Journal of Nonlinear Mathematical Physics 1998, V.5, N 1, 23–40. Article

Stochastic Cohomology of the Frame Bundle of the Loop Space

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Received January 10, 1997; Accepted September 1, 1997

Abstract

We study the differential forms over the frame bundle of the based loop space. They are stochastics in the sense that we put over this frame bundle a probability measure. In order to understand the curvatures phenomena which appear when we look at the Lie bracket of two horizontal vector fields, we impose some regularity assumptions over the kernels of the differential forms. This allows us to define an exterior stochastic differential derivative over these forms.

Introduction

Let $L_x(M)$ be the based loop space of smooth applications γ_s from the circle into M such that $\gamma_0 = \gamma_1 = x$. Let $Q \to M$ be a principal bundle over M with structure group G. $L_e(Q)$ is the set of based loop in Q over the based loop space of M. It is a based loop group bundle whose the structure group is $L_e(G)$, the based loop group of G. If $Q \to M$ is the frame bundle, $L_e(Q)$ is the frame bundle of $L_x(M)$: the structure of $L_e(Q)$ is of the main importance to study string structures (or spin structures) over the loop space ([8], [9], [38]), and has a deep place in the understanding the Dirac operator over the loop space ([38]).

Let us suppose that the loop space is simply connected, in order to avoid all torsion phenomenon. If the loop space is the space of smooth loop, there is an equivalence between the cohomology with values in Z and S^1 bundles over the loop space. Let us now endow the loop space with the Brownian bridge measure, if the manifold is supposed riemannian. The equivalence is not at all clear in the stochastic context: let us clarify what it means. In the stochastic context, the loop are only continuous. A stochastic cohomology of $L_x(M)$

is defined in [27], [29] and [30] with values in C or R: since $L_x(M)$ is supposed simply connected, we can neglect all torsion phenomenon in order to construct a S^1 bundle from a Z closed 2 form over the loop space of finite energy loops. But we have to choose distinguished paths in $L_x(M)$ in order to shrink a loop in a constant loop: let $l_t(\gamma)_s$ such a distinguished path. The law of $l_t(\gamma)_.$ is not absolutely continuous with respect of the law of γ. So we have to consider special type of forms in order to overcome the problem: this avoids to use a Z stochastic cohomology of the brownian bridge, by considering only examples.

The goal of this paper is to do a stochastic cohomology of the frame bundle of $L_x(M)$, to construct the stochastic forms which allow to consider a string structure over $L_x(M)$. Namely, we have already constructed stochastic bundles over $L_e(Q)$ by starting from a given deterministic form over this set, and the goal of this paper is to give a stochastic meaning to this form [34].

As in [34], we define a measure over $L_e(Q)$, by putching together measures in the fiber: the fiber is a continuous loop group. We start with the equation in the fiber

$$dg_s = dB_s g_s. \tag{0.1}$$

In [34], we have studied the equation

$$dg_s = B_s ds g_s. \tag{0.2}$$

We choose this equation in order to reflect the fiber structure of $L_e(Q)$, the only obstacle to the trivialization being the holonomy over a loop in the basical manifold. Namely, we can consider the Albeverio-Hoegh-Krohn quasi invariance formulas under the right translation $g_. \to g_.K_.$. If $K_.$ is deterministic in C^1, the quasi-invariance density belongs to all the L^p in the first case, while it belongs only to L^1 in the second case, if $K_.$ is C^2.

This allows us to define a tangent space of $L_e(Q)$ by using an infinite dimensional connection and to get horizontal vector fields and vertical vector fields. We meet the following paradoxe: the big difference between the Sobolev Calculus over the loop group and the Sobolev Calculus over the loop space of a riemannian manifold is the following: in the first case, the tangent vector fields are stable by Lie Bracket, in the second case no. Apparently, if we follow this remark, we have to separate the treatment of the horizontal component and of the vertical component of a form, in order to define a stochastic exterior derivative over $L_e(Q)$. Let us recall namely that, in order to define some cohomology groups over the loop spaces, we have imposed in [27] some regularity assumptions over the kernels of the associated forms, in order to simplify the treatment of the anticipative Stratonovitch integrals which appears in the definition of the exterior stochastic differential. These conditions lead to needless complications in the case of loop groups [15]. But in our situation, we cannot neglect the curvature phenomena which appear: we are obliged to treat the horizontal and the vertical components in the same manner, in order to define some stochastic cohomology groups of $L_e(Q)$. The Carey-Murray [38] form is closed for this stochastic cohomology (If the first Pontryaguin class of Q vanishes), because it is a mixture between a basical iterated integral and the canonical 2 form over a loop group: this gives the second aspect of the construction of the string bundle in our stochastic situation.

Moreover, this Calculus depends apparently of the connection over the frame bundle $L_e(Q) \to L_x(M)$. But we show that the functional spaces which are got with some regularity assumptions over the kernels are independant of this connection.

Stochastic cohomology of the loop space of the bundle

Let $Q \to M$ be a principal bundle with a compact connected structural Lie goup G. We suppose that M is endowed with a Riemannian metric: there exists a heat semi-group over M and a brownian bridge measure $dP_{1,x}$ associated to the riemannian metric. It is a measure over the based continuous loop space.

Over G, we consider the following stochastic differential equation:

$$dg_s = dB_s g_s; \qquad g_0 = e, \tag{1.1}$$

where B_s is a brownian motion independant of the law of the loop γ over M over Lie G.

We get a law Q which can be desintegrated over the pinned path space of paths in the group joining e to g ([17], [2], [3]). We get a space of continuous paths in G $L_g(G)$ endowed with a law Q_g. The non pinned based path group is denoted $P(G)$.

We put over the bundle $Q \to M$ a connection ∇^Q: τ_s^Q the parrallel transport for a loop γ_s is therefore almost surley defined for the connection ∇^Q. We denote by $L_e(Q)$ the space of loop $q_.$ in Q such that $q_s = \tau_s^Q g_s$, $g_1 = (\tau_1^Q)^{-1}$. We get the following commutative diagramm [38]:

$$\begin{array}{ccc} L_e(Q) & \to & P(G) \\ \downarrow & & \downarrow \\ L_x(M) & \to & G \end{array} . \tag{1.2}$$

The map from P(G) to G is the map which to $g_.$ associates g_1. The map from $L_e(Q)$ to $L_x(M)$ is the projection map. The map f from $L_x(M)$ to G is the map which to a stochastic loop $\gamma_.$ associates $(\tau_1^Q)^{-1}$. The map from $L_e(Q)$ to $P(G)$ is the map which to $q_.$ associates $g_.$. It is nothing else than f^*.

Over $L_e(Q)$, we put the measure:

$$dP_{tot} = dP_{1,x} \otimes dQ_{(\tau_1)^{-1}}. \tag{1.3}$$

Let us analyze a little bit more the $L_e(G)$ bundle $P(G) \to G$. If $g_1 \in G_i$ is a small open neighborhood of G, we can choose a section $g_{i,s}(g_1)$ of this bundle which is jointly smooth in s and in g_1. It checks the following property: $g_{i,0}(g_1) = e$; $g_{i,1}(g_1) = g_1$; $g_{i,s}(g_1) \in G$. This means that the transition functions of $P(G)$ can be choosen to take their values in the smooth based loop space og G, $L_e^\infty(G)$. Since G is a compact manifold, we can choose a connection over the bundle $P(G) \to G$ whose the structural group is reduced to $L_e^\infty(G)$. Let us call ∇^∞ this connection: if $g_1 \in G_i$, the connection one form is a smooth path in the Lie algebra of G starting from 0 and arriving at 0 $K_{i,s}(g_1)$, which depends smoothly from $g_1 \in G_i$ and which is a one form in g_1.

The obstruction to trivialize $L_e(Q)$ over $L_x(M)$ lies in $(\tau_1^Q)^{-1}$: if $(\tau_1^Q)^{-1} \in G_i$, there is a local slice of $L_e(Q)$ which is $g_{i,.}((\tau_1^Q)^{-1})$. We look at the left transformation $g_. \to (g_{i,.}((\tau_1^Q)^{-1}))^{-1} g_.$. Modulo this transformation, the bridge between e and $(\tau_1^Q)^{-1}$ is transformed into the bridge between e and e. Let us recall namely the purpose of the quasi-invariance formula from Albeverio-Hoegh-Krohn [4]: if k_s is a deterministic C^1 path in the group G, the law of $g_.k_.$ and the law of $k_.g_.$ are quasi-invariant with respect to the law of (1.1). Moreover the density of quasi-invariance belong to all the L^p and can be desintegrated along the appropriate bridge. We denote by $J_r(k)$ and by $J_l(k)$ the right quasi-invariance density and the left quasi-invariance density [4], [17].

Therefore if $(\tau_1^Q)^{-1} \in G_i$:

$$dP_{tot} = dP_{1,x} \otimes J_l(g_{i,}((\tau_1^Q)^{-1}))dQ_e. \tag{1.4}$$

$J_l(g_{i,.}((\tau_1^Q)^{-1})$ belongs to all the L^p and is bounded in L^p when $(\tau_1^Q)^{-1}$ describes G_i. (1.4) produces a stochastic trivialization of our bundle.

Let us recall that a vector field over $L_x(M)$ is given by [7], [20]

$$X_t = \tau_t H_t \qquad X_0 = X_1 = 0, \tag{1.5}$$

where τ_t is the parallel transport associated to the Levi-Civita connection and $H_.$ is a finite energy path in T_x. We choose as Hilbertian norm of $X_.$ the norm

$$\|X\|^2 = \int_0^1 \|H'_s\|^2 ds = \| H\|^2. \tag{1.6}$$

Let us recall that a right vector field over $L_e(G)$ is given by $X_t^r = g_t K_t$ where K_t is a finite energy path with end points equal to 0 in the Lie algebra of G which checks $\int_0^1 \|K'_s\|^2 ds = \|K\|^2 < \infty$. A left vector field over $L_e(G)$ is given by $K_s g_s = X_s^l$ where K_s checks the same condition (See [34]).

We pullback the connection ∇^∞ to be a connection over the stochastic bundle $L_e(Q) \to L_x(M)$. If $s \to K_{i,s}(g_1)(dg_1)$ is the connection form for $g_1 \in G_i$, the connection form of the pullback connection still denoted ∇^∞ is $s \to K_{i,s}((\tau_1^Q)^{-1})(\langle d(\tau_1^Q)^{-1}, .\rangle)$. For that, we recall that:

$$\langle d\tau_1^Q, X\rangle = \tau_1^Q \int_0^1 (\tau_s^Q)^{-1} R^Q(d\gamma_s, X_s)\tau_s^Q, \tag{1.7}$$

where R^Q is the curvature tensor of ∇^Q.

We define as tangent space of the total space $L_e(Q)$ the orthonormal sum of the horizontal vector fields and vertical vector fields. In the trivialization given by (1.4), the horizontal vector fields are given by:

$$X^H(H)_s = \tau_s H_s - K_{i,s}((\tau_1^Q)^{-1})\langle d(\tau_1^Q)^{-1}, X\rangle g_{i,s} \tag{1.8}$$

and the vertical vector fields are given by $q_s K_s = X^V(K)_s$. We choose as Hilbert norm of $X^H(H)$ the quantity $\|H\|^2$ and of $X^V(K)$ the quantity $\|K\|^2$. These vector fields are consistently defined (See [34]). Let us recall the following theorem [34]:

Theorem I.1. *Let us consider a cylindrical functional $F(q_{s_1}, \ldots, q_{s_r})$ over $L_e(Q)$. Then there exists a functional* $\operatorname{div} X^H(H)$ *and a functional* $\operatorname{div} X^V(K)$ *which belong to all the L^p such that for deterministic H and K:*

$$E_{tot}\left[\langle dF, X^H(H)\rangle\right] = E_{tot}\left[F \operatorname{div} X^H(H)\right] \tag{1.9}$$

and such that

$$E_{tot}\left[\langle dF, X^V(K)\rangle\right] = E_{tot}\left[F \operatorname{div} X^V(K)\right]. \tag{1.10}$$

Let us introduce over $T^H(L_e(Q))$ and $T^V(L_e(Q))$ a connection:

$$\nabla X^V(K) = X^V(\nabla K) \tag{1.11}$$

and

$$\nabla X^H(H) = X^H(\nabla H), \tag{1.12}$$

where ∇K_s is the H-derivative of K_s in the fixed Lie algebra of G and ∇H_s is the H-derivative in the fixed tangent space at x of M. The integration by parts (1.9) and (1.10) allow to define consistently these derivatives.

If $K_s = \sum k_s^i k_i$ where k_i is an orthonormal basis of the Lie algebra of G, we get:

$$\nabla_X K_s = \sum < dk_s^i, X > k_i. \tag{1.13}$$

The same holds for $H_s = \sum h_s^i e_i$ where e_i is a fixed basis of the tangent space of M at x.

Let us consider a n cotensor ω over $L_e(Q)$. Let us recall that $\nabla\omega$ is defined as follows:

$$\nabla\omega(X_1, .., X_{n+1}) = \langle d(\omega(X_1, \ldots, X_n)), X_{n+1}\rangle - \sum_{i=1}^{n} \omega(X_1, \ldots, \nabla_{X_{n+1}} X_i, \ldots, X_n). \tag{1.14}$$

This allows us to define iteratively the k covariant derivative of a n form σ. Let us describe a bit the situation: a n form is a n antisymmetric tensor over the tangent Hilbert space of $q_.$, which has a priori two types of behaviour:

- The horizontal contribution.

- The vertical contribution.

These contributions have two different behaviours:

$$\begin{gathered}\sigma\left(X^H(H_1), \ldots, X^H(H_n), X^V(K_1), \ldots, X^V(K_m)\right) = \\ \int\int \sigma^{n,m}(s_1, \ldots, s_n; t_1, \ldots, t_m) H'_{1,s_1} \ldots H'_{n,s_n} K'_{1,t_1} \ldots K'_{m,t_m} ds_1 \ldots ds_n dt_1 \ldots dt_m,\end{gathered} \tag{1.15}$$

where $\sigma^{n,m}$ is a kernel which checks the antisymmetric conditions due to the antisymmetric conditions over σ. The covariant derivatives of σ have too two different contributions which are due to the vertical and horizontal vector fields. In order to simplify the exposure, we won't do in the formulas the difference between the two type of contributions: a form σ is given by kernels $\sigma(s_1, \ldots, s_n)$ whose the covariant derivatives with respect to the connection ∇ are given by kernels $\sigma(s_1, \ldots, s_n; t_1, \ldots, t_k)$. Moreover $\int \sigma(s_1, \ldots, s_n; t_1, \ldots, t_k) ds_i = 0$ and $\int \sigma(s_1, \ldots, s_n; t_1, \ldots, t_k) dt_j = 0$ since we work over the loop space.

Let us define the Nualart-Pardoux constants of σ. Let K be a connected component of $[0,1]^n \times [0,1]^k$ where we had removed the diagonals. We define the first Nualart-Pardoux constant as $C(p,n,k)(Q)$ by the smallest constant such that:

$$\begin{gathered}\|\sigma(s_1, \ldots, s_n; t_1, \ldots, t_k) - \sigma(s'_1, \ldots, s'_n; t'_1, \ldots, t'_k)\|_{L^p} \leq \\ C(p,n,k)(Q)\left(\sum \sqrt{| s_i - s'_i |} + \sum \sqrt{| t_i - t'_i |}\right)\end{gathered} \tag{1.16}$$

over any K.

The second Nualart-Pardoux constan $C'(p,n,k)(Q)$ is the smallest one such that for all s_i and all t_j:

$$\|\sigma(s_1,\ldots,s_n;t_1,\ldots,t_k)\|_{L^p} \leq C'(p,n,k)(Q). \tag{1.17}$$

Definition I.2. *A n form is said smooth in the Nualart-Pardoux sense if the collection of $C(p,n,k)(Q)$ and $C'(p,n,k)(Q)$ is $finite$.*

We have a theorem whose the proof is the analoguous of the proof of the theorem I.2 of [27].

Theorem I.3. *If σ is a n form which is smooth in the Nualart-Pardoux sense and if σ' is a n' form which is smooth in the Nualart-Pardoux sense, $\sigma \wedge \sigma'$ is a $n+n'$ form which is still smooth in the Nualart-Pardoux sense.*

Over $P(G)$, we can consider the brownian motion measure: g_1 is free. The tangent space of a path g_s is given by the set of vector of the shape $g_s K_s = X_s$ where K_0 is equal to 0 and K_1 is free. It is endowed with the Hilbert structure $\int_0^1 \|K'_s\|^2 ds$. We can repeat the previous considerations and give the definition of a form which is smooth in the Nualart-Pardoux sense over $P(G)$: its Nualart-Pardoux constants are called $C(p,n,k)(G)$ and $C'(p,n,k)(G)$. We choose the same connection than in Q for the definition of iterated covariant derivatives of a form over $P(G)$.

In the same way, over $L_x(M)$, we can consider the brownian bridge measure. The tangent space of a loop is the space of $\tau_s H_s$, $H_0 = H_1 = 0$ and we choose the Hilbert structure $\int_0^1 \|H'\|^2 ds$. We choose the same connection as before in order to iterate the covariant derivatives of a form. If σ is a n form, we can define its Nualart-Pardoux constant $C(p,n,k)(M)$ and $C'(p,n,k)(M)$.

Theorem I.4. *Let $\sigma(M)$ be a n form over $L_x(M)$ which belongs to all the Nualart-Pardoux spaces. Then $\pi^*\sigma(M) = \sigma(Q)$ is a n form over $L_e(Q)$ which belongs to all the Nualart-Pardoux spaces.*

The proof is clear: the Nualart-Pardoux constants are the same. It is not the same for the next theorem:

Theorem I.5. *Let $\sigma(G)$ be a n form over $P(G)$ which belongs to all the Nualart-Pardoux spaces. Then $(f^*)^*\sigma(G) = \sigma(Q)$ is a n form which belongs to all the Nualart-Pardoux spaces over $L_e(Q)$.*

Proof. Since the functional $\gamma_. \to h(\tau_1^Q)$ belongs to all the Nualart-Pardoux spaces over $L_x(M)$ if h is smooth, because the covariant derivatives of τ_1^Q are given by iterated integrals (See (1.7)), we can work in a region where $L_e(Q)$ is trivial, by using a partition of unity over G associated to the G_i. $L_e(Q)$ is locally a product, and we can speak of basical and (left or right) vertical vector fields. By the lemma A.2. of the appendix, the Nualart-Pardoux norms in terms of basical and right vertical vector fields are equivalent to the Nualart-Pardoux norms in term of the right vertical vector fields and the horizontal vector fields.

It remains to show that $(f^*)^*\sigma$ has locally Nualart-Pardoux constants for the basical and the right vertical vector fields which are finite.

The vertical derivative are given by the vertical derivatives over a pinned path group: the basical one are view by using the derivatives of $g_{i,.}((\tau_1^Q)^{-1})$ which check the Nualart-Pardoux conditions, because (τ_1^Q) checks the Nualart-Pardoux conditions and the $g_{i,s}$ are smooth in τ_1^Q and s together. It remains to solve the problem that we don't consider the form σ over $P(G)$ but the form over $L_{(\tau_1^Q)^{-1}}(G)$ isomorphic to $L_e(G)$ by the map $g_. \to g_{i,.}((\tau_1^Q)^{-1})g_.$ This leads to the vector field $\langle dg_{i,.}((\tau_1^Q)^{-1}, X\rangle g_.$ which is a left vector field over $P(G)$. But a map which checks the Nualart-Pardoux conditions over $P(G)$ for right vector fields checks still the Nualart-Pardoux conditions for left vector fields (See lemma A.3): if we had consider as trivialization the couple of $L_x(M) \times P(G)$ with the trivialization map $(\gamma_., g_.) \to (\gamma_., g_{i,.}((\tau_1^Q)^{-1})g_.)$, the proof would be finished. But a map over $P(G)$ can be reduced into a map over $L_e(G)$ if it satisfies the Nualart-Pardoux conditions (See [27] beginning of the chapter II for the Riemannian case which is more complicated). So if $\sigma(\gamma_., g_.)$ checks the Nualart-Pardoux conditions over $L_x(M) \times P(G)$, it checks still the Nualart-Pardoux conditions over $L_x(M) \times L_e(G)$: the map which is associated is the map $(\gamma_., g_.) \to (g_1)$ which gives $L_x(M) \times L_e(G)$ as a finite codimensional manifold of $L_x(M) \times P(G)$. $\square$

Example. Let $Q \to M$ a principal bundle with structure group G which is supposed simply connected simply laced [39]. Over $P(G)$ we consider the form which at the level of the Lie algebra of $P(G)$ is equal for right vector fields to:

$$c(X,Y) = \frac{1}{8\pi^2}\int_0^1 \langle X_s, dY_s\rangle - \langle Y_s, dX_s\rangle. \tag{1.18}$$

The form $(f^*)^*c$ satisfies over $L_e(Q)$ the Nualart-Pardoux conditions by the theorem I.5.

Let μ be the form over $L_x(M)$:

$$\mu = \frac{1}{8\pi^2}\int_{0<u<s<1} \langle (\tau_s^Q)^{-1}R^Q(d\gamma_s, .)\tau_s^Q \wedge (\tau_u^Q)^{-1}R^Q(d\gamma_u, .)\tau_u^Q\rangle. \tag{1.19}$$

It satisfies the Nualart-Pardoux conditions over $L_x(M)$. Therefore $\pi^*\mu$ satisfies the Nualart-Pardoux conditions over $L_e(Q)$. Let ν be a form such that $d\nu = p_1^Q$, the first Pontryaguin class of the bundle Q which is supposed to be zero in class. Let $\tau(\nu)$ be the two form over $L_x(M)$

$$\tau(\nu) = \int_0^1 \nu(d\gamma_s, ., .). \tag{1.20}$$

It satisfies to the Nualart-Pardoux conditions. Therefore the Carey-Murray two form [8] $F_Q = (f^*)^*c - \pi^*(\mu + \tau(\nu))$ satisfies to the Nualart-Pardoux conditions over the big space $L_e(Q)$.

Theorem I.6. *The space of n form which are smooth in the Nualart-Pardoux sense is independent of the connection ∇^∞.*

Proof. Let us consider two connections ∇^∞ and ∇'^∞. After using a partition of unity associated to the open neighborhoods G_i, since τ_1^Q satisfies to the Nualart-Pardoux conditions, we come back to a situation where the bundle $L_e(Q)$ is trivialized. The Nualart-Pardoux constants of σ with respect of the connection ∇^∞ can be estimated in terms of the basical Nualart-Pardoux constants, which can be estimated themselves in terms of the Nualart-Pardoux constants with respect of the connection ∇'^∞ (See lemma A.2). Therefore the result. $\square$

Let us recall that the exterior derivative of a $n-1$ form σ is defined as follows:

$$\begin{aligned} d\sigma(X_1,\dots,X_n) &= \sum(-1)^{i-1}\langle d\sigma(X_1,\dots,X_{i-1},X_{i+1}\dots X_n),X_i\rangle \\ &+\sum_{i<j}(-1)^{i+j}\sigma\left([X_i,X_j],X_1,\dots,X_{i-1},X_{i+1},\dots,X_{j-1},X_{j+1},\dots,X_n\right). \end{aligned} \tag{1.21}$$

Our goal is to define an exterior derivative over $l_e(Q)$. (1.21) shows that we need to compute some Lie brackets.

- Let us compute the Lie bracket of two (right) vertical vector fields. It is nothing else than $g_t[K_t^1,K_t^2]$ if $X_t^V(K^i)=g_tK_t^i$ for deterministic process K_t^i in the Lie algebra of G.

- In order to compute the Lie bracket of two horizontal vector fields, we work in a local trivialization of $L_e(Q)$. The first horizontal vector fild is given by

$$X^H(H^1)_t=\tau_tH_t^1-K_{i,t}\left(\langle d(\tau_1^Q)^{-1},\tau_.H_.^1\rangle\right)g_t \tag{1.22}$$

  and the second one is given by

$$X^H(H^2)_t=\tau_tH_t^2-K_{i,t}\left(\langle d(\tau_1^Q)^{-1},\tau_.H_.^2\rangle\right)g_t \tag{1.23}$$

  for deterministic H_t^1 and H_t^2. We have:

$$\begin{aligned} [X^H(H^1),X^H(H^2)]_t &= \tau_t\int_0^t\tau_s^{-1}R(d\gamma_s,\tau_sH_s^2)\tau_sH_t^1 \\ &-\langle d(K_{i,t}(\langle d(\tau_1^Q)^{-1},\tau_.H_.^1\rangle),\tau_.H^2\rangle g_t \\ &+\langle K_{i,t}(\langle d(\tau_1^Q)^{-1},\tau_.H_.^1\rangle)K_{i,t}(\langle d(\tau_1^Q)^{-1},\tau_.H_.^2\rangle)\rangle g_t \\ &+\text{antisymmetry}=X^H[\tau_.H_.^1,\tau_.H_.^2]+R^\infty(H_.^1,H_.^2)g_. \end{aligned} \tag{1.24}$$

  In other terms, the Lie bracket of horizontal vector fields is not an horizontal vector field associated to the generalized vector field $[\tau_.H_.^1,\tau_.H_.^2]$: some curvature phenomenon appears, which leads to some extra (left) vertical fields over the fiber (and not some (right) vector fields).

- The Lie bracket of an horizontal vector field $X^H(H)$ (H deterministic) and of a vertical vector field $X^V(K)$ (K deterministic) is equal to zero.

We are ready to state the following theorem:

Theorem I.7. *Let σ be an n form which is smooth in the Nualart-Pardoux sense over $L_e(Q)$. Then $d\sigma$ is a $n+1$ form which is smooth in the Nualart-Pardoux sense over $L_e(Q)$ and its Nualart-Pardoux constants can be estimated in terms of the Nualart-Pardoux constants of σ.*

Proof. Only the contribution of the Lie bracket in (1.21) gives any problem. Since the Lie bracket of two (right) vertical vector fields is still a (right) vertical field, only the contribution of the Lie bracket of two horizontal vector fields put any problem.

We treat first the contribution of $X^H[\tau_. H^1_., \tau_. H^2_.]$: this leads to a Stratonovitch integral in $d\gamma_s$: the lemma A.2 of [27] and more precisely the lemma A.1 of this work allow to show that this contribution satisfies to the Nualart-Pardoux conditions.

We consider now the contribution of $R_t^\infty(H^1_., H^2_.)g_t$ where R_t is a process with finite energy in the Lie algebra, which satisfies to the Nualart-Pardoux conditions because $K_{i,t}$ is smooth in t and because τ_1^Q and τ_s satisfy to the Nualart-Pardoux conditions simultaneously. But

$$R_t^\infty(H^1_., H^2_.)g_t = g_t\left(g_t^{-1}R_t^\infty(H^1, H^2)g_t\right) \tag{1.25}$$

which is a generalized (right) vector field over $L_e(G)$. The proposition A.4 allows to show that this contribution satisfies still the Nualart-Pardoux conditions: the only stochastic integral which appears does not occur from the derivative in time t of R_t^∞ but of the time differential element of g_t^{-1} and of g_t. □

Example. Let $\sigma(G)$ be a n form over $P(G)$ which belongs to all the Nualart-Pardoux spaces over $P(G)$. $d\sigma(G)$ is a $n+1$ form over $P(G)$ which satisfies to the Nualart-Pardoux conditions. We get

$$d(f^*)^*\sigma(G) = (f^*)^*d\sigma(G). \tag{1.26}$$

Namely this property is true if we consider finite energy loop over Q, and reflects some algebraic identities between iterated integrals; these algebraic identities remain true in the stochastic context.

If we consider a n form $\sigma(M)$ over $L_x(M)$ which satisfies to the Nualart-Pardoux conditions, $d\sigma(M)$ is a $n+1$ form over $L_x(M)$ which satisfies to the Nualart-Pardoux conditions, and we have clearly:

$$\pi^*d\sigma(M) = d\pi^*\sigma(M). \tag{1.27}$$

In particular if the first Pontryaguin class of the bundle Q is equal to zero, we can use the result of [8]

$$dF_Q = 0 \tag{1.28}$$

because dF_Q is equal to zero over the finite energy loop space of Q: dF_Q is given by iterated integrals: these formulas remain true in the stochastic context.

Appendix: anticipative stratonovitch integrals

Let us recall the following fact: if $X_s = \tau_s H_s$; $H_0 = H_1 = 0$ is a deterministic vector field (This means, it corresponds to the deterministic vector field H_s), we get the following integration by parts formula, for a cylindrical functional F:

$$E[\langle dF, X\rangle] = E[F \operatorname{div} X]. \tag{a.1}$$

$\operatorname{div} X$ is defined by the formula:

$$\operatorname{div} X = \int_0^1 \langle \tau_s H_s', \delta\gamma_s\rangle + \frac{1}{2}\int_0^1 \langle S_{X_s}, \delta\gamma_s\rangle. \tag{a.2}$$

S is the Ricci tensor and δ the Ito integral with respect to the Levi-Civita connection.

If $K_s g_s$ is a left vector field ($K_0 = K_1 = 0$; K_s deterministic) over the basical loop group (or any pinned path space in the group), we get over the loop group an integration by parts formula analoguous to (a.1), but this time

$$\operatorname{div}(K_. g_.) = \int_0^1 \langle K_s', \delta B_s\rangle \tag{a.3}$$

if $g_.$ is given by the equation (1.1).

If $g_s K_s$ is a right vector field over the basical loop group ($K_.$ deterministic; $K_0 = K_1 = 0$), we get:

$$\operatorname{div}(g_. K_.) = \int_0^1 \langle g_s K_s' g_s^{-1}, \delta B_s\rangle. \tag{a.4}$$

We have:

Proposition A.1. *Let $u(s,t;u,\tilde{s})$ a random variable with value in $T_x(M)$ which satisfies to the Nualart-Pardoux conditions over $L_e(Q)$, the both type of derivatives included and $s,t,u,\tilde{s}$ included. Then the anticipative Stratonovitch integral:*

$$\int_s^t \langle \tau_u u(s,t;u\tilde{s}), d\gamma_u\rangle = I(s,t;\tilde{s}) \tag{a.5}$$

satisfies to the Nualart-Pardoux conditions, the both type of derivatives included, and $s,t,\tilde{s}$ included.

Proof. We integrate by part in order to compute $E[(I(s,t;\tilde{s})^p]$ for some even integer p. $u(s,t;u,\tilde{s})$ is a vector field over the loop space if $\int_s^t u(s,t;u,\tilde{s})du = 0$. If the previous inequality is not checked, we can remove the average of u in order to recognize a vector field over the based loop space.

Let us put:

$$X_{s,t,\tilde{s}}(u) = \tau_u \int_0^u u(s,t;v,\tilde{s})dv. \tag{a.6}$$

In the definition of the divergence, we have to add the counterterm $\frac{1}{2}\int_0^1 \langle S_{X_{s,t,\tilde{s}}(u)}, \delta\gamma_u\rangle$ which is only apparently an anticipative integral, by integrating by part, in order to recognize the beginning of a curved Skorohod integral (See [24], [25]). The second counterterm we have to add is the integral of the kernels of some H-derivative of $u(s,t;u,\tilde{s})$ in order to recognize a complete Skorohod integral. For that, we begin by studying $E[(I(s,t;\tilde{s})^p]$ for the discrete approximation of the anticipative integral by Riemann sum of $\dfrac{1}{t_{i+1}-t_1}\displaystyle\int_{t_i}^{t_{i+1}} u(s,t;v,\tilde{s})dv$ for a suitable subdivision of $[0,1]$. We find a sum of integral over $[s,t]^k$ of polynomial expression in the derivatives of u and of the derivatives of $\tau_.$ and of $d\gamma_u$, with possible contraction over the diagonals. Let us recall that:

$$\nabla_X \tau_s = \tau_s \int_0^s \tau_u^{-1} R(d\gamma_u, X_u)\tau_u. \tag{a.7}$$

We work in a small trivialization of the bundle such that we can speak of basical derivatives (associated to basical vector fields) instead of horizontal vector derivatives (associated to horizontal vector fields): the derivatives which appear after integrating by parts in $E[(I(s,t;\tilde{s})^p]$ are basical derivatives, and not the horizontal derivatives which are given by the Nualart-Pardoux norms. This leads apparently to a problem, which is solved by the Lemma A.2: we postpone the proof of this lemma later.

Lemma A.2. *Let be a local trivialization of the bundle $L_e(Q))$, such that we can speak of a basical vector field and of a (right) vertical vector field. We can speak of the Nualart-Pardoux constants of a form σ for the basical vector fields and the (right) vertical vector field. They can be estimated in terms of the Nualart-Pardoux constants in terms of horizontal and (right) vertical vector fields. The converse is true.*

In the previous discussion, we did not speak of the fact we have removed to $u(s,t;u,\tilde{s})$ the quantity $\dfrac{1}{t-s}\displaystyle\int_s^t u(s,t;u,\tilde{s})du$ as well for the higher derivatives, because some auxiliary terms which arise from the derivative of the parallel transport can appear. We find after this remark a finite sum of integrals over $[s,t]^k$ of polynomial expressions in the basical derivatives of u with possible contraction over the diagonals, and some expressions in the basical derivatives of the parallel transport and of the curvature tensor. It is possible to divide this integral by a power k' of $t-s$, but we have always $k'+p/2 \leq k$: namely the division by $t-s$ appears by an operation of averaging in order to recognize a vector field over the loop space and not from an integration by parts, which leads to at most $p/2$ contractions.

We deduce that the discrete approximation of the integral converges in all the L^p to $I(s,t;\tilde{s})$, this from the regularity assumption over the kernels of $u(s,t;u,\tilde{s})$. Moreover $\|I(s,t;\tilde{s})\|_{L^p}^p$ is a sum of iterated integrals of the basical kernels of $u(s,t;u,\tilde{s})$ over $[s,t]^k$ with contraction of the basical kernels and half-limits over the diagonals. Therefore $\|I(s,t;\tilde{s})\|_{L^p}$ is bounded by the Nualart-Pardoux Sobolev norms of the vector fields by the lemma A.2.

In order to check the regularity assumption in $s, t, \tilde{s}$, we suppose in order to simplify that $s_1 < \tilde{s}_{1,1} < \ldots < \tilde{s}_{n,1} < t_1$ and that $s_2 < \tilde{s}_{1,2} \ldots < \tilde{s}_{n,2} < t_2$. We splitt the integral between s_1 and t_1 into smaller integrals over the intervals defined by the contiguous time of the subdivision, as it was already done in the proof of the lemma A.2 of [27].

We consider the integral of $u(s_1, t_1; u, \tilde{s}_1)$ and of $u(s_2, t_2; u, \tilde{s}_2)$ over $[\tilde{s}_{i,1}, \tilde{s}_{i+1,1}] \cap [\tilde{s}_{i,2}, \tilde{s}_{i+1,2}]$. We substract the necessary counterterm in order to get an anticipative integral over the based loop space, and we get the expectation of an integral power of it as before. The main remark is that we remain in the same connected component of the parameter set outside the diagonals in the integrals which are got. We get an estimate of the L^p norm in term of $\sqrt{| \tilde{s}_{i,1} - \tilde{s}_{i,2} |} + \sqrt{| \tilde{s}_{i+1,1} - \tilde{s}_{i+1,2} |}$. If we integrate outside the intersections, the distance between the extremities of the considered intervals is smaller than the sum of the distance between $\tilde{s}_{i,1}$ and $\tilde{s}_{i,2}$. We get an estimate in terms of the Nualart-Pardoux constants of the second type (1.17) and $\sqrt{| \tilde{s}_{i,1} - \tilde{s}_{i,2} |} + \sqrt{| \tilde{s}_{i+1,1} - \tilde{s}_{i+1,2} |}$.

In order to finish the proof, let us precise the effect of the operation $u(s, t; u; \tilde{s}) \to u(s, t; u, \tilde{s}) - \frac{1}{t-s} \int_s^t u(s, t; u, \tilde{s}) du$ over I in order to get a tangent vector over the based loop space. It has only the consequence to substract to the initial anticipative integral the non-anticipative Stratonovitch integral $\int_s^t < \tau_u C, d\gamma_u >$ for a suitable C.

For the derivative of I, we deduce from the previous discussion, that we can take derivative under the sign $\int$, for the vertical and horizontal vector fields. We conclude by using the fact that:

$$\nabla_X d\gamma_s = \tau_s H'_s ds \tag{a.8}$$

if $X_s = \tau_s H_s$. □

Proof of the lemma A.2. The lemma A.2 will be proved if, after choosing a trivialization which does not affect the Nualart-Pardoux conditions as we will see later, we prove the following fact: a functional which belongs to all the Nualart-Pardoux spaces for the vertical derivatives along right vector fields $g_. K_.$ belongs to all the Nualart-Pardoux spaces for the vertical derivatives along (left) vector fields $K_. g_.$, and its Nualart-Pardoux constants can be estimated in term of the Nualart-Pardoux constants for the first type of vector fields. Namely K_s depends in the definition of an horizontal vector field only on $(\tau_1^Q)^{-1}$ and its derivatives, which satisfy to the Nualart-Pardoux conditions if we consider horizontal vector fields.

A vector field $g_s K_s$ corresponds to the vector field $g_s K'_s g_s^{-1}$ over the leading brownian motion. Let us suppose we can get a prolongation over the path group of our functional which checks the (right) Nualart-Pardoux conditions and whose the (right) Nualart-Pardoux norms over the whole path group can be estimated into the Nualart-Pardoux norms over the pinned path group. We get therefore a functional over the leading brownian motion, which belongs to all the Sobolev spaces. The flat derivative of g_s checks s included the Nualart-Pardoux conditions: the flat Nualart-Pardoux norms can be estimated in terms of the Nualart-Pardoux norms over the pinned loop group. Moreover a (left) vector field $K_. g_.$ gives the vector $[K_s, dB_s] + K'_s ds$, which is a generalized flat vector field. We can repeat therefore the proof of the theorem A.1 of [27] in order to conclude. Let us precise this statement.

Let us precise the prolongation: we do in order to simplify as we were working over the based loop group. We consider a path in G with the condition that g_1 is closed from e. We associate to a path g_s the loop $g_s \exp[-s \operatorname{Log} g_1]$. The vector field $g_s K_s$ is transformed into the vector field:

$$\begin{aligned} &g_s \exp[-s \operatorname{Log} g_1] \left(\exp[s \operatorname{Log} g_1] K_s \exp[-s \operatorname{Log} g_1]\right) \\ &\quad + g_s \exp[-s \operatorname{Log}_1] \left(\exp[s \operatorname{Log} g_1] \frac{\partial}{\partial g_1} \exp[-s \operatorname{Log} g_1] K_1]\right). \end{aligned} \tag{a.9}$$

Therefore a vector field $g_s K_s$ is transformed into

$$g_s \exp[s \operatorname{Log} g_1] \exp[s \operatorname{Log}_1] \left(K_s \exp[-s \operatorname{Log} g_1] + \frac{\partial}{\partial g_1} \exp[-s \operatorname{Log} g_1] K_1\right). \tag{a.10}$$

g_1 satisfies to the Nualart-Pardoux conditions and the law of $g_s \exp[-s \operatorname{Log} C]$ is absolutely continuous with a density which belongs to the L^p with respect to the law of g_s. It follows than the functional over the path group:

$$F_{tot}(g_.) = F\left(g_. \exp[-. \operatorname{Log} g_1]\right) \phi(g_1). \tag{a.11}$$

where $\phi(g_1)$ is a cutoff functional destinated to ensure the existence of $\operatorname{Log} g_1$ belongs to all the (right) vertical Nualart-Pardoux Sobolev spaces, and its vertical Nualart-Pardoux constants can be estimated by the vertical Nualart-Pardoux constants over the pinned loop group.

Let us now repeat the scheme of the proof of the theorem A.1. of [27]. Let $\sigma_n = \sigma(B_{u_1}, \ldots, B_{u_n})$, $0 = u_1 < \ldots < u_n = 1$, $u_{i+1} - u_i = 1/n$ for a dyadic subdivision of length 2^k. Let $F_n = E[F|\sigma_n]$. It is a functional which depends only from a finite number of flat variables. Since $dB_s = dg_s g_s^{-1}$, F_n belongs to all the Sobolev spaces for $g_.$ related to the (left) vector fields $K_. g_.$. The flat kernel of F_n are given by:

$$\frac{1}{\prod(u_{k_{i+1}} - u_{k_i})} \int \int_{\prod[u_{k_i}, u_{k_{i+1}}]} E[k(t_1, \ldots, t_r)|\, \sigma_n] dt_1 \ldots dt_n. \tag{a.12}$$

k denotes a flat kernel of F. $F_n = G_n(g_.)$. We will show that G_n is a Cauchy sequence for the Sobolev spaces relatively to the left vector fields $K_. g_.$ (We call them the left Sobolev spaces). The kernel associated to G_n are Stratonovitch integrals in dB_s. We use for this the formula:

$$\Delta B_{u_i} = \int_{u_i}^{u_{i+1}} dg_s g_s^{-1}. \tag{a.13}$$

The derivative of dg_s is given by $K'_s g_s ds + K_s dg_s$ and the derivative of g_s^{-1} is given by $-g_s^{-1} K_s$. The kernel of the derivative of G_n are iterated Stratonovitch integrals with frozen time: we integrate expressions in the flat derivatives of F_n and algebraic expressions in g_s, g_s^{-1} and dB_s which are non anticipative.

It remains to pass at the limit: we see that the half limits over the diagonals of the flat kernels of F appear when we go to the limit. In order to pass at the limit, there are two procedures as in the proof of the theorem A.1 of [27]. Π_n is the procedure of conditional expectation over the σ-algebra σ_n and χ_n the procedure of averaging. Let $\operatorname{Ker} F$ be a flat kernel of F. The associated flat kernel of F_n is $\Pi_n \chi_n \operatorname{Ker} F$. We get:

$$\Pi_n \chi_n \operatorname{Ker} F - \Pi_{n'} \chi_{n'} \operatorname{Ker} F = \Pi_n (\chi_n - \chi_{n'}) \operatorname{Ker} F + (\Pi_n - \Pi_{n'}) \chi_{n'} \operatorname{Ker} F. \tag{a.14}$$

$\Pi_n \operatorname{Ker} F$ satisfies to the Nualart-Pardoux conditions with the same constants than $\operatorname{Ker} F$. We can apply the Kolmogorov lemma. Let us denote by U any connected component of the complement of the diagonals. We get:

$$\operatorname{Sup}_{U \times U} \frac{\| E[k(t_1, \ldots, t_r)|F_n] - E[k(t'_1, \ldots, t'_r)|F_n] \|_{L^p}}{\sum | t_i - t'_i |^\alpha} < \infty \tag{a.15}$$

for a certain $\alpha < 1/2$.

The previous L^p norms are smaller than the L^p norms which are got, when we don't take any conditional expectation. When we take the L^p norm of the difference of the Stratonovitch integral of $\Pi_n \chi_n \operatorname{Ker} F$ and of $\Pi_{n'} \chi_{n'} \operatorname{Ker} F$, we get iterated integrals without the stochastic term dB_s with some half limits of the kernels of $F_n - F_{n'}$ over the diagonals. We splitt it into an expression polynomial in $(\Pi_n - \Pi_{n'}) \chi_n \operatorname{Ker} F$ and a polynomial expression in $(\chi_n - \chi_{n'}) \Pi_{n'} \operatorname{Ker} F$. The first type of expression goes uniformly to 0 in all the L^p by using the Kolmogorov lemma when $n \to \infty$ and $n' \to \infty$. It is the same for the second type of expressions, by using the criterium of continuity of the kernels of F. G_n is a Cauchy sequence in the (left) Sobolev spaces.

Moreover, by the Kolmogorov lemma:

$$E\left[k(s_1, \ldots, s_r)|\sigma_n\right] - k(s_1, \ldots, s_r) \to 0 \tag{a.16}$$

uniformly in $(s_1, \ldots, s_r)$ outside the diagonals in all the L^p. The derivatives of G_n tend to the Stratonovitch integral which are got formally when we replace the flat dH_s by $[K_s, dB_s] + K'_s ds$.

It remains to restricts the functional G over the based loop group as well as its kernels. It is the purpose of the quasi-sure analysis: we consider the measure

$$f \to E[Gf(g_1)] \tag{a.17}$$

which has a density. The (left) Nualart-Pardoux Sobolev norms for the pinned loop group are estimated in terms of the (left) Nualart-Pardoux Sobolev norms over the path group. In order to show that , let us consider the vector fields $X^1_. = K^1_. g_.$,..., $X^r_. = K^r_. g_.$. We get the following integration by parts formula for any integer p:

$$\begin{aligned} E\Big[|G(s_1, \ldots, s_r) - \tilde{G}(\tilde{s}_1, \ldots, \tilde{s}_r)|^p \langle d\langle \ldots \langle df(g_1), X^r \rangle \ldots \rangle X^1 \rangle\Big] = \\ E[\xi f(g_1)], \end{aligned} \tag{a.18}$$

where ξ is a polynomial expression in $G(s_1, \ldots, s_r) - \tilde{G}(s_1, \ldots, s_r)$ and its (left) kernels integrated and the divergence of the vector fields X and their derivatives. We apply the lemme A.2 of [27] in order to conclude.

We have proved the lemma A.2 for functionals: for forms, we associate to a form over (right) vertical vector fields a form over flat vector fields, and after we do the transformation $dH_s \to [K, s, dB_s] + K'_s ds$ in order to get a form over the (left) vertical vector fields: we get Stratonovitch iterated integrals, and we apply the lemma A.2 of [27] in order to conclude.

The last point it remains to clarify is that the operation of trivialization in order to come to a product situation has no effect over the right vertical Nualart-Pardoux Sobolev norms. If $(\tau_1^Q)^{-1} \in G_i$, we can find a mollifer $f(g_1)$ which belongs to all the Nualart-Pardoux spaces with compact support in a small neighborhood of G_i. We put over the path group

$$F_{tot}(g_.) = F\left(g_. \exp[-s \operatorname{Log}[\tau_1^Q g_1]]\right) f(g_1). \tag{a.19}$$

We enlarge by this the functional over the vertical pinned path going from e to $(\tau_1^Q)^{-1}$ to a functional over the total space of the path group which checks still the Nualart-Pardoux conditions: its right Nualart-Pardoux norms can be estimated in term of the Nualart-Pardoux norms of the non extended functional, since (τ_1^Q) satisfies to the Nualart-Pardoux conditions. We perform after the transformation:

$$F_{tot}(g_.) \to F_{tot}\left(g_{i,.}((\tau_1^Q)^{-1})g_.\right). \tag{a.20}$$

The left Nualart-Pardoux constants of the new global functional can be estimated in term of the right Nualart-Pardoux constants of $F_{tot}(g_.)$. Since the law og $g_i.\left((\tau_1^Q)^{-1}\right)g_.$ is equivalent to the law of $g_.$ with a density which belongs to all the L^p, we deduce that the global Nualart-Pardoux constants of $F_{tot}\left(g_{i.}\left((\tau_1^Q)^{-1}g_.\right)\right)$ can be estimated in terms of the right Nualart-Pardoux constants of $F(g_.)$. □

We had shown too the following lemma:

Lemma A.3. *A functional over a trivialization which checks the (right) Nualart-Pardoux conditions for (right) vertical vector fields over $L_e(G)$ $(orP(G))$ checks still the (left) Nualart-Pardoux conditions for (left) vertical vector fields.*

We get the proposition:

Proposition A.4. *Let $u(s,t;u\tilde{s})$ a random application with values in LieG which belongs to the (right) Nualart-Pardoux spaces over the total space, $s,t,\tilde{s}$ included. Let $I(s,t,\tilde{s})$ the anticipative Stratonovitch integral:*

$$I(s,t;\tilde{s}) = \int_s^t \langle g_u u(s,t;u,\tilde{s}), dg_u\rangle. \tag{a.21}$$

$I(s,t;\tilde{s})$ checks the Nualart-Pardoux conditions over the total space $L_e(Q)$.

Proof. We begin to write $dg_u = dB_u g_u$, such that we come back to a Stratonovitch integral

$$\int_s^t \langle \tilde{u}(s,t;u,\tilde{s}), dB_u\rangle, \tag{a.22}$$

where $\tilde{u}$ checks still the Nualart-Pardoux conditions. We extend $\tilde{u}(s,t;u,\tilde{s})$ over the path group, such that it checks still the Nualart-Pardoux conditions over the path group. We use the isometry given in the beginning of the proof of the lemma A.2. We get $\tilde{u}$ which depends on B and γ which checks the Nualart-Pardoux conditions in B and γ. We come back to the flat case and to a flat Stratonovitch integral. We can use the results of [27]:

$I(s,t;\tilde{s})$ extended satisfies to the Nualart-Pardoux conditions in B and γ, for the basical vector field in γ. this from the lemma A.2 of [27]. Then $I(s,t;\tilde{s})$ extended satisfies to the Nualart-Pardoux conditions in $g_.$ and $\gamma_.$, for basical vector fields in γ. By the lemma A.2, it satisfies to the Nualart-Pardoux conditions in $g_.$ and $\gamma_.$ for horizontal vector fields, which are intrisically defined. $\square$

Acknowledgments

We thank the warm hospitality of the Max Planck Institut for Mathematics in Bonn where this work was done.

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