

Exact Solutions of a Quark Plasma Equilibrium in the Abelian Dominance Approximation

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Abstract

Stationary kinetic equations for a quark plasma (QP) in the Abelian dominance approximation are reduced to the nonlinear system of A_2 -periodic Toda chains (with elliptic operator). Using solutions of this system, which are found with the help of the first integrals and Hirota's method, such characteristics of QP as the distribution function and the potential of the self-consistent field are constructed.

1 Introduction

In the last few years in connection with the experiments on a search for a quark-gluon plasma in ion-ionic collisions, the intensive theoretical investigations of different properties of this new phase of matter are performed. As one of the approaches to a plasma dynamic description, the kinetic theory is used through short times available for equilibration in these collisions. In the precise investigation of QP, which takes into account the spinor and quantum effects, the gauge-covariant Wigner operator $\hat{W}(x, p)$ [1] for quarks interacting via the gauge field $\mathbf{A}_a^\mu(x, p)$, where p is a quark 4-momentum, x is a spatial-time state, is used. From the Dirac's field equation, the equation of motion for $\hat{W}(x, p)$ is derived. With the help of power expansions in \hbar , the quasiclassical limit [1] of these exact operator equations is defined.

The problem of construction of exact solutions even for the quasiclassical stationary non-Abelian kinetic equations with a self-consistent field is a rather complicated one. Therefore, we shall restrict our consideration to the simplest and most natural approximation of a quasiclassical kinetic theory, namely, the Abelian dominance approximation [1, 2]. It is shown in this paper that this approximation possesses interesting and physically nontrivial stationary states described by a system of elliptic equations with exponential nonlinearities on the right-hand side.

In section 2, it is demonstrated that one of the possible reductions of this system is the A_2 -periodic Toda chain [3] (with an elliptic operator). In section 3, two first integrals for the one-dimensional A_2 -periodic Toda chain are obtained. With the help of these integrals exact solutions expressed in terms of elementary functions are derived. In sections 4, 5 we obtain exact solutions of the above-mentioned system for the two-dimensional case. In

section 4 based on the analysis of one-dimensional solutions, replacement of required functions is proposed. It allows us to write the system of A_2 -periodic Toda chains in a bilinear form and use Hirota's method [4] for obtaining exact solutions. Two- and three-parameter solutions are defined in an explicit form. Moreover, further exact real solution is separated from a three-parameter one by taking complex vectors. In section 5, we simplify significantly a bilinear system and obtain a six-parameter solution. Unfortunately, the application of this method to the system of A_2 -periodic Toda chains makes it impossible to obtain an exact solution in a fairly general form, but it merely indicates an algorithm for constructing a chain of particular exact solutions of increasing complexity.

Note that it is known one paper [5] only, in which Hirota's method is used, to obtain exact solutions of two nonlinear equations. By this method, here a further nontrivial example is introduced.

Finally, we recover from the constructed exact solutions such plasma characteristics as distribution functions and the self-consistent field.

2 Reduction to the A_2 -periodic Toda chain (with an elliptic operator)

We use the metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and choose units such that $\hbar = c = 1$.

In the model of the Abelian dominance approximation, the components f_j of the Wigner function for quarks satisfy the following system with the self-consistent field $\mathbf{F}^{\mu\nu}(x) = \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu$, $\mathbf{A}^\mu = (A_1^\mu, A_2^\mu)$ [1]

$$(p\partial_x - g\varepsilon_j \mathbf{F}_{\mu\nu} p^\nu \partial_p^\mu) f_j(x, p) = \frac{1}{4} i g \varepsilon_j \mathbf{F}_{\mu\nu} [\sigma^{\mu\nu}, f_j(x, p)], \quad (1)$$

$$\partial_\mu \mathbf{F}^{\mu\nu}(x) = -g \sum_{j=1}^3 \varepsilon_j \int dP \cdot \text{Sp}(\gamma^\nu f_j), \quad j = 1, 2, 3, \quad (2)$$

where $dP = 1/(2\pi)^4 2\Theta(p_0) \delta(p^2 - m^2) d^4p$; $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$, γ^μ are Dirac's matrices; ε_j are the elementary weight vectors of $SU(3)$; the functions f_j are 4×4 matrices in spinor space. The constant of interaction g is defined as a phenomenological parameter. In equation (1), we have omitted the collision term.

Assuming that the temperature of quarks is sufficiently high, we shall seek for stationary solutions of system (1), (2) in the form

$$f_j(x, p) = f_j^{(+)}(x, p) + f_j^{(-)}(x, p), \quad (3)$$

where

$$f_j^{(\pm)}(x, p) = \left(I + \frac{1}{4} p_\mu \gamma^\mu \right) \exp \left\{ (-\beta u_\nu^{(\pm)}) (p^\nu - g \varepsilon_j \mathbf{A}^\nu) \right\}$$

are the distribution functions in each counter-flux. The constant 4-vectors $u_\nu^{(\pm)}$ are 4-velocities of each flux, which are in the form

$$u_\nu^{(\pm)} = (u_0, 0, 0, \pm u_3), \quad (u_0)^2 - (u_3)^2 = 1, \quad (4)$$

here I is the identity matrix; β is a positive constant, related to a plasma temperature.

Using the relation $[\sigma^{\mu\nu}, \gamma^\lambda] = 2i(g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu)$, after substitution (3), we check that the functions $f_j(x, p)$ satisfy the equation (1) if the following identity is realized

$$u_\mu^{(\pm)} \cdot \frac{\partial \mathbf{A}^\nu}{\partial x_\mu} = 0.$$

Let $\mathbf{A}^\nu = \mathbf{A}^\nu(x \equiv x^1, y \equiv y^1)$ (here x, y are not 4-vectors!), then, with regard to (4), we verify that the last equality is satisfied. We impose the condition of Lorentz gauge for the potential $\mathbf{A}^\mu(x) : \partial_\mu \mathbf{A}^\mu(x) = 0$. Substituting (3) in (2) and integrating with respect to 4-momentum, we reduce system (2) to the form

$$\begin{aligned} \Delta \mathbf{A}^0(x, y) &= \sum_{j=1}^3 g \varepsilon_j S^0 \left\{ \exp \beta g u_\mu^{(+)}(\varepsilon_j \mathbf{A}^\mu) + \exp \beta g u_\mu^{(-)}(\varepsilon_j \mathbf{A}^\mu) \right\}, \\ \Delta \mathbf{A}^3(x, y) &= \sum_{j=1}^3 g \varepsilon_j S^3 \left\{ (\exp \beta g u_\mu^{(+)}(\varepsilon_j \mathbf{A}^\mu) - \exp \beta g u_\mu^{(-)}(\varepsilon_j \mathbf{A}^\mu)) \right\}, \\ \mathbf{A}^1 &= \mathbf{A}^2 = 0, \end{aligned} \quad (5)$$

where $S^0 = (m^2 u^0)/(4\beta\pi^3)K_2(\beta m)$, $S^3 = (u^3 S^0)/u^0$, $K_2(\beta m)$ is the modified Bessel's function. Here we use the relation: $\text{Sp}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$.

Thus, the integro- differential system (1), (2) is reduced to the nonlinear system of elliptic equations for the potentials $\mathbf{A}^0 = (A_1^0, A_2^0)$ and $\mathbf{A}^3 = (A_1^3, A_2^3)$. Further, we rewrite system (5) having used the following notations

$$\begin{aligned} \Phi_1 &\equiv \beta g u_\mu^{(+)}(\varepsilon_1 \mathbf{A}^\mu), & \Phi_2 &\equiv \beta g u_\mu^{(+)}(\varepsilon_2 \mathbf{A}^\mu), \\ \Phi_3 &\equiv \beta g u_\mu^{(-)}(\varepsilon_1 \mathbf{A}^\mu), & \Phi_4 &\equiv \beta g u_\mu^{(-)}(\varepsilon_2 \mathbf{A}^\mu). \end{aligned}$$

Then (5) is reduced to the form

$$\begin{aligned} \Delta \Phi_1 &= a(2 \exp \Phi_1 - \exp \Phi_2 - \exp(-\Phi_1 - \Phi_2)) + \\ &\quad b(2 \exp \Phi_3 - \exp \Phi_4 - \exp(-\Phi_3 - \Phi_4)), \\ \Delta \Phi_2 &= a(2 \exp \Phi_2 - \exp \Phi_1 - \exp(-\Phi_1 - \Phi_2)) + \\ &\quad b(2 \exp \Phi_4 - \exp \Phi_3 - \exp(-\Phi_3 - \Phi_4)), \\ \Delta \Phi_3 &= b(2 \exp \Phi_1 - \exp \Phi_2 - \exp(-\Phi_1 - \Phi_2)) + \\ &\quad a(2 \exp \Phi_3 - \exp \Phi_4 - \exp(-\Phi_3 - \Phi_4)), \\ \Delta \Phi_4 &= b(2 \exp \Phi_2 - \exp \Phi_1 - \exp(-\Phi_1 - \Phi_2)) + \\ &\quad a(2 \exp \Phi_4 - \exp \Phi_3 - \exp(-\Phi_3 - \Phi_4)), \end{aligned} \quad (6)$$

where $a \equiv (mg)^2/(2\pi)^3 K_2(\beta m)$, $b = a((u^0)^2 + (u^3)^2)$; $a, b > 0$, $a \neq b$.

System (6) represents the system of two interacting A_2 - periodic Toda chains, where the wave operator is replaced by the elliptic one [3]. The connection between Φ_i and the potential \mathbf{A}_μ is given by the relations

$$\begin{aligned} A_1^0 &= \frac{1}{2\beta g u^0}(\Phi_1 - \Phi_2 + \Phi_3 - \Phi_4), & A_2^0 &= \frac{\sqrt{3}}{2\beta g u^0}(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4), \\ A_1^3 &= \frac{1}{2\beta g u^3}(\Phi_3 - \Phi_4 + \Phi_2 - \Phi_1), & A_2^3 &= \frac{\sqrt{3}}{2\beta g u^3}(\Phi_3 + \Phi_4 - \Phi_1 - \Phi_2). \end{aligned} \quad (7)$$

Next we consider reductions of system (6) to more simple equations. Let $\Phi_1 = \Phi_3$ and $\Phi_2 = \Phi_4$, then (6) is reduced to the A_2 -periodic Toda chain with the elliptic operator

$$\begin{aligned}\Delta\Phi_1 &= (a+b)(2\exp\Phi_1 - \exp\Phi_2 - \exp(-\Phi_1 - \Phi_2)) , \\ \Delta\Phi_2 &= (a+b)\{2\exp\Phi_2 - \exp\Phi_1 - \exp(-\Phi_1 - \Phi_2)\}.\end{aligned}\tag{8}$$

In this case,

$$A_1^3 = A_2^3 = 0, \quad A_1^0 = 1/(\beta gu^0)(\Phi_1 - \Phi_2), \quad A_2^0 = \sqrt{3}/(\beta gu^0)(\Phi_1 + \Phi_2).$$

Exact solutions of system (8) (for a wave operator) were obtained on the basis of the group approach in [3].

The reduction $\Phi_1 = \Phi_2 = \Phi_3 = \Phi_4 \equiv \Phi$ leads system (6) to the equation of Bullough-Dodd type (BD) [6]

$$\Delta\Phi(x, y) = (a+b)(\exp\Phi - \exp\{-2\Phi\}).\tag{9}$$

Potentials (7) have the form:

$$A_1^0 = A_1^3 = A_2^3, \quad A_2^0 = (2\sqrt{3})/(\beta gu^0)\Phi.$$

Note that there is a further reduction, which transforms (8) to Eq. (9) : $\Phi_1 = -2\Phi_2$. Then the potential \mathbf{A}_μ becomes

$$A_1^0 = -3/(\beta gu^0)\Phi_2, \quad A_2^0 = -\sqrt{3}/(\beta gu^0)\Phi_2, \quad A_1^3 = A_2^3 = 0.$$

Using (8), we turn to the construction of some exact solutions of system (1), (2).

3 First integrals. One-dimensional solution

We transform (8) to a more suitable form by setting

$$\Phi_1 = 2u - v, \quad \Phi_2 = 2v - u, \quad (x', y') = \sqrt{a+b}(x, y),\tag{10}$$

where u and v are new unknown functions. Then (8) may be written as

$$\begin{aligned}\Delta u(x', y') &= \exp(2u - v) - \exp(-u - v) , \\ \Delta v(x', y') &= \exp(2v - u) - \exp(-u - v) .\end{aligned}\tag{11}$$

Later we omit primes of the arguments. In the one-dimensional case, system (11) turns to

$$\begin{aligned}u''(x) &= \exp(2u - v) - \exp(-u - v) \equiv \mathcal{A} , \\ v''(x) &= \exp(2v - u) - \exp(-u - v) \equiv \mathcal{B}.\end{aligned}\tag{12}$$

Further we construct the simplest exact solution of this system.

It is easy to check that system (12) has two first integrals

$$\mathcal{P}^2 - \mathcal{P}\mathcal{Q} + \mathcal{Q}^2 - \mathcal{D} = 3C_1 ,\tag{13}$$

$$\mathcal{P}\mathcal{Q}^2 - \mathcal{P}^2\mathcal{Q} + \mathcal{Q}\mathcal{A} - \mathcal{P}\mathcal{B} = C_2,\tag{14}$$

where $\mathcal{P} = u'(x)$, $\mathcal{Q} = v'(x)$; $\mathcal{D} = \mathcal{D}(u, v) = \exp(2u - v) + \exp(-u - v) + \exp(2v - u)$; C_1, C_2 are arbitrary constants.

Integral (13) may be interpreted as the energy for (12). It is interesting to note that the system (12) is Hamilton's with Hamiltonian

$$H = \frac{1}{3}(p_1^2 + p_1 p_2 + p_2^2) - \mathcal{D}(q_1, q_2),$$

where (p_i, q_i) are generalized impulses and coordinates : $p_1 = 2\mathcal{P} - \mathcal{Q}$, $p_2 = 2\mathcal{Q} - \mathcal{P}$; $q_1 = u$, $q_2 = v$.

We assume that $C_2 = 0$. This condition admits the existence of reduction of the solutions of system (12) to a solution of Eq.(9). Solving (13) with respect to Q and substituting in (14), we get the cubic equation for $z = \mathcal{P}^2$:

$$z^3 + \{2(\mathcal{B} - \mathcal{D} - 3C_1) - \mathcal{A}\}z^2 + \{(\mathcal{D} - \mathcal{B} + 3C_1)^2 + \mathcal{A}(\mathcal{D} - \mathcal{B} + 3C_1) + \mathcal{A}^2\}z = \mathcal{A}^2(\mathcal{D} + 3C_1).$$

We define a real solution of the above-mentioned equation with the help of Cardano's formula [7]

$$\mathcal{P}^2 = d + \sqrt[3]{-q/2 + \sqrt{(q/2)^2 + (p/3)^3}} + \sqrt[3]{-q/2 - \sqrt{(q/2)^2 + (p/3)^3}}, \quad (15)$$

where

$$\begin{aligned} d &= 2C_1 + \exp(2u - v) + \exp(-u - v); \\ p &= -3\exp(u - 2v) - 3C_1(\exp(2u - v) + \exp(-u - v)) - 3C_1^2; \\ q &= \exp 3(u - v) + \exp(-3v) - \exp 3u - \exp(-3u) + 2 + \\ &\quad + 6C_1 \exp(u - 2v) + 3C_1^2(\exp(2u - v) + \exp(-u - v)) + 2C_1^3. \end{aligned}$$

Integrals (13), (14) allow us to turn from (12) to the following system

$$u'(x) = \mathcal{P}(u, v), \quad v'(x) = \mathcal{Q}(u, v), \quad (16)$$

where $\mathcal{P} = \mathcal{P}(u, v)$ is in the form of (15) and $\mathcal{Q}(u, v)$ results from $\mathcal{P}(u, v)$ with the replacement $u \rightarrow v, v \rightarrow u$. Autonomy of system (16) enables us to reduce (16) to one nonlinear equation

$$\frac{dv(u)}{du} = \frac{\mathcal{Q}(u, v)}{\mathcal{P}(u, v)}, \quad (17)$$

which may be written as

$$\frac{dv}{du} = \frac{\mathcal{P}^2}{\mathcal{A}} + \frac{(\mathcal{B} - \mathcal{D} - 3C_1)}{\mathcal{A}}$$

and, after the substitutions $u = \ln \theta, v = \ln \rho$, it becomes

$$\frac{\theta}{\rho} \frac{d\rho(\theta)}{d\theta} = -\frac{1}{\theta^3 - 1} - \frac{C_1 \theta \rho}{\theta^3 - 1} + \frac{1}{\theta^3 - 1} \left(\sqrt[3]{S + \sqrt{S^2 + T^3}} + \sqrt[3]{S - \sqrt{S^2 + T^3}} \right), \quad (18)$$

where

$$S = \left(\frac{1}{2}(\theta^6 + 1) - (C_1^3 + 1)\theta^3 \right) \rho^3 - \frac{3}{2}C_1^2 \left(\theta^2 + \frac{1}{\theta} \right) \theta^3 \rho^2 - 3C_1\theta\rho - \frac{1}{2}\theta^3(\theta^3 + 1),$$

$$T = -\theta^3 - C_1\theta(\theta^3 + 1)\rho - C_1^2\theta^2\rho^2.$$

Next we introduce a new function $\mu = \mu(\theta)$, which satisfies the cubic equation

$$(\mu + C_1\theta\rho)^3 + 3T(\mu + C_1\theta\rho) - 2S = 0$$

and define the equation connecting ρ with μ and θ

$$\rho^3 - \frac{3C_1\theta\rho}{(\theta^3 - 1)^2}(\mu^2 - (\theta^3 + 1)\mu + \theta^3) - \frac{1}{(\theta^3 - 1)^2}(\mu^3 - 3\theta^3\mu + \theta^3(\theta^3 + 1)) = 0.$$

The real solution [7] of this equation admits a more simple form if $C_1 = -1$. In this case, we obtain

$$\rho(\theta) = \frac{\mu(\theta - 1) + \theta^3 - \theta}{\theta^3 - 1}. \quad (19)$$

After substitution (19) and simple transformations, we lead (18) to the Riccati equation

$$\mu'(\theta) = f(\theta)\mu^2 + g(\theta)\mu + h(\theta),$$

where

$$f(\theta) = 1/(\theta^4 - \theta), \quad g(\theta) = (2\theta^3 - 1)/(\theta^4 - \theta), \quad h(\theta) = -2\theta^2/(\theta^3 - 1).$$

Since $f + g + h = 0$, we can define the solution of this equation [8]

$$\mu(\theta) = \frac{C - \theta^2}{C - 1/\theta},$$

where C is an arbitrary constant. From the connection (19), we define

$$\rho(\theta) = \frac{C - 1}{C - 1/\theta}, \quad \text{or} \quad v(u) = \ln \frac{C - 1}{C - \exp(-u)}. \quad (20)$$

Substituting (20) in the first equation of (16) (after the replacement $u = \ln \rho$), we obtain the following quadrature

$$x - x_0 = \int \left(\frac{C}{C - 1}\theta^4 - \frac{2}{C - 1}\theta^3 - 3\theta^2 + \frac{2C}{C - 1}\theta - \frac{1}{C - 1} \right)^{-1/2} d\theta, \quad (21)$$

here x_0 is an arbitrary constant.

We consider the case $C = 0$. It follows from (20) that $v = u$ (that is a reduction to the BD equation) and the solution of this equation is

$$u = \ln \left(1 - \frac{3}{2 \cosh^2 \frac{\sqrt{3}}{2}(x - x_0)} \right).$$

Here we used the equality

$$2\theta^3 - 3\theta^2 + 1 = (\theta - 1)^2(2\theta + 1).$$

Now we assume that $C \neq 0$ and rewrite (21) in the form

$$\int \left(\theta^4 - \frac{2}{C} \theta^3 - 3 \frac{C-1}{C} \theta^2 + 2\theta - \frac{1}{C} \right)^{-1/2} d\theta = \sqrt{\frac{C}{C-1}} (x - x_0), \quad (22)$$

here we suppose that $C \neq 1$ and $C/(C-1) > 0$. Using the solution of an algebraic equation of fourth power in Ferrari's representation [7], it can be shown that the subradical expression in integral (22) admits the following factorization

$$\theta^4 - \frac{2}{C} \theta^3 - 3 \frac{C-1}{C} \theta^2 + 2\theta - \frac{1}{C} = (\theta - 1)^2 \left(\theta^2 + 2 \frac{C-1}{C} \theta - \frac{1}{C} \right).$$

This fact allows us to determine this integral in terms of elementary functions and thus to define the simplest exact solution of system (11)

$$\begin{aligned} u &= \ln \left(1 - \frac{3r}{h \exp \xi + (r+1) + \frac{1}{4} \exp(-\xi)} \right), \\ v &= \ln \left(1 - \frac{3(r-1)}{h \exp \xi + (r-2) + \frac{1}{4} \exp(-\xi)} \right), \end{aligned} \quad (23)$$

where $\xi = \sqrt{3}(x - x_0)$, $h = r^2 - r + 1$, $r \equiv (C-1)/C$. Then we define the characteristics QP in terms of (10), (7), and (3)

$$\begin{aligned} A_1^0 &= \frac{3}{\beta g u^0} \ln \frac{\Gamma - 3}{\Gamma}, \quad A_2^0 = \frac{\sqrt{3}}{\beta g u^0} \ln \frac{(\Gamma - 3r)^2}{\Gamma(\Gamma - 3)}, \quad A_1^3 = A_2^3 = 0, \\ f_1^{(\pm)} &= \left(I + \frac{1}{4} p_\mu \gamma^\mu \right) \exp(-\beta u_\nu^{(\pm)} p^\nu) \left\{ \frac{(\Gamma - 3)(\Gamma - 3r)}{\Gamma^2} \right\}, \\ f_2^{(\pm)} &= \left(I + \frac{1}{4} p_\mu \gamma^\mu \right) \exp(-\beta u_\nu^{(\pm)} p^\nu) \left\{ \frac{\Gamma(\Gamma - 3r)}{(\Gamma - 3)^2} \right\}, \\ f_3^{(\pm)} &= \left(I + \frac{1}{4} p_\mu \gamma^\mu \right) \exp(-\beta u_\nu^{(\pm)} p^\nu) \left\{ \frac{\Gamma(\Gamma - 3)}{(\Gamma - 3r)^2} \right\}, \end{aligned}$$

where

$$\Gamma = h \exp \left(\sqrt{3(a+b)}(x - x_0) \right) + (r+1) + 1/4 \exp \left(-\sqrt{3(a+b)}(x - x_0) \right).$$

The lack of this solution is impossibility of its reduction to a solution of the BD equation. The reason for this fact will be explained below.

4 Hirota's method. A three-parameter solution

To construct a less trivial exact solution of system (11), we use Hirota's method [4]. In accordance with this method, we shall seek solutions in the form

$$u = \ln(1 - G/F), \quad v = \ln(1 - Q/P), \quad (24)$$

where F, G, P and Q are certain functions of x and y . Substituting (24) in (11), we obtain the equations for F, G, P , and Q

$$\begin{aligned} (P - Q) \left\{ \frac{1}{2} (2FG - G^2) D_{\Delta}^2 F \circ F + \frac{1}{2} F^2 D_{\Delta}^2 G \circ G - F^2 D_{\Delta}^2 F \circ G \right\} = \\ = P(F - G) \{-3F^2 G + 3FG^2 - G^3\}, \\ (F - G) \left\{ \frac{1}{2} (2PQ - Q^2) D_{\Delta}^2 P \circ P + \frac{1}{2} P^2 D_{\Delta}^2 Q \circ Q - P^2 D_{\Delta}^2 P \circ Q \right\} = \\ = F(P - Q) \{-3P^2 Q + 3PQ^2 - Q^3\}. \end{aligned}$$

Here D_{Δ}^2 is Hirota's bilinear operator, which acts as follows:

$$\begin{aligned} D_{\Delta}^2 F \circ G &\equiv (D_x^2 + D_y^2) F \circ G \equiv \\ &\equiv \left\{ \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^2 + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^2 \right\} F(x, y) G(x', y') \Big|_{x=x', y=y'} \end{aligned}$$

or

$$D_{\Delta}^2 F \circ G = F \Delta G + G \Delta F - 2 \nabla F \cdot \nabla G.$$

The analysis of solution (23) and the well-known bilinear system for the BD equation [9] suggest the following decomposition of two last equations

$$\begin{aligned} (P - Q)(F - G/2) D_{\Delta}^2 F \circ F &= (F - G) \{3FPG - PG^2 - (2Q - G)F^2\}, \\ (P - Q) \left\{ D_{\Delta}^2 F \circ G - \frac{1}{2} D_{\Delta}^2 G \circ G \right\} &= (F - G) \{3PG - (2Q - G)G\}, \\ (F - G)(P - Q/2) D_{\Delta}^2 P \circ P &= (P - Q) \{3PQF - FQ^2 - (2G - Q)P^2\}, \\ (F - G) \left\{ D_{\Delta}^2 P \circ Q - \frac{1}{2} D_{\Delta}^2 Q \circ Q \right\} &= (P - Q) \{3FQ - (2G - Q)Q\}. \end{aligned} \tag{25}$$

In the case of the reduction $F = P, G = Q$, system (25) transfers to bilinear equations for the BD equation [9].

We shall seek a solution of system (25) as expansions in positive and negative powers of ε . After all procedures have been implemented, we then set the parameter ε equal to unity.

Let

$$\begin{aligned} G &\equiv G_1 = b, \quad F \equiv F_1 = \frac{1}{\varepsilon} f_{-1} + a + \varepsilon f_1, \\ Q &\equiv Q_1 = c, \quad P \equiv P_1 = \frac{1}{\varepsilon} p_{-1} + d + \varepsilon p_1. \end{aligned} \tag{26}$$

Here a, b, c , and d are certain constants and $f_{\pm 1}, p_{\pm 1}$ are unknown functions of x and y . Substituting (26) in (25) and equating the coefficients of the corresponding powers of ε , we obtain the bilinear system for $f_{\pm 1}$ and $p_{\pm 1}$. If we now set $f_{\pm 1} = A_{\pm 1} \exp(\pm \eta)$, $p_{\pm 1} = C_{\pm 1} \exp(\pm \eta)$, where $A_{\pm 1}, C_{\pm 1}$ are constants and $\eta \equiv \mathbf{k} \cdot \mathbf{r} = k_x x + k_y y$, then the received

system is reduced to an algebraic system for the coefficients $A_{\pm 1}, C_{\pm 1}, a, b, c, d$, and the vector \mathbf{k} . Solving this algebraic system, we obtain the simplest solution of (11)

$$u = \ln \left(1 - \frac{b}{\frac{1}{3}A_1(b^2 - 3ab + 3a^2) \exp(\eta - \eta^0) + a + \frac{1}{4A_1} \exp\{-(\eta - \eta^0)\}} \right),$$

$$v = \ln \left(1 - \frac{2b - 3a}{\frac{1}{3}A_1(b^2 - 3ab + 3a^2) \exp(\eta - \eta^0) + b - 2a + \frac{1}{4A_1} \exp\{-(\eta - \eta^0)\}} \right),$$

where η_0 is a constant, $\mathbf{k}^2 = 3$. Let $b/a = 3$, then we obtain the solution of Eq.(9). If $b/a = (3r)/(r + 1)$, we get a solution defined with the help of first integrals. This parametrization excepts the possibility of reduction to a solution of the BD equation. The obtained solution depends on two arbitrary parameters: the ratio b/a and components of the vector \mathbf{k} , normalized by the condition $\mathbf{k}^2 = 3$.

We obtain a less trivial solution by choosing F, G, P and Q in the form

$$G_2 = \frac{1}{\varepsilon} g_{-1} + b + \varepsilon g_1, \quad F_2 = \frac{1}{\varepsilon^2} f_{-2} + \frac{1}{\varepsilon} f_{-1} + a + \varepsilon f_1 + \varepsilon^2 f_2,$$

$$Q_2 = \frac{1}{\varepsilon} q_{-1} + c + \varepsilon q_1, \quad P_2 = \frac{1}{\varepsilon^2} p_{-2} + \frac{1}{\varepsilon} p_{-1} + d + \varepsilon p_1 + \varepsilon^2 p_2, \quad (27)$$

where b and c are constants. We shall seek a solution for the following choice of functions:

$$f_{\pm 1} = A_{\pm 1}(\exp(\pm \eta_1) + \exp(\pm \eta_2)), \quad f_{\pm 2} = A_{\pm 2}(\exp\{\pm(\eta_1 + \eta_2)\}),$$

$$g_{\pm 1} = B_{\pm 1}(\exp\{\pm \eta_1\} + \exp\{\pm \eta_2\}), \quad a = a_0 + a_1 \exp(\eta_1 - \eta_2) + a_{-1} \exp(\eta_2 - \eta_1),$$

$$p_{\pm 1} = C_{\pm 1}(\exp(\pm \eta_1) + \exp(\pm \eta_2)), \quad p_{\pm 2} = C_{\pm 2} \exp\{\pm(\eta_1 + \eta_2)\},$$

$$q_{\pm 1} = S_{\pm 1}(\exp\{\pm \eta_1\} + \exp\{\pm \eta_2\}), \quad d = d_0 + d_1 \exp(\eta_1 - \eta_2) + d_{-1} \exp(\eta_2 - \eta_1),$$

where $a_0, d_0, A_{\pm 1}, A_{\pm 2}, B_{\pm 1}, a_{\pm 1}, C_{\pm 1}, C_{\pm 2}, S_{\pm 1}$, and $d_{\pm 1}$ are unknown constants; $\eta_i = \mathbf{k}_i \mathbf{r}, i = 1, 2$.

Substituting (27) in (25) and equating the coefficients of the corresponding powers of ε , we obtain an algebraic system (which, being cumbersome, is not given). This system is strongly overdetermined. From the simplest equations of this algebraic system, we determine the required constants:

$$a_0 = \frac{1}{3}(2b - c) - \frac{2}{3}\varphi\psi s, \quad d_0 = \frac{1}{3}(2c - b) - \frac{2}{3}\varphi\psi s, \quad a_{\pm 1} = d_{\pm 1} = \frac{1}{12}\varphi s,$$

$$C_{\pm 1} = \frac{c + b - s}{c - 2b + 2s} A_{\pm 1}, \quad A_{\pm 2} = C_{\pm 2} = 3A_{\pm 1}^2 \frac{1}{\varphi(c - 2b + 2s)},$$

$$B_{\pm 1} = 2A_{\pm 1} + C_{\pm 1} = \frac{3b}{c + b - s} A_{\pm 1}, \quad S_{\pm 1} = 2C_{\pm 1} + A_{\pm 1} = \frac{3c}{c - 2b + 2s} A_{\pm 1},$$

where $A_{\pm 1}$ is obtained from the relation $A_1 A_{-1} = 1/36 \varphi\psi(c - 2b + 2s)s$. Here

$$\varphi \equiv ((\mathbf{k}_1 + \mathbf{k}_2)^2 - 3)/(\mathbf{k}_1 - \mathbf{k}_2)^2, \quad \psi \equiv ((\mathbf{k}_1 - \mathbf{k}_2)^2 - 3)/(\mathbf{k}_1 + \mathbf{k}_2)^2,$$

$$s \equiv \sqrt{b^2 - bc + c^2}, \quad \mathbf{k}_1^2 = \mathbf{k}_2^2 = 3.$$

Substituting the obtained coefficients in the remaining equations of the algebraic system, we verify that they all vanish identically. The solution of the BD equation is obtained by the reduction $b = c$.

The obtained solution depends on three arbitrary parameters: the ratio b/c and two parameters from the normalization $\mathbf{k}_1^2 = \mathbf{k}_2^2 = 3$.

Another real solution can be obtained from (27) by taking complex vectors $\mathbf{k}_1 \equiv \mathbf{p} + i\mathbf{q} = \bar{\mathbf{k}}_2$ with the condition $\mathbf{p}^2 - \mathbf{q}^2 = 3$, $(\mathbf{p}, \mathbf{q}) = 0$ [10]. Then $\varphi = -(4\mathbf{p}^2 - 3)/4\mathbf{q}^2$, $\psi = -(4\mathbf{q}^2 + 3)/4\mathbf{p}^2$ and the solution has the form

$$u = \ln \left(1 - \frac{2B_{-1} \cos(\mathbf{q}, \mathbf{r}) \exp(-\mathbf{p}\mathbf{r}) + b + 2B_1 \cos(\mathbf{q}, \mathbf{r}) \exp(\mathbf{p}\mathbf{r})}{A_{-2} \exp(-2\mathbf{p}\mathbf{r}) + 2A_{-1} \cos(\mathbf{q}, \mathbf{r}) \exp(-\mathbf{p}\mathbf{r}) + a_0 + 2a_1 \cos 2(\mathbf{q}, \mathbf{r}) + 2A_1 \cos(\mathbf{q}, \mathbf{r}) \exp(\mathbf{p}\mathbf{r}) + A_2 \exp(2\mathbf{p}\mathbf{r})} \right),$$

$$v = \ln \left(1 - \frac{2S_{-1} \cos(\mathbf{q}, \mathbf{r}) \exp(-\mathbf{p}\mathbf{r}) + c + 2S_1 \cos(\mathbf{q}, \mathbf{r}) \exp(\mathbf{p}\mathbf{r})}{C_{-2} \exp(-2\mathbf{p}\mathbf{r}) + 2C_{-1} \cos(\mathbf{q}, \mathbf{r}) \exp(-\mathbf{p}\mathbf{r}) + d_0 + 2d_1 \cos 2(\mathbf{q}, \mathbf{r}) + 2C_1 \cos(\mathbf{q}, \mathbf{r}) \exp(\mathbf{p}\mathbf{r}) + C_2 \exp(2\mathbf{p}\mathbf{r})} \right).$$

Based on (10), (7), and (3), we recover the characteristics of QP, corresponding to this solution.

The obtained solutions are periodic in structure. This fact, possibly, is of main practical value. A spatially-periodic equilibrium may be the finite result of filamentating instability [11], which leads to stratification of the originally homogeneous quark flux on layers current or threads.

5 Six-parameter solution

Analysis of solutions (26) and (27) shows that the following equality is obeyed

$$F - G = P - Q. \quad (28)$$

Expressing P and substituting in (25), we obtain the more simple system of bilinear equations for F, G , and Q

$$\begin{aligned} D_\Delta^2 F \circ F &= 2F(2G - Q) - 2G(G - Q), \\ D_\Delta^2 F \circ G - \frac{1}{2} D_\Delta^2 G \circ G &= 3FG - (2G - Q)G, \\ D_\Delta^2 F \circ Q - D_\Delta^2 G \circ Q + \frac{1}{2} D_\Delta^2 Q \circ Q &= 3FQ - (2G - Q)Q. \end{aligned} \quad (29)$$

The third equation in (25) vanishes identically. On the basis of system (29), we assume that the following solution in this chain is the form

$$\begin{aligned} G_3 &= \frac{1}{\varepsilon^2} g_{-2} + \frac{1}{\varepsilon} g_{-1} + b + \varepsilon g_1 + \varepsilon^2 g_2, \\ Q_3 &= \frac{1}{\varepsilon^2} q_{-2} + \frac{1}{\varepsilon} q_{-1} + c + \varepsilon q_1 + \varepsilon^2 q_2, \\ F_3 &= \frac{1}{\varepsilon^3} f_{-3} + \frac{1}{\varepsilon^2} f_{-2} + \frac{1}{\varepsilon} f_{-1} + a + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3, \end{aligned} \quad (30)$$

where

$$\begin{aligned}
f_{\pm 3} &= A_{\pm 3} \exp\{\pm(\eta_1 + \eta_2 + \eta_3)\}, \\
f_{\pm 2} &= A_{\pm 12} \exp\{\pm(\eta_1 + \eta_2)\} + A_{\pm 13} \exp\{\pm(\eta_1 + \eta_3)\} + A_{\pm 23} \exp\{\pm(\eta_2 + \eta_3)\}, \\
f_{\pm 1} &= D_{\pm 1} \exp\{\pm\eta_1\} + D_{\pm 2} \exp\{\pm\eta_2\} + D_{\pm 3} \exp\{\pm\eta_3\} + \\
&\quad + D_{\pm 13} \exp\{\pm(\eta_1 + \eta_3 - \eta_2)\} + D_{\pm 23} \exp\{\pm(\eta_2 + \eta_3 - \eta_1)\} + \\
&\quad + D_{\pm 12} \exp\{\pm(\eta_1 + \eta_2 - \eta_3)\}, \\
g_{\pm 2} &= B_{\pm 12} \exp\{\pm(\eta_1 + \eta_2)\} + B_{\pm 13} \exp\{\pm(\eta_1 + \eta_3)\} + B_{\pm 23} \exp\{\pm(\eta_2 + \eta_3)\}, \\
g_{\pm 1} &= B_{\pm 1} \exp\{\pm\eta_1\} + B_{\pm 2} \exp\{\pm\eta_2\} + B_{\pm 3} \exp\{\pm\eta_3\}, \\
q_{\pm 2} &= S_{\pm 12} \exp\{\pm(\eta_1 + \eta_2)\} + S_{\pm 13} \exp\{\pm(\eta_1 + \eta_3)\} + S_{\pm 23} \exp\{\pm(\eta_2 + \eta_3)\}, \\
q_{\pm 1} &= S_{\pm 1} \exp\{\pm\eta_1\} + S_{\pm 2} \exp\{\pm\eta_2\} + S_{\pm 3} \exp\{\pm\eta_3\}, \\
a &= a_0 + a_{12} \exp(\eta_1 - \eta_2) + a_{13} \exp(\eta_1 - \eta_3) + a_{23} \exp(\eta_2 - \eta_3) + \\
&\quad + a_{-12} \exp(\eta_2 - \eta_1) + a_{-13} \exp(\eta_3 - \eta_1) + a_{-23} \exp(\eta_3 - \eta_2),
\end{aligned}$$

analogously for b and c ; $\eta_i = \mathbf{k}_i \cdot \mathbf{r}$, $i = 1, 2, 3$.

Substituting (30) in to system (29) and equating the coefficients of the corresponding powers of ε , we obtain, as in the previous case, an overdetermined algebraic system for unknown constants. From the simplest equations, we find

$$\begin{aligned}
S_{\pm ij} &= 2B_{\pm ij} - 3A_{\pm ij}, \quad c_{\pm ij} = 2b_{\pm ij} - 3a_{\pm ij}, \\
S_i &= \frac{A_{ij}B_{il} + A_{il}B_{ij} - 3A_{ij}A_{il}}{\varphi_{jl}A_3}, \quad B_i = \frac{B_{il}B_{ij} - A_{ij}B_{il} - A_{il}B_{ij}}{\varphi_{jl}A_3}, \\
D_i &= \frac{9(2B_{ij}B_{il} - 3A_{ij}B_{il} - 3A_{il}B_{ij} + 3A_{ij}A_{il})}{\varphi_{jl}A_3(k_j + k_l)^2(k_j - k_l)^2} - \frac{A_{ij}A_{il}}{A_3}\psi_{jl}, \\
D_{ij} &= \frac{3A_{ij}^2 + B_{ij}^2 - 3A_{ij}B_{ij}}{12A_3}, \quad D_{-ij} = \frac{D_{il}D_{lj}}{A_3} \cdot \left(\frac{\psi_{ij}}{\varphi_{ij}}\right)^2, \quad S_{-i} = \frac{D_{jl}S_i}{A_3} \cdot \frac{\psi_{ij}\psi_{il}}{\varphi_{ij}\varphi_{il}}, \\
B_{-i} &= \frac{B_i}{S_i}S_{-i}, \quad D_{-i} = \frac{D_i}{S_i}S_{-i}, \quad a_{ij} = \frac{D_{il}A_{ij}}{A_3} \cdot \frac{\psi_{jl}}{\varphi_{jl}}, \quad a_{-ij} = \frac{D_{jl}A_{ij}}{A_3} \cdot \frac{\psi_{il}}{\varphi_{il}}, \\
b_{\pm ij} &= \frac{B_{ij}}{A_{ij}}a_{\pm ij}, \quad A_{-ij} = A_{ij} \frac{D_{il}D_{lj}}{A_3^2} \cdot \left(\frac{\psi_{ij}}{\varphi_{ij}}\right)^2 \frac{\psi_{il}\psi_{lj}}{\varphi_{il}\varphi_{lj}}, \quad B_{-ij} = \frac{B_{ij}}{A_{ij}}A_{-ij}, \\
A_{-3} &= \frac{D_{12}D_{13}D_{23}}{A_3^2} \cdot \left(\frac{\psi_{12}\psi_{13}\psi_{23}}{\varphi_{12}\varphi_{13}\varphi_{23}}\right)^2, \quad \mathbf{k}_i^2 = 3,
\end{aligned}$$

where

$$\varphi_{ij} \equiv ((\mathbf{k}_i + \mathbf{k}_j)^2 - 3)/(\mathbf{k}_i - \mathbf{k}_j)^2, \quad \psi_{ij} \equiv ((\mathbf{k}_i - \mathbf{k}_j)^2 - 3)/(\mathbf{k}_i + \mathbf{k}_j)^2;$$

$$i, j, l = 1, 2, 3, \quad i \neq j, j \neq l, l \neq i;$$

all values are symmetric in lower indices. The coefficients a_0, b_0 , and c_0 are determined from the system of linear algebraic equations

$$\begin{aligned}
& A_3\{a_0(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)^2 - 2b_0 + c_0\} + A_{12}\{D_3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)^2 - 3D_3 + B_3 + S_3\} + \\
& + A_{13}\{D_2(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2)^2 - 3D_2 + B_2 + S_2\} + \\
& + A_{23}\{D_1(\mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1)^2 - 3D_1 + B_1 + S_1\} = B_{12}S_3 + B_{13}S_2 + B_{23}S_1, \\
& b_0A_3\{(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)^2 - 3\} + B_{12}\{(D_3 - B_3)(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)^2 - 3D_3 + 2B_3 - S_3\} + \\
& + B_{13}\{(D_2 - B_2)(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2)^2 - 3D_2 + 2B_2 - S_2\} + \\
& + B_{23}\{(D_1 - B_1)(\mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1)^2 - 3D_1 + 2B_1 - S_1\} = 0, \\
& c_0A_3\{(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)^2 - 3\} + B_{12}(2D_3 + S_3 - 2B_3)(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)^2 + \\
& + B_{13}(2D_2 + S_2 - 2B_2)(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2)^2 + B_{23}(2D_1 + S_1 - 2B_1)(\mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1)^2 + \\
& + A_{12}(3B_3 - 3D_3 - 2S_3)(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)^2 + A_{13}(3B_2 - 3D_2 - 2S_2)(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2)^2 + \\
& + A_{23}(3B_1 - 3D_1 - 2S_1)(\mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1)^2 = (2B_{12} - 3A_{12})(3D_3 - 2B_3 + S_3) + \\
& + (2B_{13} - 3A_{13})(3D_2 - 2B_2 + S_2) + (2B_{23} - 3A_{23})(3D_1 - 2B_1 + S_1).
\end{aligned}$$

We define coefficients of the function P from (28). It is now necessary to substitute the obtained coefficients in the remaining equations of the algebraic system. However, because expressions are very cumbersome and complicated, we restricted ourselves to the simplest equations. They all became identities. The obtained solution depends on six arbitrary parameters: B_{ij}/A_{ij} and the vectors \mathbf{k}_i , where $\mathbf{k}_i^2 = 3$. Let $B_{ij}/A_{ij} = 3$, then we obtain a solution of Eq. (9).

Unfortunately, the form of (26), (27), and (30) gives no way of constructing the general structure of coefficients of the solution of system (11).

Finally, we obtain the characteristics of QP corresponding to the solutions of system (11) in the form

$$\begin{aligned}
& \mathbf{A}^1 = \mathbf{A}^2 = \mathbf{A}^3 = 0, \\
& A_1^0 = \frac{3}{\beta g u^0} \ln \left\{ \frac{F_N - G_N + Q_N}{F_N} \right\}, \quad A_2^0 = \frac{\sqrt{3}}{\beta g u^0} \ln \left\{ \frac{(F_N - G_N)^2}{F_N(F_N - G_N + Q_N)} \right\}, \\
& f_1^{(\pm)} = \left(I + \frac{1}{4} p_\mu \gamma^\mu \right) \exp(-\beta u_\nu^{(\pm)} p^\nu) \cdot (F_N - G_N + Q_N) \cdot (F_N - G_N)/F_N^2, \\
& f_2^{(\pm)} = \left(I + \frac{1}{4} p_\mu \gamma^\mu \right) \exp(-\beta u_\nu^{(\pm)} p^\nu) \cdot F_N(F_N - G_N)/(F_N - G_N + Q_N)^2, \\
& f_3^{(\pm)} = \left(I + \frac{1}{4} p_\mu \gamma^\mu \right) \exp(-\beta u_\nu^{(\pm)} p^\nu) \cdot F_N(F_N - G_N + Q_N)/(F_N - G_N)^2,
\end{aligned}$$

where, for $N = 1, 2, 3$, we use (26), (27), and (30) respectively, and set the parameter ε equal to unity.

In conclusion, we note that the obtained solutions (and corresponding characteristics of plasma \mathbf{A}_μ and f_j) cannot be realized in the complete space \mathbf{R}^3 , since they possess singularities. These singularities have a purely mathematical nature. In regions near the singular surfaces, there is a breakdown of the condition of applicability of the Abelian

dominance approximation model [1]. To "smooth" the singularity, it is necessary to use more accurate models.

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