Dynamic Asset Allocation under Stochastic Volatility - Theory and Practice

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Abstract

This study develops inter-temporal dynamic asset allocation with stochastic volatility (DAASV) models. The DAASV models integrate the stochastic volatility feature inherent in asset returns into the allocation procedure. By applying the DAASV, an investor can more efficiently diversify the unsystematic risks, so as to achieve better performance. We demonstrate that the DAASV models dominate the traditional mean-variance portfolio models by using Taiwan equity market empirical data. Finally, we show that under the consideration of trade-off between transaction costs and rebalancing timing, an optimal asset allocation rebalancing frequency can be derived.

Keywords: asset allocation, stochastic volatility, transaction cost.

1. Introduction

Asset allocation is in itself a practical issue, it has long been a topic of great interest for practitioners and academics, especially for the institutional investors who routinely possess a large number of assets. Markowitz (1952) pioneered developing the mean-variance (MV) optimization framework for portfolio selection. The MV model is assumed to be a suitable technique to obtain efficient investments in modern portfolio theory. However, the original MV model was built under a one-period framework; moreover, many empirical studies have shown that the most important assumption - fixed mean and volatility of asset return- in the MV framework of Markowitz (1952) is not supported.[1] The over-simplified one-period and constant mean and volatility assumption of asset return inherent in the MV model results in its failure for practical asset allocation application.

This study develops inter-temporal dynamic asset allocation with stochastic volatility (DAASV) models, which integrate different stochastic volatility features of assets into the allocation procedure. The DAASV models relax the limitation in the traditional MV model such as the parameters of asset return are symmetrically distributed. Consequently, investors can more efficiently diversify the unsystematic risks by applying the DAASV, so as to achieve better investment performance.

To capture the practical stochastic volatility features of asset returns, we propose two DAASV models. The first is the DAASV-G model, which allows the volatility of asset return to follow a geometric Brownian motion. The second is the DAASV-M model, which allows the volatility of asset return to follow a CIR (Cox, Ingersoll, and Ross, 1985) mean-reverting stochastic process. However, as the constant volatility assumption removed, an analytic solution to the optimal asset allocation weights is no longer exists. By applying the Monte Carlo simulation technique together with a regional enumeration searching, we can effectively derive the optimal solution for DAASV models.

The rest of this paper is organized as follows. Section 2 demonstrates the basic mean-variance portfolio framework and the DAASV models. Section 3 presents the empirical analyses of different asset allocation models. We demonstrate that the DAASV

[1] Blattberg and Gonedes (1974), Castanias (1979), and Christie (1982) showed that the variances change over time. Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987), and Stein and Stein (1991), to name a few, developing stochastic volatility option pricing schemes to conquer the constant volatility problem. Few studies have examined the asset allocation problem by considering the first two moments of asset returns are varying over time. However, these studies only explore the case of one risky asset. The interested readers can refer to Gomes (2002), and Han (2004)

models dominate the traditional mean-variance portfolio models by using Taiwan equity market empirical data. Section 4 exploits the effect of transaction cost to asset allocation. We also prove in this section that under the consideration of trade-off between transaction costs and rebalancing timing, an optimal asset allocation rebalancing frequency can be derived. We finally draw conclusions in section 5.

2. The models

2.1 The mean-variance framework

Markowitz (1952) pioneered developed a mean-variance (MV) efficient portfolio selection model. The MV model derive the weight of each candidate asset by minimize the volatility of the portfolio subject to a predetermined required return. The formulation is as follows

$$\begin{aligned} \text{Min} \quad & \sigma_{Pt}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{it}x_{jt}\sigma_{ijt} \\ \text{S.T.} \quad & \sum_{i=1}^{n} x_{it}E(r_{it}) = R_t^* \end{aligned} \tag{1}$$

where σ_{Pt}^2 is the total variance of portfolio with n candidate assets at time t; σ_{ijt} is the covariance of asset i and j; $E(r_{it})$ is the expected return of asset i at time t; R_t^* is the predetermined required return for investing the portfolio at time t; x_{it} is the proportion invested into asset i at time t, which is the solution to an asset allocation we are going to solve. We allow the investor to short sell assets, the restriction on the proportion of each asset i at time t is $-1 \le x_{it} \le 1$.

Assume the investment horizon is T, say 2 years, and an investor rebalance his/her portfolio every 6 months (rebalance frequency Δt =0.5 years), then the investor will adjust the portfolio 4 (=2/0.5) times in the whole period of 2 years horizon. If T= Δt , then the investor adopts the so called buy-and-hold strategy, since he/she only adjust the portfolio once at the beginning of investment.

2.2 The DAASV models

Under the MV framework of Markowitz (1952), we only need to know the first two moments of asset returns so as to find the solution of optimal asset allocation of a portfolio. However, to integrate the practical stochastic volatility features inherent in assets into the asset allocation procedure, we develop the DAASV models as follows.

$$\frac{dS_{it}}{S_{it}} = \mu_{it}dt + \sigma_{it}dZ_{it} \tag{2}$$

$$d\sigma_{it} = \alpha_{it}\sigma_{it}dt + \eta_{it}\sigma_{it}dW_{it} \tag{3}$$

$$d\sigma_{it} = a_{it}(b_{it} - \sigma_{it})dt + \xi_{it}\sqrt{\sigma_{it}}dW_{it} \tag{4}$$

where S_{it} is the market price of candidate asset i at time t; μ_{it} is the drift rate of asset i at time t; σ_{it} is the volatility rate of asset i at time t; α_{it} is the drift rate of volatility of asset i at time t; η_{it} is the volatility rate of volatility of asset i at time t when the volatility of asset return follows a geometric Brownian motion; a_{it} is the convergence speed parameter of volatility of asset i at time t; b_{it} is the long-term mean of volatility of asset i at time t; η_{it} is the volatility rate of volatility of asset i at time t; ξ_{it} is the volatility rate of volatility of asset i at time t when the volatility of asset return follows a geometric Brownian motion; dZ_{it} and dW_{it} are standard Wiener processes.

Equation (2) is the stochastic process of assets, which follows the geometric Brownian motion. As the volatility of asset return follows equation (3), the geometric Brownian motion, we call it the DAASV-G model. If the volatility of asset return follows equation (4), the CIR stochastic process, we call it the DAASV-M model.

2.3 Performance evaluation criteria

To gauge the performance of different asset allocation models, we apply the Sharpe ratio as the performance evaluation measure.

$$\frac{\tilde{\mu}_{Pt} - r_{ft}}{\tilde{\sigma}_{Pt}} \tag{5}$$

where $\tilde{\mu}_{Pt} = E(\sum_{i=1}^{n} x_{it}r_{it})$, $\tilde{\sigma}_{Pt} = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_{it}x_{jt}\sigma_{ijt})$, r_{ft} is the risk-free rate of return at time t. The economic meaning of Sharpe ratio is the excess return over risk-free rate of return per unit portfolio risk (volatility). A portfolio performs better than others as it has a larger Sharpe ratio than the others.

3. Empirical analyses

To test the performance of different asset allocation models, we use the Taiwan equity market data for empirical analyses. The date set is from TEJ, the leading financial data bank of Taiwan, and the empirical analyses period is from March 10, 2004 to March 10, 2006. The candidate assets include 50 stocks, which are from the top 10 largest firm value companies of 5 industries, showed in Table 1.

【Insert Table 1 about here】

From Table 2, under the scenario of no transaction cost, we know that the DAASV-M consistently performs best among the asset allocation models for different rebalancing frequencies, than is the DAASV-G.

【Insert Table 2 about here】

It is apparently that the DAASV-M, stochastic volatility with mean-reverting feature can capture the real asset stochastic processes better than the other asset allocation model. However, since all the DAASV models dominate the original MV model proposed by Markowitz (1952), we can conclude that it is necessary to incorporate the stochastic volatility characteristics of asset return so as to improve the investment performance.

4. The transaction cost effects and optimal rebalancing frequency

4.1 The transaction cost effects

Transaction cost is critical to the practical asset allocation procedure. Theoretically, as the rebalancing frequency increases the performance of asset allocation will increase as well. However, the statement mentioned above will come into existence only in the world of no transaction cost. As we consider the transaction cost of rebalancing portfolio, more adjustment to the portfolio, more transaction fee and tax the investor has pay to the stock broker and the government. By considering the trade-off of the rebalancing and transaction cost effects, we infer that there must be an optimal rebalancing frequency exists for the dynamic asset allocation model.

【Insert Table 3 about here】

From Table 3 we can tell that the result is similar to Table 2, the DAASV-M consistently performs best among the asset allocation models for different rebalancing frequencies, than is the DAASV-G. However, the Sharpe ratios are systematically smaller than those in Table2. One thing worth noticing is that the Sharpe ratios no longer increase as the rebalancing frequencies rise.

4.2 Optimal rebalancing frequencies

We infer from last section that there must be optimal rebalancing frequencies exist for different asset allocation models, as we consider the transaction cost and adjustment effects. From the result of Figures 1 and 2, we prove that my hypothesis is hold. As we neglect the effects of transaction cost, from Figure 1 we see that the performance of every asset allocation models increase when the rebalancing frequencies rise. However, as we take into account of the trade-off between transaction costs and rebalancing timing, we can see from Figure 2, optimal asset allocation rebalancing frequencies do exist in all the three asset allocation models.

【Insert Figures 1 and 2 about here】

5. Conclusions

We provide the DAASV models which can effectively integrate the practical stochastic volatility features in asset returns. By applying DAASV, we prove in this study that an investor can consistently acquire a better investment performance than the traditional mean-variance portfolio model. Among the DAASV models, the one with mean-reverting stochastic volatility feature, the DAASV-M model, is more suitable for the Taiwan equity market.

On the other hand, we prove that optimal rebalancing frequencies do exist for all kind of asset allocation models. By applying our procedure, investors can find the optimal rebalancing frequency for their asset allocation so as to achieve a better investment performance.

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Table 1
The candidate stock number codes of Taiwan equity market

Finance industry	Food industry	Plastics industry	Electrical industry	Traditional industry
2881	1201	1301	2303	1402
2882	1216	1303	2330	1440
2883	1203	1304	2357	1409
2886	1210	1308	2376	1408
2887	1217	1309	2382	2501
2888	1218	1310	2409	2511
2891	1215	1312	2412	2526
5820	1207	1314	2454	2002
2825	1229	1326	3009	5007
2851	1234	6505	5387	2006

The date set is from TEJ, the leading financial data bank of Taiwan, and the empirical analyses period is from March 10, 2004 to March 10, 2006. The candidate assets include 50 stocks, which are from the top 10 largest firm value companies of 5 industries.

Table 2
The Sharpe rations performance of different asset allocation models without transaction cost

	MV	DAASV-G	DAASV-M
BH	5.2129	6.5449	6.8334
Rebalancing 2 times	5.3357	6.5721	6.9689
Rebalancing 4 times	5.5306	6.5966	7.0736
Rebalancing 8 times	6.3180	6.7494	7.7077
Rebalancing 24 times	6.5605	6.6801	7.7509

1. The investment horizon is 2 years.
2. The required return R_t^* =0.03.
3. BH represents the buy-and-hold strategy.
4. Rebalancing 2 times means the investor adjust his/her portfolio once a year, at the beginning of investment and at the end of fist year.
5. Rebalancing 4 times means the investor adjust his/her portfolio once every half years.
6. Rebalancing 8 times means the investor adjust his/her portfolio once every 3 months.
7. Rebalancing 24 times means the investor adjust his/her portfolio once every month.
8. We simulate the daily return with 10,000 paths each day for each asset and repeat 100 trials. The Sharpe ratios reported in this table are the mean values of simulations.

Table 3
The Sharpe rations performance of different asset allocation models with transaction cost

	MV	DAASV-G	DAASV-M
BH	2.6827	3.7575	4.8238
Rebalancing 2 times	2.7637	4.2260	4.8624
Rebalancing 4 times	3.4287	4.7312	4.8669
Rebalancing 8 times	4.0982	5.1824	5.3547
Rebalancing 24 times	1.8030	3.0502	3.8334

The settings are as Table 2

Figure 1
The Sharpe ratios of different asset allocation models without transaction cost

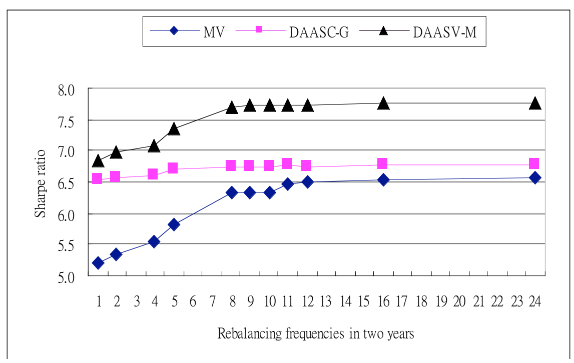

Figure 2
The Sharpe ratios of different asset allocation models with transaction cost