$$B = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{4.74}$$

If $P_1, \dots, P_N$, $Q_1, \dots, Q_N \in R\langle p, q\rangle$ preserve the Hamiltonian structure B:

$$JBJ^{\dagger} = B, \tag{4.75}$$

then $p_1, \dots, p_n$, $q_1, \dots, q_N \in R\langle P, Q\rangle$. (The adjoint $J^{\dagger}$ of J in formula (4.75) is taken in the noncommutative sense defined in [7].)

We conclude this Section by mentioning two other versions of Conjecture $\mathcal{K}$:

Conjecture $\mathcal{K}^{var}$. If R is commutative and $F_1, \dots, F_m \in R\langle x\rangle$ are such that the matrix $J^{var}(F) \in \mathrm{Mat}_m(R\langle x\rangle)$ [7] is invertible, where

$$J^{var}(F)_{ij} = \frac{\delta F_i}{\delta x_j} \ \in \ R\langle x\rangle, \tag{4.76}$$

$$\frac{\delta F_i}{\delta x_j}\chi \equiv \frac{\partial^{\sim} F_i}{\partial x_j}(\chi) \ (\mathrm{mod}\ [R\langle x\rangle, R\langle x\rangle]), \qquad \forall\, \chi \in R\langle x\rangle, \tag{4.77}$$

then the map $F : R^m \to R^m$ is a polynomial automorphism.

Conjecture C–$\mathcal{K}$. (A noncommutative analog of the Quantization Conjecture C–Q). Let $f : k^m \to k^m$ be a polynomial map with an invertible Jacobian (resp. automorphism). Then one can find a set of polynomials $F_1, \dots, F_m \in k\langle x\rangle$ with an invertible Jacobian (in either of the two meanings, (4.71) or (4.76)) (resp. automorphism) such that $f_i(x) = F_i(x)$, $i = 1, \dots, m$, when all the x_i's are allowed to commute between themselves.

The Conjecture C–$\mathcal{K}$ is obviously true for tame automorphisms (generated by $GL_m(k)$ and triangular maps), and thus is true for $m = 2$ ([4, 6]). The same conclusion applies to the (i) part of Conjecture C–Q for the case $N = 1$.

References

[1] Björk J.-E., Rings of Differential Operators, North-Holland Publ. Co., Amsterdam, 1979.

[2] Van Den Essen A. (ed.), Automorphisms of Affine Spaces, Kluwer Academic Publ., Dordrecht, 1995.

*[3] Joseph A., Derivations of Lie Brackets and Canonical Quantisation, *Comm. Math. Phys.*, 1970, V.17, 210–232.

[4] Jung H.W.E., Über Ganze Birationale Transformationen der Ebene, *J. Reine Angew. Math.*, 1942, V.184, 161–174.

[5] Keller O.H., Ganze Gremona-Transformationen, *Monats. Math. Phys.*, 1939, V.47, 299–306.

[6] van der Kulk W., On Polynomial Rings in Two Variables, *Nieuw Archief voor Wiskunde*, 1953, V.3, 33-41.

[7] Kupershmidt B.A., KP or mKP: Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems, *Amer. Math. Soc., Providence* (to appear).

[8] Messiah A., Mecanique Quantique, Dunod, Paris, 1958; English translation: John Wiley & Sons, New York, 1963.

[9] Wollenberg L.S., Derivations of the Lie Algebra of Polynomials Under Poisson Bracket, *Proc. Am. Math. Soc.*, 1969, V.20, 315–320.

The * attached to a citation marks the publisher that demands, as a condition of printing a paper in the publisher's journal, that all authors of creative works surrender and hand in to the publisher the copyright to the fruits of their labors. The publishers so noted do not include those in Cuba, North Korea, and other savage places, where such a policy is not a matter of free choice but is state-mandated.