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The Integrability of Lie-invariant Geometric Objects Generated by Ideals in the Grassmann Algebra

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Abstract

We investigate closed ideals in the Grassmann algebra serving as bases of Lie-invariant geometric objects studied before by E.Cartan. Especially, the E.Cartan theory is enlarged for Lax integrable nonlinear dynamical systems to be treated in the frame work of the Wahlquist Estabrook prolongation structures on jet-manifolds and Cartan-Ehresmann connection theory on fibered spaces. General structure of integrable one-forms augmenting the two-forms associated with a closed ideal in the Grassmann algebra is studied in great detail. An effective Maurer-Cartan one-forms construction is suggested that is very useful for applications. As an example of application the developed Lie-invariant geometric object theory for the Burgers nonlinear dynamical system is considered having given rise to finding an explicit form of the associated Lax type representation.

1 General setting

It is well known [1, 4] that motion planning, numerically controlled machining and robotics are just a few of many areas of manufacturing automation in which the analysis and representation of swept volumes plays a crucial role. The swept volume modeling is also an important part of task-oriented robot motion planning. A typical motion planning problem consists in a collection of objects moving around obstacles from an initial to a final configuration. This may include in particular, solving the collision detecting problem.

When a solid object undergoes a rigid motion, the totality of points through which it passed constitutes a region in space called the swept volume. To describe the geometrical structure of the swept volume we pose this problem as one of geometric study of some manifold swept by surface points using powerful tools from both modern differential geometry and nonlinear dynamical systems theory [2-4, 7, 8] on manifolds. For some special cases of the euclidean motion in the space $\mathbf{R}^3$ one can construct a very rich hydrodynamic system [1] modelling a sweep flow, which appears to be a completely integrable Hamiltonian system having a special Lax type representation. To describe in detail these and other properties of swept volume dynamical systems in this article we develop differential-geometric Cartan's theory of Lie-invariant geometric objects generated by closed ideals in the Grassmann algebra as well as investigate some special examples of euclidean motions in $\mathbf{R}^3$ leading to Lax type integrable dynamical systems on functional manifolds.

Let a Lie group G act on an analytical manifold Y in the transitive way, that is the action $G\times Y \xrightarrow{\rho} Y$ generates some nonlinear exact representation of the Lie group G on the manifold Y. In the frame of the Cartan's differential geometric theory, the representation $G \times Y \xrightarrow{\rho} Y$ can be described by means of a system of differential 1-forms

$$\bar{\beta}^j := dy^j + \sum_{i=1}^{r} \xi_i^j \bar{\omega}^i(a;da) \in \Lambda^1(Y \times G) \tag{1}$$

in the Grassmann algebra $\Lambda(Y \times G)$ on the product $Y \times G$, where $\bar{\omega}^i(a;da) \in T_a^*(G)$, $i = \overline{1, r = \dim G}$ is a basis of left invariant Cartan's forms of the Lie group G at a point $a \in G$, $y := \{y^j : j = \overline{1, n = \dim Y}\} \in Y$ and $\xi_i^j : Y \times G \to \mathbf{R}$ are some smooth real valued functions. The following Cartan's theorem is basic in describing a geometric object invariant with respect to the mentioned above group action $G \times Y \xrightarrow{\rho} Y$:

Theorem 1.(E.Cartan). *The system of differential forms (1) is a system of an invariant geometric object if and only if the following conditions are fulfilled:*

i) the coefficients $\xi_i^j \in C^\infty(Y;\mathbf{R})$ for all $i = \overline{1,r}$, $j = \overline{1,n}$, are some analytical functions on Y;

ii) the differential sysetm (1) is completely integrable within the Frobenius-Cartan criterium.

The Theorem 1 says that the differential system (1) can be written down as

$$\bar{\beta}^j := dy^j + \sum_{i=1}^{r} \xi_i^j(y) \bar{\omega}^i(a;da), \tag{2}$$

where one-forms $\{\bar{\omega}^i(a;da) : i = \overline{1,r}\}$ satisfy the standard Maurer-Cartan equations

$$\bar{\Omega}^j := d\bar{\omega}^j + \frac{1}{2} \sum_{i,k=1}^{r} c_{ik}^j \bar{\omega}^i \wedge \bar{\omega}^k := 0 \tag{3}$$

for all $j = \overline{1,r}$ on G, coefficients $c_{ik}^j \in \mathbf{R}$, $i,j,k = \overline{1,r}$, being the corresponding structure constants of the Lie algebra $\mathcal{G}$ of the Lie group G.

Let us consider here a case when the set of canonical Maurer-Cartan one-forms $\{\bar{\omega}^i(a;da) \in T_a^*(G) : i = \overline{1,r}\}$ is defined via the scheme:

$$\begin{array}{ccccc} T^*(M \times Y) & \xrightarrow{s^*} & T^*(\bar{M}) & \xleftarrow{\mu^*} & T^*(G \times Y) \\ \downarrow & & \downarrow & & \downarrow \\ M \times Y & \xleftarrow{s} & \bar{M} & \xrightarrow{\mu} & G \times Y \end{array} \tag{4}$$

where M is a given smooth finite-dimensional manifold with some submanifold $\bar{M} \subset M$ imbedded into it as $s : \bar{M} \to M \times Y$, and $\mu : \bar{M} \to G \times Y$ is some smooth mapping into $G \times Y$. Under the mappings scheme (4) the expression (3) takes the following form:

$$s^*\Omega^j\Big|_{\bar{M}} := \mu^*\bar{\Omega}^j\Big|_{\bar{M}} :\Rightarrow 0 \tag{5}$$

for all $j = \overline{1,r}$ upon the integral submanifold $\bar{M} \subset M$, where $\Omega^j \in \Lambda^2(M)$, $j = \overline{1,r}$, is some *a priori* given system of 2-forms on M.

Assume further that a set $\{\alpha_j \in \Lambda^2(M) : j = \overline{1,m_\alpha}\}$ is a basis of two-forms $\{\Omega^j \in \Lambda^2(M) : j = \overline{1,r}\}$, generating the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$. The ideal $\mathcal{I}(\alpha)$ should be completely integrable within the Cartan criterium, because due to the set of equations $d\Omega^j \in \mathcal{I}(\Omega)$, $j = \overline{1,r}$, following from (3), giving $\mathcal{I}(\Omega) \equiv 0$ on $\bar{M}$, from the scheme (4) it follows that $d\mathcal{I}(\alpha) \subset \mathcal{I}(\alpha)$ since $s^*\mathcal{I}(\alpha) = \mu^*\mathcal{I}(\bar{\Omega})$.

To define now a criterium for a Lie group action $G \times Y \stackrel{\rho}{\to} Y$ to generate a representation of the Lie group G, we need to build the ideal $\mathcal{I}(\alpha,\beta) \subset \Lambda(M \times Y)$, corresponding to (2) and (5), for a some set of forms $\beta^j \in \Lambda^1(M \times Y)$, $j = \overline{1,n}$, where $s^*\beta_j := \mu^*\bar{\beta}_j \in \Lambda(\bar{M} \times Y)$, $j = \overline{1,n}$, and to insist it to be closed in $\Lambda(M \times Y)$, that is $d\mathcal{I}(\alpha,\beta) \subset \mathcal{I}(\alpha,\beta)$, or

$$d\beta^j = \sum_{k=1}^{m_\alpha} f_k^j \alpha^k + \sum_{i=1}^{n} g_i^j \wedge \beta^i \tag{6}$$

for all $j = \overline{1,n}$ and some $f_k^j \in \Lambda^0(M \times Y)$, $k = \overline{1,m_\alpha}$, $g_i^j \in \Lambda^1(M \times Y)$, $i,j = \overline{1,n}$. The condition (6) assures that there exist some smooth submanifold $\bar{M}(Y) \subset M \times Y$, on which a nonlinear Lie group G representation acts exactly. Thereby we have stated that the following theorem is valid [4].

Theorem 2. *The system $\{\beta\}$ of Cartan's one-forms $\beta^j \in \Lambda^1(M \times Y)$, $j = \overline{1,n}$, generated by the mapping scheme (4), describes an exact nonlinear Lie group G representation on a manifold Y if and only if the adjoint ideal $\mathcal{I}(\alpha,\beta)$ generated by the system $\{\beta\}$ and a basic system $\{\alpha\}$ of the "curvature" 2-forms $\Omega^j \in \Lambda^2(M)$, $j = \overline{1,r}$, of (5), is closed together with the corresponding ideal $\mathcal{I}(\alpha) = \mathcal{I}(\alpha,0)$ in the Grassmann algebras $\Lambda(M \times Y)$ and $\Lambda(M)$ correspondingly.*

Going out of the results stated above, it is naturally to make some specialization of Cartan's geometric construction by means of the theory of principal fiber bundles [5]. To proceed with, let us try to interpret the Cartan differential system $\{\beta\}$ on $M \times Y$ as one generating a linear $(r \times r)$ - matrix adjoint representation [6,10] of the Lie algebra $\mathcal{G}$, putting the functions $\xi_i^j(y) := \sum_{k=1}^{r} c_{ik}^j y^k$, $i,j = \overline{1,r}$, when $\dim Y = n := r$:

$$\beta^j :\Rightarrow dy^j + \sum_{i,k=1}^{r} c_{ik}^j y^k b^i(z) \in \Lambda^1(M \times Y), \tag{7}$$

where $z \in M$, and 1-forms $b^i(z)$ on M satisfy the necessary embedding conditions $s^*b^i = \mu^*\bar{\omega}^i$ upon $\bar{M} \subset M$ for all $i,j = \overline{1,r}$ in accordance with the scheme (4).

The Lie group G acts on the linear r-dimensional space Y by the usual left shifts as follows: $Y \times G \ni y \times a \xrightarrow{\rho} ay \in Y$ for all $a \in G$. Whence we can easily deduce the following infinitesimal shifts in the Lie group G:

$$da_k^j + \sum_{s=1}^{r} c_{si}^j b^s(z) a_k^i \in \Lambda^1(M \times G).$$

These expressions ultimately engender the next $\mathcal{G}$-valued Ad-invariant 1-form ω on $M \times G$ via the isomorphic mapping $\rho^* : \Lambda^1(M \times Y) \to \Lambda^1(M \times G) \otimes \mathcal{G}$:

$$\{\beta\} :\xrightarrow{\rho^*} \omega := a^{-1}da + Ad_{a^{-1}}\Gamma(z), \tag{8}$$

where the one-forms matrix $\Gamma(z) := \|\Gamma_k^j(z)\|$, $j,k = \overline{1,r}$, belongs to the $(r\times r)$-matrix representation of the Lie algebra $\mathcal{G}$ due to construction: $\|\Gamma_k^j(z)\| := \|\sum_{i=1}^{r} c_{ik}^j b^i(z)\| \in T^*(M)\otimes\mathcal{G}$. The results above one can naturally interpret as a way of defining [5, 7] some $\mathcal{G}$-valued connection Γ upon a principal fibered space $P(M;G)$, carrying the $\mathcal{G}$-valued connection 1-form (8). The corresponding Cartan's 1-forms determine the horizontal subspace of the parallel transporting vectors of the fiber bundle $P(M;G,Y)$ associated with $P(M;G)$ according to the general theory [5] of fibered spaces with connections.

Thus, we have built the $\mathcal{G}$-valued connection 1-form (8) at a point $(z,a) \in P(M;G)$ as $\omega := \bar{\omega}(a) + Ad_{a^{-1}}\Gamma(z)$, where $\bar{\omega}(a) \in T^*(G) \otimes \mathcal{G}$ is the standard Maurer-Cartan left-invariant $\mathcal{G}$- valued 1-form on the Lie group G. The connection 1-form (8) is vanishing upon the above mentioned horizontal subspace, consisting of vector fields on $P(M;G)$, which generate a Lie group G representation on the space Y . This means, that this horizontal subspace necessarily defines a completely integrable differential system on $P(M;G)$, or equivalently, the corresponding curvature $\Omega \in \Lambda^2(M)\otimes\mathcal{G}$ of the connection Γ is vanishing upon the integral submanifold $\bar{M} \subset M$:

$$\Omega := d\omega + \omega\wedge\omega = Ad_{a^{-1}}(d\Gamma(z) + \Gamma(z)\wedge\Gamma(z)) = \frac{1}{2}Ad_{a^{-1}}\sum_{j=1}^{m}\Omega_{jk}dz^j\wedge dz^k\Big|_{\bar{M}} \Rightarrow 0. \tag{9}$$

from where we obtain

$$\begin{aligned} \Omega_{ij}(z) &:= \frac{\partial\Gamma_j(z)}{\partial z_i} - \frac{\partial\Gamma_i(z)}{\partial z_j} + [\Gamma_i(z), \Gamma_j(z)], \\ \Gamma(z) &:= \sum_{j=1}^{m}\Gamma_j(z)dz^j := \sum_{j=1}^{m}\sum_{k=1}^{r}\Gamma_j^k(z)dz^jA_k. \end{aligned} \tag{10}$$

The vanishing curvature Ω (9) upon the submanifold $\bar{M} \subset M$ is easily explained by means of the followimg commuting diagram:

$$\begin{array}{ccccccc} T^*(G) & \xrightarrow{\mu^*} & T^*(P(\bar{M};G)) & \xleftarrow{s^*} & T^*(P(M;G)) & \xleftarrow{\rho^*} & T^*(P(M;G,Y)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G & \xleftarrow{\mu} & P(\bar{M};G) & \xrightarrow{s} & P(M;G) & \xrightarrow{\rho} & P(M;G,Y) \end{array} \tag{11}$$

We can now derive from (12), that due to (8)

$$\rho^*\{\beta\} = \omega, \quad s^*\Gamma_k^j = \sum_{i=1}^{r} c_{ik}^j \mu^*\bar{\omega}^i \;\Rightarrow\; s^*\Omega = \mu^*\bar{\Omega} = 0, \tag{12}$$

giving rise to the implication (9) upon $\bar{M}$.

Thus, if some integrable ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$ is *a priori* given on the manifold M, we can take the corresponding to (9) equation in $\Lambda(M)$:

$$\sum_{j,k=1}^{m} \Omega_{jk} dz^j \wedge dz^k \subset \mathcal{I}(\alpha) \otimes \mathcal{G}$$

both as determining the $\mathcal{G}$-valued 1-forms $\Gamma_j(z) \in T^*(M) \otimes \mathcal{G}$, $j = \overline{1,m}$, and as determining a Lie algebra structure of $\mathcal{G}$, taking into account the holonomy Lie group reduction theorem of Ambrose, Singer and Loos [9, 10]. Namely, the holonomy Lie algebra $\mathcal{G}(h) \subset \mathcal{G}$ being generated by covariant derivatives composition of the $\mathcal{G}$-valued curvature form $\Omega \in T^*(M) \otimes \mathcal{G}$:

$$\mathcal{G}(h) := span_{\mathbf{R}}\{\nabla_1^{j_1}\nabla_2^{j_2}\ldots\nabla_n^{j_n}\Omega_{si} \in \mathcal{G} : \ j_k \in \mathbf{Z}_+,\ s,i,k = \overline{1,n}\} \tag{13}$$

where, by definition, the covariant derivative $\nabla_j : \Lambda(M) \to \Lambda(M)$, $j = \overline{1,n}$, is given as follows

$$\nabla_j := \partial/\partial z^j + \Gamma_j(z). \tag{14}$$

If the identity $\mathcal{G}(h) \equiv span_{\mathbf{R}}\{\Omega_{sl} \in \mathcal{G} : \ s,l = \overline{1,n}\}$ takes place, that is the inclusion $[\mathcal{G}(h), \mathcal{G}(h)] \subset \mathcal{G}(h)$ is reached, the holonomy Lie algebra $\mathcal{G}(h)$ is called perfect. Thus, we can formulate the following equivalence theorem.

Theorem 3. *Given a closed ideal $\mathcal{I}(\alpha)$ on a manifold M, $d\mathcal{I}(\alpha) \subset \mathcal{I}(\alpha)$, its 1-forms augmentation $\mathcal{I}(\alpha,\beta)$ on $M \times Y$ by means of a special set $\{\beta\}$ of 1-forms*

$$\{\beta\} := \left\{\beta^j = dy^j + \sum_{k=1}^{n} \xi_k^j(y) b^k(z) : \ b^j(z) \in T^*(M),\ j = \overline{1,n}\right\}, \tag{15}$$

compatible with the scheme (11), is integrable within Frobenius-Cartan criterium if and only if there exists some Lie group G action on Y, such that the adjoint connection (8) on a fibered space $P(M;G)$ with the structure group G is vanishing upon the integral submanifold $\bar{M} \subset M$ of the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$. The latter can serve as the algorithm of determining the structure of the Lie group G basing on the holonomy Lie algebra reduction theorem of Ambrose-Singer-Loos [9, 10].

If the conditions of Theorem 3 are fulfilled, the set of 1-forms $\{\beta\}$ (15) generates a representation of the Lie group G upon the analytical manifold Y according to the Cartan theorem 1. The Lie algebra $\mathcal{G}$ of the Lie group G can be reduced to the holonomy Lie algebra $\mathcal{G}(h)$, generated via (13) by the curvature 2-form Ω of the connection Γ on the principal fiber bundle $P(M;G)$ built above.

2 An effective Maurer-Cartan one-forms construction

To proceed further in study of the integrability of Lie-invariant geometric objects generated by the scheme (4) with some mapping $s : \bar{M} \to G$, one needs to have an effective way of construction corresponding to the Lie algebra $\mathcal{G} \simeq T_e^*(G)$ the Maurer-Cartan forms $\bar{\omega}^j(a;da) \in T_a^*(G) \otimes \mathcal{G}$, $j = \overline{1,r}$. Below we will describe an effective direct procedure of building these forms on G.

Let be given a Lie group G with the Lie algebra $\mathcal{G} \simeq T_e(G)$, whose basis is a set $\{A_i \in \mathcal{G} : i = \overline{1,r}\}$, where $r = \dim G \equiv \dim \mathcal{G}$. Let also a set $U_0 \subset \{a^i \in \mathbf{R} : i = \overline{1,r}\}$ be some open neighborhood of the zero point in $\mathbf{R}^r$. The exponential mapping $\exp : U_0 \to G_0$, where by definition

$$\mathbf{R}^r \supset U_0 \ni (a^1, \ldots, a^r) :\stackrel{\exp}{\longrightarrow} \exp\left(\sum_{i=1}^{r} a^i A_i\right) := a \in G_0 \subset G, \tag{16}$$

is an analytical mapping of the whole U_0 on some open neighborhood G_0 of the unity element $e \in G$. From (16) it is easy to find that $T_e(G) = T_e(G_0) \simeq \mathcal{G}$, where $e := \exp(0) \in G$. Define now the following left invariant $\mathcal{G}$- valued differential one-form on $G_0 \subset G$:

$$\bar{\omega}(a; da) := a^{-1} da = \sum_{j=1}^{r} \bar{\omega}^j(a, da) A_j \in \mathcal{G}, \tag{17}$$

where $\bar{\omega}^j(a; da) \in T_a^*(G)$, $a \in G_0$, $j = \overline{1,r}$. To build effectively the unknown forms $\{\bar{\omega}^j(a; da) : j = \overline{1,r}\}$, let us consider the following analytical one-parameter one-form $\bar{\omega}_t(a; da) := \bar{\omega}(a_t; da_t)$ on G_0, where $a_t ;= \exp\left(t \sum\limits_{i=1}^{r} a^i A_i\right)$, $t \in [0,1]$, and differentiate this form with respect to the parameter $t \in [0,1]$. We will get that

$$\begin{aligned} d\bar{\omega}_t/dt &= -\sum_{j=1}^{r} a^j A_j a_t^{-1} da_t + \sum_{j=1}^{r} a_t^{-1} a_t da^j A_j + \sum_{j=1}^{r} a_t^{-1} da_t a^j A_j \\ &= -\sum_{j=1}^{r} a^j [A_j, \bar{\omega}_t] + \sum_{j=1}^{r} A_j da_j. \end{aligned} \tag{18}$$

Having used the Lie identity $[A_j, A_k] = \sum\limits_{i=1}^{r} c_{jk}^i A_i$, $j,k = \overline{1,r}$, and the right hand side of (17) in form

$$\bar{\omega}^j(a; da) := \sum_{k=1}^{r} \bar{\omega}_k^j(a) da^k, \tag{19}$$

we ultimately obtain that

$$\frac{d}{dt}(t\bar{\omega}_i^j(ta)) = \sum_{k=1}^{r} \mathcal{A}_k^j t\bar{\omega}_i^k(ta) + \delta_i^j, \tag{20}$$

where the matrix $\mathcal{A}_i^k$, $i,k = \overline{1,r}$, is defined as follows:

$$\mathcal{A}_i^k := \sum_{j=1}^{r} c_{ij}^k a^j. \tag{21}$$

Thus, the matrix $W_i^j(t) := t\bar{\omega}_i^j(ta)$, $i,j = \overline{1,r}$, satisfies the following from (20) differential equation [6]

$$dW/dt = \mathcal{A}W + E, \quad W|_{t=0} = 0, \tag{22}$$

where $E = \|\delta_i^j\|$ is the unity matrix. The solution of (22) is representable as

$$W(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \mathcal{A}^{n-1} \tag{23}$$

for all $t \in [0,1]$. Whence, recalling the above definition of the matrix $W(t)$, we obtain easily that

$$\bar{\omega}_k^j(a) = W_k^j(t)\Big|_{t=1} = \sum_{n=1}^{\infty} (n!)^{-1} \mathcal{A}^{n-1}. \tag{24}$$

Thereby the task of finding the Maurer-Cartan one-form for a given Lie algebra $\mathcal{G}$ is solved in the effective and constructive way, being at the same time completely algebraic.

Therefore, the following theorem solves the problem of finding in an effective algebraic way corresponding to a Lie algebra $\mathcal{G}$ the left invariant one-form $\bar{\omega}(a;da) \in T_a^*(G) \otimes \mathcal{G}$ at any $a \in G$:

Theorem 4. *Let's be given a Lie algebra $\mathcal{G}$ with the structure constants $c_{ij}^k \in \mathbf{R}$, $i,j,k = \overline{1, r = dim\,\mathcal{G}}$, related to some basis $\{A_j \in \mathcal{G} : j = \overline{1,r}\}$. Then the adjoint to $\mathcal{G}$ left-invariant Maurer-Cartan one-form $\bar{\omega}(a;da)$ is built as follows:*

$$\bar{\omega}(a;da) = \sum_{k,j=1}^{r} A_j \bar{\omega}_k^j(a) da^k, \tag{25}$$

where the matrix $W := \|\bar{w}_k^j(a)\|$, $j,k = \overline{1,r}$, is given exactly as

$$W = \sum_{n=1}^{\infty} (n!)^{-1} \mathcal{A}^{n-1}, \quad \mathcal{A}_k^j := \sum_{i=1}^{r} c_{ki}^j a^i. \tag{26}$$

Below we shall try to use the experience gained above in solving an analogous problem of the theory of connections over a principal fiber bundle $P(M;G)$ as well as over associated with it a fiber bundle $P(M;Y,G)$.

3 General structure of integrable one-forms augmenting the two-forms associated with a closed ideal in the Grassmann algebra

Given two-forms generating a closed ideal $\mathcal{I}(\alpha)$ in the Grassmann algebra $\Lambda(M)$, we will denote as above by $\mathcal{I}(\alpha,\beta)$ an augmented ideal in $\Lambda(M;Y)$, where the manifold Y will be called in further the representation space of some adjoint Lie group G action: $G \times Y \xrightarrow{\rho} Y$. We can find therefore the determining relationships for the set of one-forms $\{\beta\}$ and 2-forms $\{\alpha\}$

$$\begin{aligned} &\{\alpha\} := \{\alpha^j \in \Lambda^2(M) : j = \overline{1, m_\alpha}\}, \\ &\{\beta\} := \{\beta^j \in \Lambda^1(M \times Y) : j = \overline{1, n = \dim Y}\}, \end{aligned} \tag{27}$$

satisfying such equations:

$$\begin{aligned} d\alpha^i &= \sum_{k=1}^{m_\alpha} a_k^i(\alpha) \wedge \alpha^k, \\ d\beta^j &= \sum_{k=1}^{m_\alpha} f_k^j \alpha^k + \sum_{s=1}^{n} \omega_s^j \wedge \beta^s, \end{aligned} \tag{28}$$

where $a_k^i(\alpha) \in \Lambda^1(M)$, $f_k^j \in \Lambda^0(M \times Y)$ and $\omega_s^j \in \Lambda^1(M \times Y)$ for all $i, k = \overline{1, m_\alpha}$, $j, s = \overline{1, n}$. Since the identity $d^2\beta^j \equiv 0$ takes place for all $j = \overline{1, n}$, from (28) we deduce the following relationship:

$$\sum_{k=1}^{n} \left(d\omega_k^j + \sum_{s=1}^{n} \omega_s^j \wedge \omega_k^s \right) \wedge \beta^k + \sum_{s=1}^{m_\alpha} \left(df_s^j + \sum_{k=1}^{n} \omega_k^j f_s^k + \sum_{l=1}^{m_\alpha} f_l^j a_s^l(\alpha) \right) \wedge \alpha^s \equiv 0. \tag{29}$$

As a result of (29) we obtain that

$$\begin{aligned} & d\omega_k^j + \sum_{s=1}^{n} \omega_s^j \wedge \omega_k^s \in \mathcal{I}(\alpha, \beta), \\ & df_s^j + \sum_{k=1}^{n} \omega_k^j f_s^k + \sum_{l=1}^{m_\alpha} f_l^j a_s^l(\alpha) \in \mathcal{I}(\alpha, \beta) \end{aligned} \tag{30}$$

for all $j, k = \overline{1, n}$, $s = \overline{1, m_\alpha}$. The second inclusion in (30) gives a possibility to define the 1-forms $\theta_s^j := \sum_{l=1}^{m_\alpha} f_l^j a_s^l(\alpha)$ satisfying the next inclusion:

$$d\theta_s^j + \sum_{k=1}^{n} \omega_k^j \wedge \theta_s^k \in \mathcal{I}(\alpha, \beta) \oplus \sum_{l=1}^{m_\alpha} f_l^j c_s^l(\alpha), \tag{31}$$

which we obtained having used the identities $d^2\alpha^j \equiv 0$, $j = \overline{1, m_\alpha}$, in the form $\sum_{s=1}^{m} c_s^j(\alpha) \wedge \alpha^s \equiv 0$,

$$c_s^j(\alpha) = da_s^j(\alpha) + \sum_{k=1}^{m_\alpha} a_l^j(\alpha) \wedge a_s^l(\alpha), \tag{32}$$

following from (28). Let us suppose further that as $s = s_0$ the 2-forms $c_{s_0}^j(\alpha) \equiv 0$ for all $j = \overline{1, m_\alpha}$. Then as $s = s_0$, we can define a set of 1-forms $\theta^j := \theta_{s_0}^j \in \Lambda^1(M \times Y)$, $j = \overline{1, n}$, satisfying the exact inclusions:

$$d\theta^j + \sum_{k=1}^{n} \omega_k^j \wedge \theta^k := \Theta^j \in \mathcal{I}(\alpha, \beta) \tag{33}$$

together with a set of inclusions for 1-forms $\omega_k^j \in \Lambda^1(M \times Y)$

$$d\omega_k^j + \sum_{s=1}^{n} \omega_s^j \wedge \omega_k^s := \Omega_k^j \in \mathcal{I}(\alpha, \beta) \tag{34}$$

As it follows from the general theory [5] of connections on the fibered frame space $P(M; GL(n))$ over a base manifold M, we can interpret the equations (34) as the equations defining the curvature 2-forms $\Omega_k^j \in \Lambda^2(P)$, as well as interpret the equations (33) as those, defining the torsion 2-forms $\Theta^j \in \Lambda^2(P)$. Since $\mathcal{I}(\alpha) = 0 = \mathcal{I}(\alpha, \beta)$ upon the integral submanifold $\bar{M} \subset M$, the reduced fibered frame space $P(\bar{M}; GL(n))$ will have the flat curvature and be torsion free, being as a result, completely trivialized on $\bar{M} \subset M$. Consequently, we can formulate the following theorem.

Theorem 5. *Let the condition above on the ideals $\mathcal{I}(\alpha)$ and $\mathcal{I}(\alpha, \beta)$ be fulfilled. Then the set of 1-forms $\{\beta\}$ generates the integrable augmented ideal $\mathcal{I}(\alpha, \beta) \subset \Lambda(M \times Y)$ if and only if there exists some curvature 1-form $\omega \in \Lambda^1(P) \otimes \mathcal{G}l(n)$ and torsion 1-form $\theta \in \Lambda^1(P) \otimes \mathbf{R}^n$ on the adjoint fibered frame space $P(M; GL(n))$, satisfying the inclusions*

$$\begin{aligned} &d\omega + \omega \wedge \omega \in \mathcal{I}(\alpha, \beta) \otimes \mathcal{G}l(n), \\ &d\theta + \omega \wedge \theta \in \mathcal{I}(\alpha, \beta) \otimes \mathbf{R}^n. \end{aligned} \tag{35}$$

Upon the reduced fibered frame space $P(\bar{M}; GL(n))$ the corresponding curvature and torsion are vanishing, where $\bar{M} \subset M$ is the integral submanifold of the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$.

We can see from Theorem 5 that some its conditions coincide with those of Theorem 3, concerning the properties of adjoint curvature forms $\omega \in \Lambda^1(P) \otimes \mathcal{G}$. Thus, the condition of existing some curvature 1-form $\omega \in \Lambda^1(P) \otimes \mathcal{G}$, whose curvature form $\Omega \in \Lambda^2(P) \otimes \mathcal{G}$ must necessarily vanish upon the integral submanifold of the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$. The nature of the second inclusion of (35) is at present not completely understood, namely the condition of existence of the integrable augmented ideal $\mathcal{I}(\alpha, \beta) \subset \Lambda(M \times Y)$. This problem is under started view of an article under preparation. Below we will analyse in detail some special examples [7, 8] of the construction suggested above, concerned with the integrable dynamical systems, given on some invariant jet-submanifolds.

4 The Cartan's invariant geometric object structure of Lax integrable nonlinear dynamical systems in partial derivatives

Consider at the beginning some set $\{\beta\}$ defining a Cartan's Lie group G invariant object on a manifold $M \times Y$:

$$\beta^j := dy^j + \sum_{k=1}^{r} \xi_k^j(y) b^k(z), \tag{36}$$

where $i = \overline{1, n = \dim Y}$, $r = \dim G$, satisfying the mapping scheme (4) with a chosen integral submanifold $\bar{M} \subset M$. This means, that the set (36) defines on the manifold Y a set $\{\xi\}$ of vector fields, compiling a representation $\rho : \mathcal{G} \to \{\xi\}$ of a given Lie algebra $\mathcal{G}$, that is vector fields $\xi_s := \sum_{j=1}^{n} \xi_s^j(y) \dfrac{\partial}{\partial y^j} \in \{\xi\}$, $s = \overline{1, r}$, enjoy the following Lie algebra $\mathcal{G}$ relationships

$$[\xi_s, \xi_l] = \sum_{k=1}^{r} c_{sl}^k \xi_k \tag{37}$$

for all $s,l,k=\overline{1,r}$. We can now compute the differentials $d\beta^j \in \Lambda^2(M\times Y)$, $j=\overline{1,n}$, using (36) and (37) as follows:

$$\begin{aligned} d\beta^j &= \sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\left(\beta^l-\sum_{s=1}^{r}\xi_s^l(y)b^s(z)\right)\wedge b^k(z)+\sum_{l=1}^{n}\sum_{k=1}^{r}\xi_k^j(y)db^k(z)\\ &=\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b^k(z)-\sum_{l=1}^{n}\sum_{k,s=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\xi_s^l(y)b^s(z)\wedge b^k(z)+\sum_{k=1}^{r}\xi_k^j(y)db^k(z)\\ &=\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b_k(z)+\frac{1}{2}\sum_{l=1}^{n}\sum_{k,s=1}^{r}\left[\frac{\partial\xi_k^j(y)}{\partial y^l}\xi_s^l(y)-\frac{\partial\xi_s^j(y)}{\partial y^l}\xi_k^l(y)\right]\\ &\times db^k(z)\wedge db^s(z)+\sum_{k=1}^{r}\xi_k^j(y)db^k(z) \end{aligned} \tag{38}$$

$\Rightarrow$

$$\begin{aligned} &\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b_k(z)+\frac{1}{2}\sum_{k,s=1}^{r}[\xi_s,\xi_k]^j db^k(z)\wedge db^s(z)\\ &+\sum_{k=1}^{r}\xi_k^j(y)db^k(z)\Rightarrow\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b_k(z)\\ &+\frac{1}{2}\sum_{l=1}^{n}\sum_{k,s=1}^{r}c_{ks}^l\xi_l^j db^k(z)\wedge db^s(z)+\sum_{k=1}^{r}\xi_k^j(y)db^k(z)\Rightarrow\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b_k(z)\\ &+\sum_{l=1}^{r}\xi_l^j\left(db^l(z)+\frac{1}{2}\sum_{k,s=1}^{r}c_{ks}^l db^k(z)\wedge db^s(z)\right):\in\mathcal{I}(\alpha,\beta)\subset\Lambda(M\times Y), \end{aligned}$$

where $\{\alpha\}\subset\Lambda^2(M)$ is some *a priori* given integrable system of 2-forms on M, vanishing upon the integral submanifold $\bar{M}\subset M$. It is obvious that inclusions (38) take place if and only if the following conditions are fulfilled: for all $j=\overline{1,r}$

$$db^j(z)+\frac{1}{2}\sum_{k,s=1}^{r}c_{ks}^j db^k(z)\wedge db^s(z)\in\mathcal{I}(\alpha). \tag{39}$$

The inclusions (39) mean in particular, that upon the integral submanifold $\bar{M}\subset M$ of the ideal $\mathcal{I}(\alpha)\subset\Lambda(M)$ the equalities

$$\mu^*\bar{\omega}^j\equiv s^*b^j, \tag{40}$$

are true, where $\bar{\omega}^j\in T_e^*(G)$, $j=\overline{1,r}$, are the left invariant Maurer-Cartan forms on the invariance Lie group G. Thus, due to inclusions (39) all conditions of Cartan's Theorem 1 are enjoyed, giving rise to a possibility to obtain the set of forms $b^j(z)\in\Lambda^1(M)$ in an explicit form. To do this, let us define a $\mathcal{G}$-valued curvature 1-form $\omega\in\Lambda^1(P(M;G))\otimes\mathcal{G}$ as follows

$$\omega:=Ad_{a^{-1}}\left(\sum_{j=1}^{r}A_jb^j\right)+\bar{\omega} \tag{41}$$

where $\bar{\omega} \in \mathcal{G}$ is the standard Maurer-Cartan 1-form on G, built in Chapter 2. This 1-form satisfies followed by (39) the canonical structure inclusion (9) for $\Gamma := \sum_{j=1}^{r} A_j b^j \in \Lambda^1(M) \otimes \mathcal{G}$:

$$d\Gamma + \Gamma \wedge \Gamma \in \mathcal{I}(\alpha) \otimes \mathcal{G}, \tag{42}$$

serving as a main relationships determining the form (41) in accordance with results of Chapter 3. To proceed further we need to give the set of 2-forms $\{\alpha\} \subset \Lambda^2(M)$ in explicit form.

Example 1. The Burgers dynamical system.

Let's be given the following Burgers dynamical system on a functional manifold $M \subset C^\infty(\mathbf{R};\mathbf{R})$:

$$u_t = uu_x + u_{xx}, \tag{43}$$

where $u \in M$, $t \in \mathbf{R}$ is an evolution parameter. The flow (43) on M can be recast into a set of 2-forms $\{\alpha\} \subset \Lambda^2(J(\mathbf{R}^2;\mathbf{R}))$ upon the adjoint jet-manifold $J(\mathbf{R}^2;\mathbf{R})$ as follows:

$$\begin{aligned}\{\alpha\} = \Big\{ & du^{(0)} \wedge dt - u^{(1)} dx \wedge dt = \alpha^1,\ du^{(0)} \wedge dx + u^{(0)} du^{(0)} \wedge dt \\ & + du^{(1)} \wedge dt = \alpha^2 :\ \left(x,t;u^{(0)},u^{(1)}\right)^\tau \in M^4 \subset J^1(\mathbf{R}^2;\mathbf{R})\Big\},\end{aligned} \tag{44}$$

where M^4 is some finite-dimensional submanifold in $J^1(\mathbf{R}^2;\mathbf{R}))$ with coordinates $\left(x, t, u^{(0)} = u, u^{(1)} = u_x\right)$. The set of 2-forms (44) generates the closed ideal $\mathcal{I}(\alpha)$, since

$$d\alpha^1 = dx \wedge \alpha^2 - u^{(0)} dx \wedge \alpha^1, \quad d\alpha^2 = 0, \tag{45}$$

the integral submanifold $\bar{M} = \{x, t \in \mathbf{R}\} \subset M^4$ being defined by the condition $\mathcal{I}(\alpha) = 0$. We now look for a reduced "curvature" 1-form $\Gamma \in \Lambda^1(M^4) \otimes \mathcal{G}$, belonging to some not yet determined Lie algebra $\mathcal{G}$. This 1-form can be represented using (44), as follows:

$$\Gamma := b^{(x)}(u^{(0)}, u^{(1)})dx + b^{(t)}(u^{(0)}, u^{(1)})dt, \tag{46}$$

where elements $b^{(x)}, b^{(t)} \in \mathcal{G}$ satisfy such determining equations, engendered by (42):

$$\begin{aligned}& \frac{\partial b^{(x)}}{\partial u^{(0)}} du^{(0)} \wedge dx + \frac{\partial b^{(x)}}{\partial u^{(1)}} du^{(1)} \wedge dx + \frac{\partial b^{(t)}}{\partial u^{(0)}} du^{(0)} \wedge dt \\ & \qquad + \frac{\partial b^{(t)}}{\partial u^{(1)}} du^{(1)} \wedge dt + [b^{(x)}, b^{(t)}] dx \wedge dt \equiv \Omega \\ \Rightarrow \quad & g_1(du^{(0)} \wedge dt - u^{(1)} dx \wedge dt) + g_2(du^{(0)} \wedge dx \\ & \qquad + u^{(0)} du^{(0)} \wedge dt + du^{(1)} \wedge dt) \in \mathcal{I}(\alpha) \otimes \mathcal{G}\end{aligned} \tag{47}$$

for some $\mathcal{G}$-valued functions g_1, g_2 on M. From (47) it follows that

$$\begin{aligned}& \frac{\partial b^{(x)}}{\partial u^{(0)}} = g_2, \quad \frac{\partial b^{(x)}}{\partial u^{(1)}} = 0, \quad \frac{\partial b^{(t)}}{\partial u^{(0)}} = g_1 + g_2 u^{(0)}, \\ & \frac{\partial b^{(t)}}{\partial u^{(1)}} = g_2, \quad [b^{(x)}, b^{(t)}] = -u^{(1)} g_1.\end{aligned} \tag{48}$$

The set (48) has the following unique solution

$$b^{(x)} = A_0 + A_1 u^{(0)},$$
$$b^{(t)} = u^{(1)} A_1 + \frac{u^{(0)2}}{2} A_1 + [A_1, A_0] u^{(0)} + A_2, \tag{49}$$

where $A_j \in \mathcal{G}$, $j = \overline{0,2}$, are some constant elements on M of a Lie algebra $\mathcal{G}$ under search, enjoying the next Lie structure equations:

$$\begin{aligned} &[A_0, A_2] = 0, \\ &[A_0, [A_1, A_0]] + [A_1, A_2] = 0, \\ &[A_1, [A_1, A_0]] + \frac{1}{2}[A_0, A_1] = 0. \end{aligned} \tag{50}$$

From (48) one can see that the curvature 2-form $\Omega \in span_{\mathbf{R}}\{A_1, [A_0, A_1] : A_j \in \mathcal{G}, j = \overline{0,1}\}$. Therefore, reducing via the Ambrose-Singer theorem the associated principal fibered frame space $P(M; G = GL(n))$ to the principal fiber bundle $P(M; G(h))$, where $G(h) \subset G$ is the corresponding holonomy Lie group of the connection Γ on P, we need to satisfy the following conditions for the set $\mathcal{G}(h) \subset \mathcal{G}$ to be a Lie subalgebra in $\mathcal{G}$: $\nabla_x^m \nabla_t^n \Omega \in \mathcal{G}(h)$ for all $m, n \in \mathbf{Z}_+$.

Let us try now to close the above transfinitive procedure requiring that

$$\mathcal{G}(h) = \mathcal{G}(h)_0 := span_{\mathbf{R}}\{\nabla_x^m \nabla_x^n \Omega \in \mathcal{G} : m + n = 0\} \tag{51}$$

This means that

$$\mathcal{G}(h)_0 = span_{\mathbf{R}}\{A_1, A_3 = [A_0, A_1]\}. \tag{52}$$

To enjoy the set of relations (50) we need to use expansions over the basis (52) of the external elements $A_0, A_2 \in \mathcal{G}(h)$:

$$A_0 = q_{01} A_1 + q_{13} A_3, \qquad A_2 = q_{21} A_1 + q_{23} A_3. \tag{53}$$

Substituting expansions (53) into (50), we get that $q_{01} = q_{23} = \lambda$, $q_{21} = -\lambda^2/2$ and $q_{03} = -2$ for some arbitrary real parameter $\lambda \in \mathbf{R}$, that is $\mathcal{G}(h) = span_{\mathbf{R}}\{A_1, A_3\}$, where

$$[A_1, A_3] = A_3/2; \qquad A_0 = \lambda A_1 - 2A_3, \qquad A_2 = -\lambda^2 A_1/2 + \lambda A_3. \tag{54}$$

As a result of (54) we can state that the holonomy Lie algebra $\mathcal{G}(h)$ is a real two-dimensional one, assuming the following (2×2)-matrix representation:

$$\begin{aligned} &A_1 = \begin{pmatrix} 1/4 & 0 \\ 0 & -1/4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ &A_0 = \begin{pmatrix} \lambda/4 & -2 \\ 0 & -\lambda/4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\lambda^2/8 & \lambda \\ 0 & \lambda^2/8 \end{pmatrix}. \end{aligned} \tag{55}$$

Thereby from (46), (49) and (55) we obtain the next reduced curvature 1-form $\Gamma \in \Lambda^1(M) \otimes \mathcal{G}$

$$\Gamma = (A_0 + uA_1)dx + ((u_x + u^2/2)A_1 - uA_3 + A_2)dt, \tag{56}$$

generating parallel transporting of vectors from the representation space Y of the holonomy Lie algebra $\mathcal{G}(h)$:

$$dy + \Gamma y = 0 \tag{57}$$

upon the integral submanifold $\bar{M} \subset M^4$ of the ideal $\mathcal{I}(\alpha)$, generated by the set of 2-forms (44). The result (57) means also that the dynamical system (43) is endowed with the standard Lax type representation, having the spectral parameter $\lambda \in \mathbf{R}$ necessary for its integrability in quadratures.

In the case when the condition

$$\mathcal{G}(h) = \mathcal{G}(h)_1 := span_{\mathbf{R}}\{\nabla_x^m \nabla_t^n \Omega \in \mathcal{G} :\ m + n = \overline{0,1}\}$$

is assumed satisfied, one can compute that

$$\begin{aligned}
&\mathcal{G}(h)_1 := span_{\mathbf{R}}\{\nabla_x^m \nabla_t^n g_j \in \mathcal{G} :\ j = \overline{1,2},\ m + n = \overline{0,1}\} \\
&\Rightarrow \qquad span_{\mathbf{R}}\{g_j \in \mathcal{G};\ \partial g_j/\partial x + [g_j, A_0 + A_1 u^{(0)}], \\
&\partial g_j/\partial t + [g_j, u^{(1)} A_1 + u^{(0)} A_1/2 + [A_1, A_0]u^{(0)} + A_2] \in \mathcal{G} :\ j = \overline{1,2}\} \\
&\Rightarrow \qquad span_{\mathbf{R}}\{A_1, [A_1, A_0], [[A_1, A_0], A_0], [[A_1, A_0], A_1], \\
&\qquad [A_1, A_2], [[A_1, A_0], A_2] \in \mathcal{G}\} \Rightarrow span_{\mathbf{R}}\{A_{j\neq 2} \in \mathcal{G} :\ j = \overline{1,7}\},
\end{aligned} \tag{58}$$

where, by definition,

$$\begin{aligned}
&[A_1, A_0] = A_3, \quad [A_3, A_0] = A_4, \quad [A_3, A_2] = A_7, \\
&[A_3, A_1] = A_5, \quad [A_1, A_2] = A_6.
\end{aligned} \tag{59}$$

As a result, we have the following expansions for undetermined hidden elements $A_0, A_2 \in \mathcal{G}$

$$A_0 := \sum_{j=1, j\neq 2}^{7} q_{0j} A_j, \quad A_2 := \sum_{j=1, j\neq 2}^{7} q_{2j} A_j, \tag{60}$$

where $q_{0j}, q_{2j} \in \mathbf{R}$ are some real members to be found successfully from conditions (58) and (59) as well as from the standard Jacobi identities. Having found some finite-dimensional representation of the Lie algebra $\mathcal{G}(h) = \mathcal{G}(h)_1$ (58) and substituted it into (56), we will be in a position to write down the parallel transportation equation (57) in a new Lax type form useful for the study of exact solutions to the Burgers dynamical system (43). The analogous calculations could be fulfilled effectively in cases of any other nonlinear dynamical systems [7,8], integrable by Lax on some infinite-dimensional functional spaces.

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