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Symmetry of the Schrödinger Equation with Variable Potential

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Abstract

We study symmetry properties of the Schrödinger equation with the potential as a new dependent variable, i.e., the transformations which do not change the form of the class of equations. We also consider systems of the Schrödinger equations with certain conditions on the potential. In addition we investigate symmetry properties of the equation with convection term. The contact transformations of the Schrödinger equation with potential are obtained.

1. Introduction

Let us consider the following generalization of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + \Delta\psi + W(t,\vec{x},|\psi|)\psi + V_a(t,\vec{x})\frac{\partial \psi}{\partial x_a} = 0, \tag{1}$$

where $\Delta = \dfrac{\partial^2}{\partial x_a \partial x_a}$, $a = \overline{1,n}$, $\psi = \psi(t,\vec{x})$ is an unknown complex function, $W = W(t,\vec{x},|\psi|)$ and $V_a = V_a(t,\vec{x})$ are potentials of interaction.

When $V_a = 0$ in (1), the standard Schrödinger equation is obtained. Symmetry properties of this equation were thoroughly investigated (see, e.g., [1]–[4]). For arbitrary $W(t,\vec{x})$, equation (1) admits only the trivial group of identical transformations $\vec{x} \to \vec{x}' = \vec{x}$, $t \to t' = t$, $\psi \to \psi' = \psi$ [1, 3].

In [5]–[7], a method for extending the symmetry group of equation (1) was suggested. The idea lies in the fact that, in equation (1), we assume that $W(t,\vec{x},|\psi|)$ is a new dependent variable on equal conditions with ψ. This means that equation (1) is regarded as a nonlinear equation even in the case where the potential W does not depend on ψ. Indeed, equation (1) is a set of equations when V is a certain set of arbitrary smooth functions.

2. Symmetry of the Schrödinger Equation with Potential

Using this idea, we obtain the invariance algebra of the Schrödinger equation with potential, i.e.,

$$i\frac{\partial\psi}{\partial t}+\Delta\psi+W(t,\vec{x},|\psi|)\psi=0. \tag{2}$$

Theorem 1. *Equation (2) is invariant under the infinite-dimensional Lie algebra with infinitesimal operators of the form*

$$\begin{aligned}
&J_{ab}=x_a\partial_{x_b}-x_b\partial_{x_a},\\
&Q_a=U_a\partial_{x_a}+\frac{i}{2}\dot{U}_a x_a\left(\psi\partial_\psi-\psi^*\partial_{\psi^*}\right)+\frac{1}{2}\ddot{U}_a x_a\partial_W,\\
&Q_A=2A\partial_t+\dot{A}x_c\partial_{x_c}+\frac{i}{4}\ddot{A}x_cx_c\left(\psi\partial_\psi-\psi^*\partial_{\psi^*}\right)\\
&\qquad -\frac{n\dot{A}}{2}\left(\psi\partial_\psi+\psi^*\partial_{\psi^*}\right)+\left(\frac{1}{4}\dddot{A}\,x_cx_c-2W\dot{A}\right)\partial_W,\\
&Q_B=iB\left(\psi\partial_\psi-\psi^*\partial_{\psi^*}\right)+\dot{B}\partial_W,\quad Z_1=\psi\partial_\psi,\quad Z_2=\psi^*\partial_{\psi^*},
\end{aligned} \tag{3}$$

where $U_a(t), A(t), B(t)$ are arbitrary smooth functions of t, over the index c we mean summation from 1 to n, $a,b=\overline{1,n}$, and over the repeated index a there is no summation. The upper dot stands for the derivative with respect to time.

Note that the invariance algebra (3) includes the operators of space ($U_a=1$) and time ($A=1/2$) translations, the Galilei operator ($U_a=t$), the dilation ($A=t$) and projective ($A=t^2/2$) operators.

Proof of Theorem 1. We seek the symmetry operators of equation (2) in the class of first-order differential operators of the form:

$$X=\xi^\mu(t,\vec{x},\psi,\psi^*)\partial_{x_\mu}+\eta(t,\vec{x},\psi,\psi^*)\partial_\psi+\eta^*(t,\vec{x},\psi,\psi^*)\partial_{\psi^*}+\rho(t,\vec{x},\psi,\psi^*,W)\partial_W. \tag{4}$$

Using the invariance condition [1, 8, 9] of equation (2) under operator (4) and the fact that $W=W(t,\vec{x},|\psi|)$, i.e., $\psi\dfrac{\partial W}{\partial\psi}=\psi^*\dfrac{\partial W}{\partial\psi^*}$, we obtain the system of determining equations:

$$\begin{aligned}
&\xi^j_\psi=\xi^j_{\psi^*}=0,\quad \xi^0_a=0,\ \xi^a_a=\xi^b_b,\quad \xi^a_b+\xi^b_a=0,\quad \xi^0_0=2\xi^a_a,\\
&\eta_{\psi^*}=0,\quad \eta_{\psi\psi}=0,\quad \eta_{\psi a}=(i/2)\xi^a_0,\\
&\eta^*_\psi=0,\quad \eta^*_{\psi^*\psi^*}=0,\quad \eta^*_{\psi^*a}=-(i/2)\xi^a_0,\\
&i\eta_0+\eta_{cc}-\eta_\psi W\psi+2W\xi^n_n\psi+W\eta+\rho\psi=0,\\
&-i\eta^*_0+\eta^*_{cc}-\eta^*_{\psi^*}W\psi^*+2W\xi^n_n\psi^*+W\eta^*+\rho\psi^*=0,\\
&\rho_\psi=\rho_{\psi^*}=0,
\end{aligned} \tag{5}$$

where an index j varies from 0 to n, $a,b=\overline{1,n}$, over the repeated index c we mean the summation from 1 to n, and over the indices a,b there is no summation.

We solve system (5) and obtain the following result:

$$\begin{aligned}
&\xi^0=2A,\quad \xi^a=\dot{A}x_a+C^{ab}x_b+U_a,\quad a=\overline{1,n},\\
&\eta=\frac{i}{2}\left(\frac{1}{2}\ddot{A}x_cx_c+\dot{U}_cx_c+B\right)\psi,\quad \eta^*=-\frac{i}{2}\left(\frac{1}{2}\ddot{A}x_cx_c+\dot{U}_cx_c+E\right)\psi^*,\\
&\rho=\frac{1}{2}\left(\frac{1}{2}\dddot{A}\,x_cx_c+\ddot{U}_cx_c+\dot{B}\right)-\frac{n}{2}i\ddot{A}-2W\dot{A},
\end{aligned}$$

where A, U_a, B are arbitrary functions of t, $E = B - 2in\dot{A} + C_1$, $C^{ab} = -C^{ba}$ and C_1 are arbitrary constants. The theorem is proved.

The operators Q_B generate the finite transformations:

$$\begin{cases} t' = t, \quad \vec{x}\,' = \vec{x}, \\ \psi' = \psi \exp(iB(t)\alpha), \\ {\psi^*}' = \psi^* \exp(-iB(t)\alpha), \\ W' = W + \dot{B}(t)\alpha, \end{cases} \tag{6}$$

where α is a group parameter, $B(t)$ is an arbitrary smooth function.

Using the Lie equations, we obtain that the following transformations correspond to the operators Q_a:

$$\begin{cases} t' = t, \quad x_a' = U_a(t)\beta_a + x_a, \quad x_b' = x_b \ \ (b \neq a), \\ \psi' = \psi \exp\left(\dfrac{i}{4}\dot{U}_a U_a \beta_a^2 + \dfrac{i}{2}\dot{U}_a x_a \beta_a\right), \\ {\psi^*}' = \psi^* \exp\left(-\dfrac{i}{4}\dot{U}_a U_a \beta_a^2 - \dfrac{i}{2}\dot{U}_a x_a \beta_a\right), \\ W' = W + \dfrac{1}{2}\ddot{U}_a x_a \beta_a + \dfrac{1}{4}\ddot{U}_a U_a \beta_a^2, \end{cases} \tag{7}$$

where $\beta_a (a = \overline{1, n})$ are group parameters, $U_a = U_a(t)$ are arbitrary smooth functions, there is no summation over the index a. In particular, if $U_a(t) = t$, then the operators Q_a are the standard Galilei operators

$$G_a = t\partial_{x_a} + \frac{i}{2}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right). \tag{8}$$

For the operators Q_A, it is difficult to write out the finite transformations in the general form. We consider several particular cases:
(a) $A(t) = t$. Then

$$Q_A = 2t\partial_t + x_c\partial_{x_c} - \frac{n}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2W\partial_W$$

is a dilation operator generating the transformations

$$\begin{cases} t' = t\exp(2\lambda), \quad x_c' = x_c \exp(\lambda), \\ \psi' = \exp\left(-\dfrac{n}{2}\lambda\right)\psi, \quad {\psi^*}' = \exp\left(-\dfrac{n}{2}\lambda\right)\psi^*, \\ W' = W\exp(-2\lambda), \end{cases} \tag{9}$$

where λ is a group parameter.
(b) $A(t) = t^2/2$. Then

$$Q_A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_c x_c\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{n}{2}t\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2tW\partial_W$$

is the operator of projective transformations:

$$\begin{cases} t' = \dfrac{t}{1-\mu t}, \quad x'_c = \dfrac{x_c}{1-\mu t}, \\ \psi' = \psi(1-\mu t)^{n/2} \exp\left(\dfrac{i x_c x_c \mu}{4(1-\mu t)}\right), \\ \psi^{*\prime} = \psi^*(1-\mu t)^{n/2} \exp\left(\dfrac{-i x_c x_c \mu}{4(1-\mu t)}\right), \\ W' = W(1-\mu t)^2, \end{cases} \tag{10}$$

μ is an arbitrary parameter.

Consider the example. Let

$$W = \frac{1}{\vec{x}^{\,2}} = \frac{1}{x_c x_c}. \tag{11}$$

We describe how new potentials are generated from potential (11) under transformations (6), (7), (9), (10).

(i) Q_B:

$$W = \frac{1}{x_c x_c} \to W' = \frac{1}{x_c x_c} + B(t)\alpha \to W'' = \frac{1}{x_c x_c} + B(t)(\alpha + \alpha') \to \cdots,$$

where $B(t)$ is an arbitrary smooth function, α and α' are arbitrary real parameters.

(ii) Q_a:

$$W = \frac{1}{x_c x_c} \to W',$$

$$W' = \frac{1}{(x_a - U_a(t)\beta_a)^2 + x_b x_b} + \frac{1}{4}\ddot{U}_a U_a \beta_a^2 + \frac{1}{2}\ddot{U}_a \beta_a (x_a - U_a \beta_a),$$

$$W' \to W'',$$

$$\begin{aligned} W'' = {} & \frac{1}{(x_a - U_a(t)(\beta_a + \beta'_a))^2 + x_b x_b} + \frac{1}{4}\ddot{U}_a U_a (\beta_a^2 + \beta_a'^2) \\ & + \frac{1}{2}\ddot{U}_a (\beta_a + \beta'_a)(x_a - U_a(\beta_a + \beta'_a)) + \frac{1}{2}\ddot{U}_a U_a \beta_a \beta'_a \to \cdots, \end{aligned}$$

where U_a are arbitrary smooth functions, β_a and β'_a are real parameters, there is no summation over a but there is summation over b ($b \neq a$). In particular, if $U_a(t) = t$, then we have the standard Galilei operator (8) and

$$W = \frac{1}{x_c x_c} \to W' = \frac{1}{(x_a - t\beta_a)^2 + x_b x_b} \to W'' = \frac{1}{(x_a - t(\beta_a + \beta'_a))^2 + x_b x_b} \to \cdots$$

(iii) Q_A for $A(t) = t$ or $A(t) = t^2/2$ do not change the potential, i.e.,

$$W = \frac{1}{x_c x_c} \to W' = \frac{1}{x_c x_c} \to W'' = \frac{1}{x_c x_c} \to \cdots$$

3. The Schrödinger Equation and Conditions for the Potential

Consider several examples of the systems in which one of the equations is equation (2) with potential $W = W(t, \vec{x}\,)$, and the second equations is a certain condition for the potential W. We find the invariance algebras of these systems in the class of operators

$$X = \xi^\mu(t, \vec{x}, \psi, \psi^*, W)\partial_{x_\mu} + \eta(t, \vec{x}, \psi, \psi^*, W)\partial_\psi \\ + \eta^*(t, \vec{x}, \psi, \psi^*, W)\partial_{\psi^*} + \rho(t, \vec{x}, \psi, \psi^*, W)\partial_W.$$

(i) Consider equation (2) with the additional condition for the potential, namely the Laplace equation.

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t, \vec{x}\,)\psi = 0, \\ \Delta W = 0. \end{cases} \tag{12}$$

System (12) admits the infinite-dimensional Lie algebra with the infinitesimal operators

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \\ &Q_a = U_a\partial_{x_a} + \frac{i}{2}\dot{U}_a x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}\ddot{U}_a x_a\partial_W, \quad a = \overline{1, n}, \\ &D = x_c\partial_{x_c} + 2t\partial_t - \frac{n}{2}\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2W\partial_W, \\ &A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_c x_c\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{n}{2}t\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2Wt\partial_W, \\ &Q_B = iB(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \dot{B}\partial_W, \quad Z_1 = \psi\partial_\psi, \;\; Z_2 = \psi^*\partial_{\psi^*}, \end{aligned} \tag{13}$$

where $U_a(t)$ $(a = \overline{1, n}\,)$ and $B(t)$ are arbitrary smooth functions. In particular, algebra (13) includes the Galilei operator (8).

(ii) The condition for the potential is the heat equation.

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t, \vec{x}\,)\psi = 0, \\ W_0 + \lambda\Delta W = 0. \end{cases} \tag{14}$$

The maximal invariance algebra of system (14) is

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \\ &D = 2t\partial_t + x_c\partial_{x_c} - \frac{n}{2}\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2W\partial_W, \\ &Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \quad Z_3 = it\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \partial_W. \end{aligned}$$

(iii) The condition for the potential is the wave equation.

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t, \vec{x}\,)\psi = 0, \\ \Box W = 0. \end{cases} \tag{15}$$

The maximal invariance algebra of system (15) is

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \\ &Z_3 = it\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \partial_W, \quad Z_4 = it^2\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + 2t\partial_W. \end{aligned}$$

(iv) The condition for the potential is the Hamilton-Jacobi equation.

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x})\psi = 0, \\ \dfrac{\partial W}{\partial t} - \lambda\dfrac{\partial W}{\partial x_a}\dfrac{\partial W}{\partial x_a} = 0. \end{cases} \tag{16}$$

The maximal invariance algebra is

$$P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},$$
$$Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \quad Z_3 = it(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \partial_W.$$

(v) Consider very important and interesting case in $(1+1)$-dimensional space-time where the condition for the potential is the KdV equation.

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \dfrac{\partial^2\psi}{\partial x^2} + W(t,x)\psi = 0, \\ \dfrac{\partial W}{\partial t} + \lambda_1 W\dfrac{\partial W}{\partial x} + \lambda_2\dfrac{\partial^3 W}{\partial x^3} = F(|\psi|), \quad \lambda_1 \neq 0. \end{cases} \tag{17}$$

For an arbitrary $F(|\psi|)$, system (17) is invariant under the Galilei operator and the maximal invariance algebra is the following:

$$\begin{gathered} P_0 = \partial_t, \quad P_1 = \partial_x, \quad Z = i\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right), \\ G = t\partial_x + \frac{i}{2}\left(x + \frac{2}{\lambda_1}t\right)\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{\lambda_1}\partial_W. \end{gathered} \tag{18}$$

For $F = C =$ const, system (17) admits the extension, namely, it is invariant under the algebra $\langle P_0, P_1, G, Z_1, Z_2\rangle$, where P_0, P_1, G have the form (18) and $Z_1 = \psi\partial_\psi$, $Z_2 = \psi^*\partial_{\psi^*}$.

The Galilei operator G generates the following transformations:

$$\begin{cases} t' = t, \quad x' = x + \theta t, \quad W' = W + \dfrac{1}{\lambda_1}\theta, \\ \psi' = \psi\exp\left(\dfrac{i}{2}\theta x + \dfrac{i}{\lambda_1}\theta t + \dfrac{i}{4}\theta^2 t\right), \\ \psi^{*\prime} = \psi^*\exp\left(-\dfrac{i}{2}\theta x - \dfrac{i}{\lambda_1}\theta t - \dfrac{i}{4}\theta^2 t\right), \end{cases}$$

where θ is a group parameter. Here, it is important that $\lambda_1 \neq 0$, since otherwise, system (17) does not admit the Galilei operator.

4. Finite-dimensional Subalgebras

Algebra (3) is infinite-dimensional. We select certain finite-dimensional subalgebras from it. In particular, we give the examples of functions $U_a(t)$ and $B(t)$, for which the subalgebra generated by the operators

$$P_0, \; P_a, \; J_{ab}, \; Q_a, \; Q_B, \; Z_1, \; Z_2 \tag{19}$$

is finite-dimensional.

(a) $U_a(t) = \exp(\gamma t)$. In this case, subalgebra (19) has the form

$$\begin{aligned}
&P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,\\
&Q_a = e^{\gamma t}\left(\partial_{x_a} + \frac{i}{2}\gamma x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}\gamma^2 x_a\partial_W\right), \quad a = \overline{1,n},\\
&Q_B = e^{\gamma t}\left(i\psi\partial_\psi - i\psi^*\partial_{\psi^*} + \gamma\partial_W\right).
\end{aligned}$$

(b) $U_a(t) = C_1\cos(\nu t) + C_2\sin(\nu t)$. Then subalgebra (19) has the form:

$$\begin{aligned}
&P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,\\
&Q_a^{(1)} = \cos(\nu t)\partial_{x_a} - \frac{i}{2}\nu\sin(\nu t)x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{1}{2}\nu^2\cos(\nu t)x_a\partial_W,\\
&Q_a^{(2)} = \sin(\nu t)\partial_{x_a} + \frac{i}{2}\nu\cos(\nu t)x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{1}{2}\nu^2\sin(\nu t)x_a\partial_W,\\
&X_1 = i\sin(\nu t)\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \nu\cos(\nu t)\partial_W,\\
&X_2 = i\cos(\nu t)\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \nu\sin(\nu t)\partial_W.
\end{aligned}$$

(c) $U_a(t) = C_1t^k + C_2t^{k-1} + \cdots + C_kt + C_{k+1}$. Then subalgebra (19) has the form:

$$\begin{aligned}
&P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,\\
&Q_a^{(1)} = t^k\partial_{x_a} + \frac{i}{2}kt^{k-1}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}k(k-1)t^{k-2}x_a\partial_W,\\
&Q_a^{(2)} = t^{k-1}\partial_{x_a} + \frac{i}{2}(k-1)t^{k-2}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}(k-1)(k-2)t^{k-3}x_a\partial_W,\\
&\dots\\
&Q_a^{(k)} = t\partial_{x_a} + \frac{i}{2}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right),\\
&Q_B^{(1)} = it\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \partial_W,\\
&\dots\\
&Q_B^{(2k-2)} = it^{2k-2}\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + (2k-2)t^{2k-3}\partial_W.
\end{aligned}$$

5. The Schrödinger Equation with Convection Term

Consider equation (1) for $W = 0$, i.e., the Schrödinger equation with convection term

$$i\frac{\partial\psi}{\partial t} + \Delta\psi = V_a\frac{\partial\psi}{\partial x_a}, \tag{20}$$

where ψ and V_a ($a = \overline{1,n}$) are complex functions of t and $\vec{x}$. For extension of symmetry, we again regard the functions V_a as dependent variables. Note that the requirement that the functions V_a are complex is essential for symmetry of (20).

Let us investigate symmetry properties of (20) in the class of first-order differential operators

$$X = \xi^\mu\partial_{x_\mu} + \eta\partial_\psi + \eta^*\partial_{\psi^*} + \rho^a\partial_{V_a} + \rho^{*a}\partial_{V_a^*},$$

where $\xi^\mu, \eta, \eta^*, \rho^a, \rho^{*a}$ are functions of $t, \vec{x}, \psi, \psi^*, V_a, V_a^*$.

Theorem 2. *Equation (20) is invariant under the infinite-dimensional Lie algebra with the infinitesimal operators*

$$\begin{aligned} &Q_A = 2A\partial_t + \dot{A}x_c\partial_{x_c} - i\ddot{A}x_c\left(\partial_{V_c} - \partial_{V_c^*}\right) - \dot{A}\left(V_c\partial_{V_c} + V_c^*\partial_{V_c^*}\right), \\ &Q_{ab} = E_{ab}\left(x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*}\right) \\ &\qquad - i\dot{E}_{ab}\left(x_a\partial_{V_b} - x_b\partial_{V_a} - x_a\partial_{V_b^*} + x_b\partial_{V_a^*}\right), \\ &Q_a = U_a\partial_{x_c} - i\dot{U}_a\left(\partial_{V_a} - \partial_{V_a^*}\right), \\ &Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \quad Z_3 = \partial_\psi, \quad Z_4 = \partial_{\psi^*}, \end{aligned} \tag{21}$$

where A, E_{ab}, U_a are arbitrary smooth functions of t. We mean summation over the index c and no summation over indices a and b.

This theorem is proved by analogy with the previous one.

Note that algebra (21) includes, as a particular case, the Galilei operator of the form:

$$G_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \tag{22}$$

This operator generates the following finite transformations:

$$\begin{cases} t' = t, \quad x'_a = x_a + \beta_a t, \quad x'_b = x_b \ (b \neq a), \\ \psi' = \psi, \quad \psi^{*'} = \psi^*, \\ V'_a = V_a - i\beta_a, \quad V_a^{*'} = V_a^* + i\beta_a, \end{cases}$$

where β_a is an arbitrary real parameter. Operator (22) is essentially different from the standard Galilei operator (8) of the Schrödinger equation, and we cannot derive operator (8) from algebra (21).

Consider now the system of equation (20) with the additional condition for the potentials V_a, namely, the complex Euler equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi = V_a\dfrac{\partial\psi}{\partial x_a}, \\ i\dfrac{\partial V_a}{\partial t} - V_b\dfrac{\partial V_a}{\partial x_b} = F(|\psi|)\dfrac{\partial\psi}{\partial x_a}. \end{cases} \tag{23}$$

Here, ψ and V_a are complex dependent variables of t and $\vec{x}$, F is an arbitrary function of $|\psi|$. The coefficients of the second equation of the system provide the broad symmetry of this system.

Let us investigate the symmetry classification of system (23). Consider the following five cases.

1. F is an arbitrary smooth function. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a\rangle$, where

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*}, \\ &G_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \end{aligned}$$

2. $F = C|\psi|^k$, where C is an arbitrary complex constant, $C \neq 0$, k is an arbitrary real number, $k \neq 0$ and $k \neq -1$. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, D^{(1)}\rangle$, where

$$D^{(1)} = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*} - \frac{2}{1+k}(\psi\partial_\psi + \psi^*\partial_{\psi^*}).$$

3. $F = \dfrac{C}{|\psi|}$, where C is an arbitrary complex constant, $C \neq 0$. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, Z = Z_1 + Z_2\rangle$, where

$$Z = \psi\partial_\psi + \psi^*\partial_{\psi^*}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}.$$

4. $F = C \neq 0$, where C is an arbitrary complex constant. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, D^{(1)}, Z_3, Z_4\rangle$, where

$$Z_3 = \partial_\psi, \;\; Z_4 = \partial_{\psi^*}.$$

5. $F = 0$. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, D, A, Z_1, Z_2, Z_3, Z_4\rangle$, where

$$D = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*},$$
$$A = t^2\partial_t + tx_c\partial_{x_c} - (ix_c + tV_c)\partial_{V_c} + (ix_c - tV_c^*)\partial_{V_c^*}.$$

6. Contact Transformations

Consider the two-dimensional Schrödinger equation

$$i\psi_t + \psi_{xx} = V(t, x, \psi, \psi_x, \psi_t). \tag{24}$$

We seek the infinitesimal operators of contact transformations in the class of the first-order differential operators of the form [1, 9]

$$\begin{aligned} X = {} & \xi^\nu(t, x, \psi, \psi_t, \psi_x)\partial_{x_\nu} + \eta(t, x, \psi, \psi_t, \psi_x)\partial_\psi \\ & + \zeta^\nu(t, x, \psi, \psi_t, \psi_x)\partial_{\psi_\nu} + \mu(t, x, \psi, \psi_t, \psi_x, V)\partial_V, \end{aligned} \tag{25}$$

where

$$\xi^\nu = -\frac{\partial W}{\partial \psi_\nu}, \quad \eta = W - \psi_\nu\frac{\partial W}{\partial \psi_\nu}, \quad \zeta^\nu = \frac{\partial W}{\partial x_\nu} + \psi_\nu\frac{\partial W}{\partial \psi} \tag{26}$$

for a function $W = W(t, x, \psi, \psi_x, \psi_t)$. The condition of invariance of equation (24) under operators (25), (26) implies that the unknown function W has the form

$$W = F^1(t)\psi_t + F^2(t, x, \psi, \psi_x),$$

where F^1 and F^2 are arbitrary functions of their arguments.

Then

$$\xi^0 = -F^1(t), \quad \xi^1 = -F^2_{\psi_x}(t, x, \psi, \psi_x),$$
$$\eta = F^2 - \psi_x F^2_{\psi_x}, \quad \zeta^0 = F^1_t\psi_t + F^2_t + \psi_t F^2_\psi, \quad \zeta^1 = F^2_x + \psi_x F^2_\psi,$$
$$\begin{aligned} \mu = {} & i(W_t + \psi_t W_\psi) + W_{xx} + 2W_{x\psi}\psi_x \\ & - (i\psi_t - V)\left(W_{x\psi_x} + W_\psi + \psi_x W_{\psi\psi_x}\right) + (\psi_x)^2 W_{\psi\psi} \\ & - (i\psi_t - V)\left(W_{x\psi_x} + \psi_x W_{\psi\psi_x} - (i\psi_t - V)W_{\psi_x\psi_x}\right). \end{aligned}$$

Thus, equation (24) is invariant under the infinite-dimensional group of contact transformations with the infinitesimal operators:

$$\begin{aligned}
Q_{F^1} &= -F^1\partial_t + F^1_t\psi_t\partial_{\psi_t} + iF^1_t\psi_t\partial_V,\\
Q_{F^2} &= -F^2_{\psi_x}\partial_x + (F^2 - \psi_x F^2_{\psi_x})\partial_\psi + (F^2_t + \psi_t F^2_\psi)\partial_{\psi_t}\\
&\quad + (F^2_x + \psi_x F^2_\psi)\partial_{\psi_x} + \Big\{ iF^2_t + i\psi_t F^2_\psi + F^2_{xx} + 2F^2_{x\psi}\psi_x\\
&\quad + (\psi_x)^2 F^2_{\psi\psi} - (i\psi_t - V)(2F^2_{x\psi_x} + 2\psi_x F^2_{\psi\psi_x} + F^2_\psi) + (i\psi_t - V)^2 F^2_{\psi_x\psi_x}\Big\}\partial_V,
\end{aligned}$$

where $F^1 = F^1(t)$ and $F^2 = F^2(t, x, \psi, \psi_x)$ are arbitrary functions.

Consider the special case. Let $F^1(t) = 1$, $F^2(t, x, \psi, \psi_x) = -(\psi_x)^2$. Then $W = \psi_t - (\psi_x)^2$. The operators of the contact transformations have the form

$$\begin{aligned}
Q_{F^1} &= \partial_t,\\
Q_{F^2} &= 2\psi_x\partial_x + (\psi_x)^2\partial_\psi - 2(i\psi_t - V)^2\partial_V.
\end{aligned} \tag{27}$$

The operator (27) generate the finite transformations:

$$\begin{cases}
x' = 2\psi_x\theta + x, \quad t' = t,\\
\psi' = (\psi_x)^2\theta + \psi, \quad \psi'_x = \psi_x, \quad \psi'_t = \psi_t,\\
V' = \dfrac{2i\theta(V - i\psi_t)\psi_t + V}{2\theta(V - i\psi_t) + 1}.
\end{cases} \tag{28}$$

Transformations (28) can be used for generating exact solutions of equation (24) from the known solution and for constructing nonlocal ansatzes reducing the given equation to the system of ordinary differential equations.

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