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Graded Symmetry Algebras of Time-Dependent Evolution Equations and Application to the Modified KP Equations

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Dedicated to Prof. W.I. Fushchych on the occasion of his 60th birthday

Abstract

By starting from known graded Lie algebras, including Virasoro algebras, new kinds of time-dependent evolution equations are found possessing graded symmetry algebras. The modified KP equations are taken as an illustrative example: new modified KP equations with m arbitrary time-dependent coefficients are obtained possessing symmetries involving m arbitrary functions of time. A particular graded symmetry algebra for the modified KP equations is derived in this connection homomorphic to the Virasoro algebras.

1 Introduction

Symmetries are one of the important and currently active areas in soliton theory. They are closely connected with the integrability of corresponding nonlinear equations. For a given evolution equation

$$u_t = K(t,x,u) \qquad \left(u = u(t,x), u_t \equiv \frac{\partial u}{\partial t} \right), \tag{1.1}$$

a vector field $\sigma(t,x,u)$ is called its symmetry (or generalized symmetry) if $\sigma(t,x,u)$ satisfies its linearized equation

$$\frac{d\sigma(t,x,u)}{dt} = K'[\sigma] \qquad \text{or} \qquad \frac{\partial \sigma(t,x,u)}{\partial t} = [K,\sigma], \tag{1.2}$$

where the prime means the Gateaux derivative:

$$K'[\sigma] = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} K(u+\varepsilon\sigma) \tag{1.3}$$

and the Lie product $[\cdot,\cdot]$ is defined by

$$[K,\sigma]=K'[\sigma]-\sigma'[K]=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}K(u+\varepsilon\sigma)-\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\sigma(u+\varepsilon K). \qquad (1.4)$$

Actually, the symmetries defined above are infinitesimal generators of one-parameter groups of invariant transformations of $u_t=K(t,x,u)$. If a vector field σ does not depend on the time variable t, the condition of σ being a symmetry of (1.1) becomes a very simple equality, namely $[K,\sigma]=0$, which means that a symmetry σ only needs to commute with the vector field K generating the considered equation. However if the σ depends on t, then the problem is not so simple. Some specific methods for dealing with this case were introduced, for example, in [1] – [4].

Note that the above definition of symmetry may also be viewed as

$$[u_t-K(t,x,u),\sigma(t,x,u)]=0, \qquad (1.5)$$

where the Lie product should be understood as the one in the extended vector field Lie algebra including the time variable t, not just including the space variable x, as in [5]. A Lie homomorphism $\exp(\mathrm{ad}_T)$ of the vector field Lie algebra with a suitable vector field $T=T(t,x,u)$ may be applied to the discussion of the integrability of time-dependent evolution equations. Here ad_T denotes the adjoint map of the vector field T and thus we have

$$(\mathrm{ad}_T)S=[T,S] \qquad \text{for any vector field } S=S(t,x,u).$$

Fuchssteiner observed [6] that if the Lie homomorphism $\exp(\mathrm{ad}_T)$ acts on (1.5), a new and significant result may be reached which states that a new evolution equation

$$u_t=\exp(\mathrm{ad}_T)K+\sum_{i=0}^{\infty}\frac{(\mathrm{ad}_T)^i}{(i+1)!}\partial_t T \qquad (1.6)$$

has a symmetry $\exp(\mathrm{ad}_T)\sigma(t,x,u)$, when $\sigma(t,x,u)$ is a symmetry of $u_t=K(t,x,u)$. This is because we have

$$\exp(\mathrm{ad}_T)[u_t-K(t,x,u),\sigma(t,x,u)]=[\exp(\mathrm{ad}_T)(u_t-K(t,x,u)),\exp(\mathrm{ad}_T)\sigma(t,x,u)]$$

and

$$\exp(\mathrm{ad}_T)(u_t)=u_t-\sum_{i=0}^{\infty}\frac{(\mathrm{ad}_T)^i}{(i+1)!}\partial_t T.$$

Here we require, of course, that the relevant series converge.

The present paper aims at the construction of time-dependent evolution equations which possess graded symmetry algebras and most particularly centerless Virasoro algebras. The basic tools we will adopt in this paper are the above observation by Fuchssteiner [6] and the Lax operator algebra method in [7]. The result of the analysis gives rise to various concrete realizations of graded Lie algebras. As an illustrative example, two graded Lie algebras are presented for the modified KP equations. Moreover, an application of our result for constructing evolution equations with arbitrary time varying coefficients and a graded symmetry algebra involving these arbitrary coefficients is given for the modified KP equations. Some concluding remarks are given in the last section.

2 Variable-coefficient equations from Virasoro algebras

We take the centerless Virasoro algebra:

$$\begin{cases} [K_{l_1}, K_{l_2}] = 0, \\ [K_{l_1}, \rho_{l_2}] = (l_1 + \gamma) K_{l_1+l_2}, \\ [\rho_{l_1}, \rho_{l_2}] = (l_1 - l_2) \rho_{l_1+l_2}, \end{cases} \tag{2.1}$$

in which the vector fields K_l, ρ_l do not depend explicitly on the time variable t and the γ is a fixed constant. Note that here the space variable x may belong to $\mathbb{R}^p$ or Z^p and $u(x,t)$ may belong to $\mathbb{R}^q$ generally. If we define an operator Φ as

$$\Phi K_l = K_{l+1}, \qquad \Phi \sigma_l = \sigma_{l+1}, \tag{2.2}$$

in which σ_l is a symmetry of $u_t = K_l$, then this operator Φ is hereditary over the above Virasoro algebra, i.e., it is to satisfy the equality

$$\Phi^2 [K,S] + [\Phi K, \Phi S] - \Phi\{[\Phi K, S] + [K, \Phi S]\} = 0 \tag{2.3}$$

for any vector fields K, S belonging to that Virasoro algebra (see [8] for example).

We first consider equation

$$u_t = \alpha_1(t) K_{i_1} \tag{2.4}$$

with $\alpha_1(t)$ an arbitrary function of time. Choose a key vector field as

$$T_1 = \beta_1(t) K_{i_1}, \qquad \text{where } \frac{\partial}{\partial t}\beta_1(t) = \alpha_1(t). \tag{2.5}$$

Then we have

$$\exp(\mathrm{ad}_{T_1})(u_t) = u_t + [T_1, u_t] = u_t - \alpha_1(t) K_{i_1}$$

so that an application of $\exp(\mathrm{ad}_{T_1})$ to the zero equation $u_t = 0$ yields equation (2.4) considered. Since any vector field which does not depend explicitly on t is a symmetry of $u_t = 0$, we have, in particular, the symmetries K_l, ρ_l. Further by applying $\exp(\mathrm{ad}_{T_1})$ to these symmetries, we obtain two hierarchies of symmetries of (2.4)

$$\exp(\mathrm{ad}_{T_1}) K_l = K_l, \tag{2.6}$$

$$\sigma_{l,i_1} = \exp(\mathrm{ad}_{T_1})\rho_l = \rho_l + [T_1, \rho_l] = \beta_1(t)[K_{i_1}, \rho_l] + \rho_l = \beta_1(t)(i_1+\gamma)K_{i_1+l} + \rho_l, \tag{2.7}$$

which also constitute the same Virasoro algebra as (2.1). Of course, we might generate other symmetries of (2.4) from any vector field $\rho = \rho(x,u)$ which causes the series $\exp(\mathrm{ad}_{T_1})\rho$ to converge. Here we give only two sorts of such symmetries, because any other symmetries, just based on Virasoro algebras, are not at all clear as symmetries.

We next consider the more general equation

$$u_t = \alpha_1(t) K_{i_1} + \alpha_2(t) K_{i_2} \tag{2.8}$$

with two arbitrary functions of time $\alpha_1(t)$, $\alpha_2(t)$. We choose a key vector field

$$T_2 = \beta_2(t) K_{i_2}, \qquad \text{where } \frac{\partial}{\partial t}\beta_2(t) = \alpha_2(t). \tag{2.9}$$

Now we find that an application of $\exp(\mathrm{ad}_{T_2})$ to the evolution equation (2.4) and its symmetries K_l, σ_{l,i_1} yields the considered equation (2.8), together with its following symmetries:

$$\exp(\mathrm{ad}_{T_2})K_l = K_l, \tag{2.10}$$

$$\begin{aligned}\sigma_{l,i_1i_2} &= \exp(\mathrm{ad}_{T_2})\sigma_{l,i_1} = \exp(\mathrm{ad}_{T_2})[\beta_2(t)(i_2+\gamma)K_{i_2+l}] + \beta_2(t)(i_2+\gamma)K_{i_2+l}\rho_l \\ &= \beta_1(t)(i_1+\gamma)K_{i_1+l} + \beta_2(t)(i_2+\gamma)K_{i_2+l} + \rho_l,\end{aligned} \tag{2.11}$$

which still constitute the same Virasoro algebra as (2.1).

In general, we can obtain the variable-coefficient evolution equation

$$u_t = \alpha_1(t)K_{i_1} + \alpha_2(t)K_{i_2} + \cdots + \alpha_m(t)K_{i_m} \tag{2.12}$$

with m given arbitrary time-dependent functions $\alpha_j(t)$, $1 \le j \le m$, and its two hierarchies of symmetries

$$\exp(\mathrm{ad}_{T_m})\cdots\exp(\mathrm{ad}_{T_1})K_l = K_l, \tag{2.13}$$

$$\begin{aligned}\sigma_{l,i_1\cdots i_m} &= \exp(\mathrm{ad}_{T_m})\sigma_{l,i_1\cdots i_{m-1}} = \cdots = \exp(\mathrm{ad}_{T_m})\cdots\exp(\mathrm{ad}_{T_1})\rho_l \\ &= \sum_{j=0}^{m}\beta_j(t)(i_j+\gamma)K_{i_j+l} + \rho_l,\end{aligned} \tag{2.14}$$

where $T_j = \beta_j(t)K_{i_j}$, $\frac{\partial}{\partial t}\beta_j(t) = \alpha_j(t),\ 1 \le j \le m$. The symmetries so obtained constitute a Virasoro algebra with the same commutation relations as (2.1):

$$\begin{cases} [K_{l_1}, K_{l_2}] = 0, \\ [K_{l_1}, \sigma_{l_2,i_1\cdots i_m}] = (l_1+\gamma)K_{l_1+l_2}, \\ [\sigma_{l_1,i_1\cdots i_m}, \sigma_{l_2,i_1\cdots i_m}] = (l_1-l_2)\sigma_{l_1+l_2,i_1\cdots i_m}. \end{cases} \tag{2.15}$$

This may also be directly checked. For example, we can calculate that

$$\begin{aligned}[\sigma_{l_1,i_1\cdots i_m}, \sigma_{l_2,i_1\cdots i_m}] &= \left[\sum_{j=0}^{m}\beta_j(t)(i_j+\gamma)K_{i_j+l_1} + \rho_{l_1}, \sum_{j=0}^{m}\beta_j(t)(i_j+\gamma)K_{i_j+l_2} + \rho_{l_2}\right] \\ &= \left[\sum_{j=0}^{m}\beta_j(t)(i_j+\gamma)K_{i_j+l_1}, \rho_{l_2}\right] + \left[\rho_{l_1}, \sum_{j=0}^{m}\beta_j(t)(i_j+\gamma)K_{i_j+l_2}\right] + [\rho_{l_1}, \rho_{l_2}] \\ &= (l_1-l_2)\sum_{j=0}^{m}\beta_j(t)(i_j+\gamma)K_{i_j+l_1+l_2} + (l_1-l_2)\rho_{l_1+l_2} = (l_1-l_2)\sigma_{l_1+l_2,i_1\cdots i_m}.\end{aligned}$$

Note that the symmetries K_l are generally time-independent, while at the same time, the symmetries $\sigma_{l,i_1\cdots i_m}$ include m given arbitrary functions of time and so are time-dependent in the same way as our considered equation (2.12). The symmetries $\sigma_{l,i_1\cdots i_m}$ contain the generators of Galilean invariance and invariance under scale transformations [9], [10]. On the other hand, the symmetry algebra (2.15) provides a new explicit realization of the original Virasoro algebra (2.1). We can still take the Virasoro algebra (2.15) to be a starting algebra. However, any new result is really no more than that we have already reached.

If we choose $\alpha_i(t), 1 \le i \le m$, to be polynomials in time, then the evolution equation (2.12) and its symmetries (2.14) are of the polynomial-in-time type (see [11]). Therefore, we may see that there exist higher-degree polynomial-in-time dependent symmetries for many evolution equations in $1+1$ dimensions. This is itself an interesting result in the symmetry theory of evolution equations, because a soliton equation in $1+1$ dimensions usually has only master symmetries of the first order (the reader is referred to [2] for a definition of master symmetry). Furthermore, our derivation does not refer to any particular choices of dimensions and space variables. Hence, the evolution equation (2.12) may be not only both continuous ($x \in \mathbb{R}^p$) and discrete ($x \in Z^p$), but also both $1+1$ ($p=1$) and higher dimensional ($p>1$).

It is well known that there are many integrable equations which possess a centerless Virasoro algebra (2.1) (see [12] – [17] for example). Among the most famous examples are the KdV hierarchy in the continuous case and the Toda lattice hierarchy in the discrete case. According to the result above, we can say that a KdV-type equation

$$u_t = t^{n_1}K_0 + t^{n_2}K_1 = t^{n_1}u_x + t^{n_2}(u_{xxx} + 6uu_x) \quad (n_1, n_2 \in Z/\{-1\}) \tag{2.16}$$

possesses a hierarchy of time-dependent symmetries

$$\sigma_{l,01} = \frac{t^{n_1+1}}{2(n_1+1)}K_l + \frac{3t^{n_2+1}}{2(n_2+1)}K_{l+1} + \rho_l. \tag{2.17}$$

Here, the vector fields K_l, σ_l, are defined by

$$K_l = \Phi^l u_x, \qquad \rho_l = \Phi^l\left(u + \frac{1}{2}xu_x\right), \qquad l \ge 0,$$

in which the Φ is a well-known hereditary operator

$$\Phi = \partial_x^2 + 4u + 2u_x\partial_x^{-1}.$$

They constitute a centerless Virasoro algebra (2.1) with $\gamma = \frac{1}{2}$ [15], [18] and thus so do the symmetries K_l, $\sigma_{l,01}$. The symmetries $\sigma_{l,01}$ are of the polynomial-in-time type when $n_1 \ge 0$ and $n_2 \ge 0$, and they are of the Laurent polynomial-in-time type when $n_1 \le -2$ or $n_2 \le -2$. The latter is worthy of notice, since a time-independent evolution equation does not have such symmetries [4].

We can also conclude that a Toda-type lattice equation

$$\begin{aligned}
(u(n))_t &= \begin{pmatrix} p(n) \\ v(n) \end{pmatrix}_t = K_0 + t^{n_1}K_1 + t^{n_2}K_0 \quad (n_1, n_2 \in Z/\{-1\}) \\
&= (1+t^{n_2})\begin{pmatrix} v(n) - v(n-1) \\ v(n)(p(n) - p(n-1)) \end{pmatrix} \\
&\quad + t^{n_1}\begin{pmatrix} p(n)(v(n) - v(n-1)) + v(n)(p(n+1) - p(n-1)) \\ v(n)(v(n-1) - v(n+1)) + v(n)(p(n)^2 - p(n-1)^2) \end{pmatrix}
\end{aligned} \tag{2.18}$$

possesses a hierarchy of time-dependent symmetries

$$\sigma_{l,010} = tK_l + \frac{t^{n_1+1}}{n_1+1}K_{l+1} + \frac{t^{n_2+1}}{n_2+1}K_l + \rho_l. \tag{2.19}$$

Here, the vector fields $K_l,\ \rho_l$ are defined by

$$K_l = \Phi^l K_0, \quad \rho_l = \Phi^l \rho_0, \quad K_0 = \begin{pmatrix} v - v^{(1)} \\ v(p - p^{(-1)}) \end{pmatrix}, \quad \rho_0 = \begin{pmatrix} p \\ 2v \end{pmatrix}, \quad l \geq 0,$$

in which the hereditary operator Φ is given by

$$\Phi = \begin{pmatrix} p & (v^{(1)}E^2 - v)(E-1)^{-1}v^{-1} \\ v(E^{-1}+1) & v(pE - p^{(-1)})(E-1)^{-1}v^{-1} \end{pmatrix}.$$

Here, we have used a normal shift operator E: $(Eu)(n) = u(n+1)$ and $u^{(m)} = E^m u, m \in Z$. These discrete vector fields K_l together with the discrete vector fields ρ_l constitute a centerless Virasoro algebra (2.1) with $\gamma = 1$ [13] and, thus, the symmetry Lie algebra consisting of K_l and $\sigma_{l,010}$ has the same commutation relations as that Virasoro algebra.

3 Variable-coefficient equations from general graded algebras

In this section, we consider more general algebraic structures by starting from a general graded Lie algebra. In keeping with the notation in [7], let us write a graded Lie algebra consisting of vector fields not depending explicitly on the time variable t as follows:

$$E(R) = \sum_{i=0}^{\infty} E(R_i), \qquad [E(R_i), E(R_j)] \sqsubseteq E(R_{i+j-1}), \quad i, j \geq 0, \tag{3.1}$$

where $E(R_{-1}) = 0$ and the Lie product $[\cdot,\cdot]$ is defined by (1.4). Note that such a graded Lie algebra is called a master Lie algebra in [7] since it is actually similar to a semi-graded Lie algebra under the group $< Z, * >$ with $i * j = i + j - 1$, but not a graded Lie algebra as defined in [19]. It itself looks like a $W_{1+\infty}$ algebra and includes a Virasoro algebra $E(R_0)+E(R_1)$ and a W_∞ type algebra $\sum_{i=1}^{\infty} E(R_i)$ as subalgebras. The W_∞ and $W_{1+\infty}$ type algebras broadly appear in conformal field theory and in 2-dimensional quantum gravity [20], [21]. However, here we focus on applications to symmetries of variable-coefficient evolution equations.

Consider a variable-coefficient evolution equation

$$u_t = \alpha_1(t)K_1 + \alpha_2(t)K_2 + \cdots + \alpha_m(t)K_m, \tag{3.2}$$

with m given arbitrary time-dependent functions $\alpha_j(t),\ 1 \leq j \leq m$. As in Section 2, we can first choose a key vector field to be

$$T_1 = \beta_1(t)K_1, \qquad \text{where } \frac{\partial}{\partial t}\beta_1(t) = \alpha_1(t). \tag{3.3}$$

We then observe that the application of $\exp(\mathrm{ad}_{T_1})$ to the zero equation $u_t = 0$ and its symmetries $\rho_l, \rho_l \in E(R_l)$ yields the evolution equation $u_t = \alpha_1(t)K_{i_1}$ and its symmetries

$$\sigma_1(\rho_l) = \exp(\mathrm{ad}_{T_1})\rho_l = \sum_{j=0}^{l} \frac{\beta_1^j}{j!}(\mathrm{ad}_{K_1})^j \rho_l. \tag{3.4}$$

Note that the series $\exp(\mathrm{ad}_{T_1})\rho_l$ is truncated at the $l+1$-th term. These symmetries also constitute the same graded Lie algebra as (3.1). We next choose a vector field

$$T_2 = \beta_2(t)K_2, \qquad \text{where } \frac{\partial}{\partial t}\beta_2(t) = \alpha_2(t). \tag{3.5}$$

and make an application of $\exp(\mathrm{ad}_{T_2})$ to $u_t = \alpha_1(t)K_1$ and its symmetries $\sigma_1(\rho_l)$. In this way, we obtain the following evolution equation

$$u_t = \alpha_1(t)K_1 + \alpha_2(t)K_2$$

and its symmetries

$$\sigma_{12}(\rho_l) = \exp(\mathrm{ad}_{T_2})\sigma_1(\rho_l) = \sum_{0\le j_1+j_2\le l} \frac{\beta_1^{j_1}\beta_2^{j_2}}{j_1!j_2!}(\mathrm{ad}_{K_1})^{j_1}(\mathrm{ad}_{K_2})^{j_2}\rho_l. \tag{3.6}$$

Note that here we interchanged the position of $(\mathrm{ad}_{K_1})^{j_1}$ and $(\mathrm{ad}_{K_2})^{j_2}$, because we have

$$[K_1,[K_2,\rho_l]] = [K_2,[K_1,\rho_l]].$$

In general, we can obtain the variable-coefficient evolution equation (3.2) and its following symmetries

$$\begin{aligned}\sigma_{1\cdots m}(\rho_l) &= \exp(\mathrm{ad}_{T_m})\sigma_{1\cdots m-1}(\rho_l) = \cdots = \exp(\mathrm{ad}_{T_m})\cdots\exp(\mathrm{ad}_{T_1})\rho_l \\ &= \sum_{0\le j_i+\cdots+j_m\le l} \frac{\beta_1^{j_1}\cdots\beta_m^{j_m}}{j_1!\cdots j_m!}(\mathrm{ad}_{K_1})^{j_1}\cdots(\mathrm{ad}_{K_m})^{j_m}\rho_l,\end{aligned} \tag{3.7}$$

where $\frac{\partial}{\partial t}\beta_j(t) = \alpha_j(t)$, $1\le j\le m$. These symmetries still constitute a graded Lie algebra as at (3.1), that is, we have

$$[\sigma_{1\cdots m}(\rho_{l_1}), \sigma_{1\cdots m}(\rho_{l_2})] = \sigma_{1\cdots m}([\rho_{l_1},\rho_{l_2}]), \quad \rho_{l_1}\in E(R_{l_1}), \quad \rho_{l_2}\in E(R_{l_2}). \tag{3.8}$$

Therefore, the map

$$\sigma_{1\cdots m} : E(R) \to \sigma_{1\cdots m}(E(R)), \quad \rho_l \mapsto \sigma_{1\cdots m}(\rho_l),\ \rho_l\in E(R_l), \tag{3.9}$$

is a Lie homomorphism between the graded Lie algebra (3.1) and the graded symmetry algebra

$$\sigma_{1\cdots m}(E(R)) = \sum_{j=0}^{\infty} \sigma_{1\cdots m}(E(R_j)). \tag{3.10}$$

Of course, we may also directly prove the Lie homomorphism property (3.8). We use mathematical induction to prove the required result. The proof is the following:

$$\begin{aligned}[\sigma_{1\cdots m}(\rho_{l_1}), \sigma_{1\cdots m}(\rho_{l_2})] = \Bigg[&\sum_{0\le j_1+\cdots+j_m\le l_1} \frac{\beta_1^{j_1}\cdots\beta_m^{j_m}}{j_1!\cdots j_m!}(\mathrm{ad}_{K_1})^{j_1}\cdots(\mathrm{ad}_{K_m})^{j_m}\rho_{l_1}, \\ &\sum_{0\le j_1'+\cdots+j_m'\le l_2} \frac{\beta_1^{j_1'}\cdots\beta_m^{j_m'}}{j_1'!\cdots j_m'!}(\mathrm{ad}_{K_1})^{j_1'}\cdots(\mathrm{ad}_{K_m})^{j_m'}\rho_{l_2}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \Big[\sum_{j_m=0}^{l_1}\frac{\beta_m^{j_m}}{j_m!}(\mathrm{ad}_{K_m})^{j_m}\sigma_{1\cdots m-1}(\rho_{l_1}),\sum_{j'_m}^{l_2}\frac{\beta_m^{j'_m}}{j'_m!}(\mathrm{ad}_{K_m})^{j'_m}\sigma_{1\cdots m-1}(\rho_{l_2})\Big]\\
&= \sum_{j_m=0}^{l_1}\sum_{j'_m=0}^{l_2}\frac{\beta_m^{j_m+j'_m}}{j_m!j'_m!}\Big[(\mathrm{ad}_{K_m})^{j_m}\sigma_{1\cdots m-1}(\rho_{l_1}),(\mathrm{ad}_{K_m})^{j'_m}\sigma_{1\cdots m-1}(\rho_{l_2})\Big]\\
&= \sum_{j''_m=0}^{l_1+l_2-1}\frac{\beta_m^{j''_m}}{j''_m!}\sum_{j_m+j'_m=j''_m}\frac{j''_m!}{j_m!j'_m!}\Big[(\mathrm{ad}_{K_m})^{j_m}\sigma_{1\cdots m-1}(\rho_{l_1}),(\mathrm{ad}_{K_m})^{j'_m}\sigma_{1\cdots m-1}(\rho_{l_2})\Big]\\
&= \sum_{j''_m=0}^{l_1+l_2-1}\frac{\beta_m^{j''_m}}{j''_m!}(\mathrm{ad}_{K_m})^{j''_m}\Big[\sigma_{1\cdots m-1}(\rho_{l_1}),\sigma_{1\cdots m-1}(\rho_{l_2})\Big]\\
&= \sum_{j''_m=0}^{l_1+l_2-1}\frac{\beta_m^{j''_m}}{j''_m!}(\mathrm{ad}_{K_m})^{j''_m}\sigma_{1\cdots m-1}([\rho_{l_1},\rho_{l_2}])\\
&= \sigma_{1\cdots m}([\rho_{l_1},\rho_{l_2}]).
\end{aligned}
$$

Here, in the last but one step, we have used the induction assumption. Generally, the symmetries $\sigma_{1\cdots m}(\rho_l)$ are time-independent when $l = 0$ and time-dependent when $l \geq 1$.

A graded Lie algebra has been exhibited for the time-independent KP hierarchy [1] in [2], [7], which includes a centerless Virasoro algebra [14], [22]. Therefore, we may also generate the corresponding graded Lie algebra of time-dependent symmetries for a resulting new set of variable-coefficient KP equations, and this is done in our paper [11]. In the next section, we shall go on to construct another graded Lie algebra, which is related to the *modified* KP hierarchy.

4 Application to modified KP equations

We first obtain a graded symmetry Lie algebra for modified KP equations and then give an application of the theory presented in the last section to the symmetries of modified KP equations. Let us consider the $2+1$ dimensional spectral operator L corresponding to the modified KP hierarchy:

$$L = \partial_x^2 + u\partial_x + \partial_y, \qquad u = u(t,x,y), \quad t,x,y \in \mathbb{R}. \tag{4.1}$$

Evidently, its Gateaux derivative operator reads as $L'[X] = X\partial_x$, and, thus, is injective, i.e., if $L'[X_1] = L'[X_2]$, then $X_1 = X_2$.

Choose the following polynomial differential operators in ∂_x

$$A = \sum_{k=1}^{m} a_k\partial_x^k, \qquad a_k = a_k(x,y,u),\ m \geq 1 \tag{4.2}$$

as candidates for Lax operators [7]. Then we may make the following calculation

$$
\begin{aligned}
AL &= \sum_{k=1}^{m} a_k\partial_x^k(\partial_x^2 + u\partial_x + \partial_y)\\
&= \sum_{k=1}^{m} a_k\partial_x^{k+2} + \sum_{k=1}^{m} a_k\sum_{i=0}^{k}\binom{k}{i}(\partial_x^{k-i}u)\partial_x^{i+1} + \sum_{k=1}^{m} a_k\partial_x^k\partial_y
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} a_k \partial_x^{k+2} + \sum_{k=1}^{m} a_k (\partial_x^k u)\partial_x + \sum_{k=1}^{m} a_k \sum_{i=1}^{k} \binom{k}{i} (\partial_x^{k-i} u)\partial_x^{i+1} + \sum_{k=1}^{m} a_k \partial_x^k \partial_y \\
&= \sum_{k=1}^{m} a_k \partial_x^{k+2} + \sum_{k=1}^{m} a_k (\partial_x^k u)\partial_x + \sum_{i=1}^{m} \sum_{k=i}^{m} \binom{k}{i} a_k (\partial_x^{k-i} u)\partial_x^{i+1} + \sum_{k=1}^{m} a_k \partial_x^k \partial_y \\
&= \sum_{k=1}^{m} a_k \partial_x^{k+2} + \sum_{k=1}^{m} a_k (\partial_x^k u)\partial_x + \sum_{k=1}^{m} \sum_{i=k}^{m} \binom{i}{k} a_i (\partial_x^{i-k} u)\partial_x^{k+1} + \sum_{k=1}^{m} a_k \partial_x^k \partial_y,
\end{aligned}
$$

$$
\begin{aligned}
LA &= (\partial_x^2 + u\partial_x + \partial_y) \sum_{k=1}^{m} a_k \partial_x^k = \sum_{k=1}^{m} (a_{kxx}\partial_x^k + 2a_{kx}\partial_x^{k+1} + a_k \partial_x^{k+2}) \\
&\quad + \sum_{k=1}^{m} (u a_{kx} \partial_x^k + u a_k \partial_x^{k+1}) + \sum_{k=1}^{m} (a_{ky}\partial_x^k + a_k \partial_x^k \partial_y).
\end{aligned}
$$

Here the coefficient $\binom{k}{i}$ denotes the standard binomial coefficient, i.e., $\binom{k}{i} = \frac{k!}{i!(k-i)!}$.

Further we have

$$
\begin{aligned}
[A, L] &= AL - LA \\
&= \sum_{k=1}^{m} a_k (\partial_x^k u)\partial_x + \sum_{k=1}^{m} \sum_{i=k}^{m} \binom{i}{k} a_i (\partial_x^{i-k} u)\partial_x^{k+1} \\
&\quad - \sum_{k=1}^{m} (a_{kxx} + u a_{kx} + a_{ky})\partial_x^k - \sum_{k=1}^{m} (2a_{kx} + u a_k)\partial_x^{k+1} \\
&= \sum_{k=1}^{m} a_k (\partial_x^k u)\partial_x + \sum_{k=1}^{m-1} \sum_{i=k+1}^{m} \binom{i}{k} a_i (\partial_x^{i-k} u)\partial_x^{k+1} \\
&\quad - \sum_{k=1}^{m} (a_{kxx} + u a_{kx} + a_{ky})\partial_x^k - \sum_{k=1}^{m} 2a_{kx}\partial_x^{k+1}.
\end{aligned} \tag{4.3}
$$

Now we can see that the differential operator A of form (4.2) is a Lax operator, i.e., there exists a vector field X such that $[A, L] = L'[X] = X\partial_x$ if and only if the $a_k,\ 1 \le k \le m$, satisfy the following equations

$$
\begin{cases}
a_{mx} = 0, \\
\displaystyle\sum_{i=k}^{m} \binom{i}{k-1} a_i (\partial_x^{i-k+1} u) - (a_{kxx} + u a_{kx} + a_{ky} + 2a_{k-1,x}) = 0, \\
\qquad\qquad\qquad\qquad\qquad k = m, m-1, \cdots, 2;
\end{cases} \tag{4.4}
$$

and the vector field X must be the following,

$$
X = \sum_{k=1}^{m} a_k (\partial_x^k u) - (a_{1xx} + u a_{1x} + a_{1y}). \tag{4.5}
$$

This vector field is called an eigenvector field corresponding to the Lax operator A of (4.2).

We denote by W the following space:

$$
W = \left\{ f + g \,|\, f = \sum_{i,j \ge 0} c_{ij} x^i y^j,\ c_{ij} \in C,\ g = \sum_{i,j,l \ge 0,\, k \in Z} c_{ijkl} x^i y^j \partial_x^k \partial_y^l u,\ c_{ijkl} \in C \right\}, \tag{4.6}
$$

in which $\partial_x^{-1} = \frac{1}{2}(\int_{-\infty}^x - \int_x^\infty)\,dx'$, and we introduce the inverse operator ∂_x^{-1} of ∂_x over the space W as follows, namely

$$\partial_x^{-1}h = \partial_x^{-1}(f+g) = \int_0^x f\,dx' + \frac{1}{2}\left(\int_{-\infty}^x - \int_x^\infty\right) g\,dx', \qquad h = f+g \in W. \tag{4.7}$$

Furthermore suppose that $C[y]$ denotes all polynomials in y, and $C_n[y]$ $(n \geq 0)$, all polynomials in y with degrees no greater than n. Define a group of coefficients as follows

$$a_k = a_{k1} + a_{k2}, \qquad a_{k1} = a_k|_{u=0}, \qquad a_{k2} = a_k - a_{k1}, \qquad 1 \leq k \leq m, \tag{4.8}$$

where the a_{ki} are given recursively (from m to 1) by

$$\begin{cases} a_{m1} = c_m, \qquad a_{m2} = 0; \\ a_{k-1,1} = -\frac{1}{2}(\partial_x + \partial_x^{-1}\partial_y)a_{k1} + c_{k-1}, \\ a_{k-1,2} = -\dfrac{1}{2}(\partial_x + \partial_x^{-1}\partial_y)a_{k2} - \dfrac{1}{2}\partial_x^{-1}(ua_{kx}) + \dfrac{1}{2}\partial_x^{-1}\displaystyle\sum_{i=k}^m \binom{i}{k-1} a_i(\partial_x^{i-k+1}u), \\ \qquad\qquad k = m, m-1, \cdots 2, \end{cases} \tag{4.9}$$

where $c_k \in C[y],\ 1 \leq k \leq m$. Obviously we see that $a_k = a_{k1} + a_{k2} \in W,\ 1 \leq k \leq m$. For every set of $c_k \in C[y],\ 1 \leq k \leq m$, we can uniquely determine a Lax operator $A = \sum_{k=1}^m a_k\partial_x^k = \sum_{k=1}^m (a_{k1} + a_{k2})\partial_x^k$ through (4.9), which operator is also written as $A = P(c_1, \cdots, c_m)$ in order to show the set of c_k. We denote by R and R_i the following spaces of Lax operators

$$R = \{A = P(c_1, \cdots, c_m)|\, m \geq 1,\ c_k \in C[y],\ 1 \leq k \leq m\}, \tag{4.10}$$

$$R_i = \{A = P(c_1, \cdots, c_m)|\, m \geq 1,\ c_k \in C_i[y],\ 1 \leq k \leq m\}, i \geq 0. \tag{4.11}$$

In what follows, we want to prove that $R = \sum_{i=0}^\infty R_i$ is a graded Lie algebra, namely to prove under the operation

$$[\![A, B]\!] = A'[Y] - B'[X] + [A, B], \tag{4.12}$$

where $[A, L] = L'[X],\ [B, L] = L'[Y]$, that we have

$$[\![R_i, R_j]\!] \sqsubseteq R_{i+j-1}, R_{-1} = 0,\ i, j \geq 0. \tag{4.13}$$

We recall that we already have a product property [7]

$$[[\![A, B]\!], L] = L'[\,[X, Y]\,], \tag{4.14}$$

which will be used to derive a graded symmetry algebra later on. First by (4.9), we immediately obtain the following two basic results.

Lemma 1 *Let* $A = P(c_1, \cdots, c_m) = \sum_{k=1}^m a_k\partial_x^k \in R$, *and set*

$$a_{k3} = a_{k1} - c_k = a_k|_{u=0} - c_k,\ 1 \leq k \leq m. \tag{4.15}$$

Then we have (1) A is multilinear with respect to $c_1, \cdots, c_m$; (2) $A|_{u=0} = \sum_{k=1}^m (c_k + a_{k3})\partial_x^k$; (3) if $A|_{u=0} = 0$, then $A = 0$.

Lemma 2 *Let* $A = P(c_1, \cdots, c_m) = \sum_{k=1}^{m} a_k \partial_x^k \in R_i$ *and the* a_{k3} *be defined by (4.15). Then when* $i = 0$, $a_{k3} = 0, 1 \le k \le m$; *and when* $i \ge 1$, $a_{m3} = 0$ *and* a_{k3}, $1 \le k \le m-1$, *are polynomials in* x, y *with degrees less than* i *with respect to* y.

In order to verify that $R = \sum_{i=0}^{\infty} R_i$ is a graded Lie algebra under operation (4.12), we go on to derive two other results.

Lemma 3 *If* $A \in R_0$, *then* $[A, L]|_{u=0} = X|_{u=0}\partial_x = 0$; *and if* $A \in R_i\,(i \ge 1)$, *then the coefficient of the differential operator* $[A, L]|_{u=0}$ *is a polynomial in* x, y *with degrees less than* i *with respect to* y.

Proof: Assume that

$$A = P(c_1, \cdots, c_m) = \sum_{k=1}^{m} a_k \partial_x^k.$$

By noting (4.5), we discover that

$$\begin{aligned}&[A, L]|_{u=0} = X|_{u=0}\partial_x = -(a_{1xx} + ua_{1x} + a_{1y})|_{u=0}\partial_x \\ &= -(a_{11xx} + a_{11y})\partial_x = -[a_{13xx} + (a_{13} + c_1)_y]\partial_x,\end{aligned}$$

where a_{13} is defined by (4.15). Therefore, the required result follows from Lemma 2, which completes the proof. ∎

Lemma 4 *If* $A, B \in R_0$, *then* $[A, B]|_{u=0} = 0$; *and if* $A \in R_i$, $B \in R_j$ $(i, j \ge 0,\ i + j \ge 1)$, *then the coefficients of the differential operator* $[A, B]|_{u=0}$ *are polynomials in* x, y *with degrees less than* $i + j$ *with respect to* y.

Proof: Suppose that

$$A = P(c_1, \cdots, c_m) = \sum_{k=1}^{m} a_k \partial_x^k, \qquad B = P(d_1, \cdots, d_n) = \sum_{l=1}^{n} b_l \partial_x^l.$$

Then we have

$$A|_{u=0} = \sum_{k=1}^{m} (a_{k3} + c_k)\partial_x^k = \sum_{k=1}^{m-1} a_{k3}\partial_x^k + \sum_{k=1}^{m} c_k \partial_x^k, \tag{4.16}$$

$$B|_{u=0} = \sum_{l=1}^{n} (b_{l3} + d_l)\partial_x^l = \sum_{l=1}^{n-1} b_{l3}\partial_x^l + \sum_{l=1}^{n} d_l \partial_x^l, \tag{4.17}$$

where $a_{k3} = a_k|_{u=0} - c_k,\ 1 \le k \le m,\ b_{l3} = b_l|_{u=0} - d_l,\ 1 \le l \le n$. Then we can calculate that

$$\begin{aligned}[A, B]|_{u=0} = [A|_{u=0}, B|_{u=0}] &= \left[\sum_{k=1}^{m-1} a_{k3}\partial_x^k + \sum_{k=1}^{m} c_k\partial_x^k, \sum_{l=1}^{n-1} b_{l3}\partial_x^l + \sum_{l=1}^{n} d_l\partial_x^l\right] \\ = \left[\sum_{k=1}^{m-1} a_{k3}\partial_x^k, \sum_{l=1}^{n-1} b_{l3}\partial_x^l\right] &+ \left[\sum_{k=1}^{m-1} a_{k3}\partial_x^k, \sum_{l=1}^{n} d_l\partial_x^l\right] + \left[\sum_{k=1}^{m} c_k\partial_x^k, \sum_{l=1}^{n-1} b_{l3}\partial_x^l\right].\end{aligned} \tag{4.18}$$

Based upon this equality, we obtain by Lemma 2 the required result. ∎

Theorem 1 *Let R, R_i, $i \geq 0$, be determined by (4.10),(4.11), respectively. Then the Lax operator algebra $R = \sum_{i=0}^{\infty} R_i$ forms a graded Lie algebra under the operation $[\![\cdot,\cdot]\!]$ defined by (4.12) and thus the eigenvector field algebra $E(R) = \sum_{i=0}^{\infty} E(R_i)$ forms the same graded Lie algebra under the operation $[\cdot,\cdot]$ defined by (1.4), where*

$$\begin{aligned} &E(R) = \Big\{X | L'[X] = [A, L],\ A \in R\Big\}, \\ &E(R_i) = \Big\{X | L'[X] = [A, L],\ A \in R_i\Big\}, \qquad i \geq 0. \end{aligned} \tag{4.19}$$

Proof: We first prove equality (4.13), that is,

$$[\![R_i, R_j]\!] \sqsubseteq R_{i+j-1}, \qquad R_{-1} = 0, \qquad i, j \geq 0,$$

which shows that $R = \sum_{i=0}^{\infty} R_i$ is a graded Lie algebra. Let $A \in R_i$, $B \in R_j$ $(i, j \geq 0)$ and $X \in E(R_i)$, $Y \in E(R_j)$ be the eigenvector fields of A, B, respectively, namely $[A, L] = L'[X] = X\partial_x$, $[B, L] = L'[Y] = Y\partial_x$. Obviously we have

$$[\![A, B]\!] = A'[Y] - B'[X] + [A, B] \in R,$$

i.e., $[\![A, B]\!]$ possesses form (4.2), and thus we may assume that

$$[\![A, B]\!] = \sum_{r=1}^{s} e_r \partial_x^r = P(f_1, \cdots, f_s).$$

We observe that

$$\begin{aligned} &[\![A, B]\!]|_{u=0} = (A'[Y] - B'[X] + [A, B])|_{u=0} \\ &= A'[Y|_{u=0}]|_{u=0} - B'[X|_{u=0}]|_{u=0} + [A, B]|_{u=0}. \end{aligned} \tag{4.20}$$

When $i + j = 0$, i.e., $i = j = 0$, it follows from the above equality, Lemma 3 and Lemma 4 that $[\![A, B]\!]|_{u=0} = 0$. Thus by Lemma 1, we obtain $[\![A, B]\!] = 0$, i.e. $[\![A, B]\!] \in R_{-1}$. Now we assume that $i + j \geq 1$. Note that $A'[Y|_{u=0}]|_{u=0}$ and $B'[X|_{u=0}]|_{u=0}$ are linear with $Y|_{u=0}$ and $X|_{u=0}$, respectively. It follows similarly from (4.20), Lemma 3 and Lemma 4 that the coefficients of the differential operator $[\![A, B]\!]|_{u=0}$ are polynomials in x, y with degrees less than $i + j$ with respect to y. It means by Lemma 1 that $e_r|_{u=0} = e_{r3} + f_r$, $1 \leq r \leq s$, are polynomials in x, y with degrees less than $i + j$ with respect to y. On the other hand, by Lemma 2, the degree of $e_{r3} = e_r|_{u=0} - f_r$ with respect to y is less than the maximum degree of the f_l, $1 \leq l \leq s$, with respect to y. Therefore, $f_r \in C_{i+j-1}[y]$, $1 \leq r \leq s$, which means that $[\![A, B]\!] \in R_{i+j-1}$. In conclusion, we see that relation (4.13) holds.

The second result of the theorem is obvious, since we have (4.14), i.e., $[[\![A, B]\!], L] = [X, Y]\partial_x$ when $[A, L] = X\partial_x$ and $[B, L] = Y\partial_x$. Therefore, the proof is complete. ▮

By noting (4.16), (4.17) and (4.18), we may obtain from (4.20) that

$$[\![P(c_1, \cdots, c_m), P(d_1, \cdots, c_n)]\!] = P\left(f_1, \cdots, f_{m+n-3}, \frac{n}{2} c_{my} d_n - \frac{m}{2} c_m d_{ny}\right), \tag{4.21}$$

where $P(c_1, \cdots, c_m) = \sum\limits_{k=1}^{m} a_k \partial_x^k = \sum\limits_{k=1}^{m} (a_{k1} + a_{k2}) \partial_x^k$, the a_{k1} and the a_{k2} being defined by (4.9). However, it is very complicated to obtain the explicit expressions for polynomials f_i. It follows from (4.21) that

$$\left[\sigma^{\{m\}}(f), \sigma^{\{n\}}(g)\right] = \sigma^{\{m+n-2\}}\left(\frac{n}{2} f_y g - \frac{m}{2} f g_y\right), \qquad m, n \geq 1,$$

where $\sigma^{\{0\}}(f) = 0$, is a Lie product. The special case of $m = n = 2$ leads to an interesting Virasoro-type Lie algebra

$$\left[\sigma^{\{2\}}(f), \sigma^{\{2\}}(g)\right] = \sigma^{\{2\}}(f_y g - f g_y).$$

Now let us choose the specific Lax operators. If we choose these as

$$A_m = P(\underbrace{0, \cdots, 0}_{m}, 1) = \sum_{k=1}^{m+1} a_k^{\{m\}} \partial_x^k, \qquad m \geq 0, \tag{4.22}$$

$$B_{in} = P(\underbrace{0, \cdots, 0}_{n}, y^i) = \sum_{l=1}^{n+1} b_l^{\{in\}} \partial_x^l, \qquad i \geq 1,\ n \geq 0, \tag{4.23}$$

then the corresponding eigenvector fields read as

$$X_m = [A_m, L]\partial_x^{-1} = \sum_{k=1}^{m+1} a_k^{\{m\}} (\partial_x^k u) - \left(a_{1xx}^{\{m\}} + u a_{1x}^{\{m\}} + a_{1y}^{\{m\}}\right), \quad m \geq 0, \tag{4.24}$$

$$Y_{in} = [B_{in}, L]\partial_x^{-1} = \sum_{l=1}^{n+1} b_l^{\{in\}} (\partial_x^l u) - \left(b_{1xx}^{\{in\}} + u b_{1x}^{\{in\}} + b_{1y}^{\{in\}}\right), \quad i \geq 1,\ n \geq 0. \tag{4.25}$$

By Lemma 1, we see that

$$R_0 = \mathrm{Span}\{A_m |\, m \geq 0\}, \qquad R_i = \mathrm{Span}\{B_{jn} |\, n \geq 0,\ 0 \leq j \leq i\}, \quad i \geq 1; \tag{4.26}$$

$$E(R_0) = \mathrm{Span}\{X_m |\, m \geq 0\}, \qquad E(R_i) = \mathrm{Span}\{Y_{jn} |\, n \geq 0,\ 0 \leq j \leq i\}, \quad i \geq 1. \tag{4.27}$$

The equations $u_t = X_m,\ m \geq 0$, constitute the integrable modified KP hierarchy. By Theorem 1, this modified KP hierarchy has two Virasoro algebras

$$< R_0 + R_1, [\![\cdot, \cdot]\!] > \quad \text{and} \quad < E(R_0) + E(R_1), [\cdot, \cdot] > \tag{4.28}$$

and two graded Lie algebras

$$< R = \sum_{i=0}^{\infty} R_i, [\![\cdot, \cdot]\!] > \quad \text{and} \quad < E(R) = \sum_{i=0}^{\infty} E(R_i), [\cdot, \cdot] > . \tag{4.29}$$

In particular, the equation $u_t = X_2$ is exactly the normal modified KP equation introduced in [23] (see (4.35) below) and the space $E(R_1)$ includes all the master symmetries presented in [24].

Through Theorem 1, we find at once that every modified KP equation $u_t = X_i\,(i \geq 0)$, X_i given by (4.24), possesses a hierarchy of common time-independent symmetries $\{X_m\}_{m=0}^{\infty}$ and infinitely many hierarchies of polynomial-in-time dependent symmetries

$$\left\{ \sigma_i(Y_{kn}) = \sum_{j=0}^{k} \frac{t^j}{j!} (\mathrm{ad}_{X_i})^j Y_{kn} \right\}_{n=0}^{\infty}, \qquad k \geq 1. \tag{4.30}$$

Further by applying the theory presented in the last section , we obtain the following consequence. A modified KP equation with m given *arbitrary* functions $\alpha_i(t),\ 1 \leq i \leq m$,

$$u_t = \alpha_1(t) X_{i_1} + \alpha_2(t) X_{i_2} + \cdots + \alpha_m(t) X_{i_m} \tag{4.31}$$

has a graded symmetry algebra

$$< \sigma_{i_1 i_2 \cdots i_m}(E(R)) = \sum_{i=0}^{\infty} \sigma_{i_1 i_2 \cdots i_m}(E(R_i)), [\cdot,\cdot] > . \tag{4.32}$$

The map $\sigma_{i_1 i_2 \cdots i_m}$ is defined by

$$\begin{aligned} &\sigma_{i_1 i_2 \cdots i_m}(\rho_i) = \exp(\mathrm{ad}_{X_{i_1}}) \cdots \exp(\mathrm{ad}_{X_{i_m}}) \rho_i \\ &= \sum_{0 \leq j_1 + j_2 + \cdots + j_m \leq i} \frac{\beta_1^{j_1} \beta_2^{j_2} \cdots \beta_m^{j_m}}{j_1! j_2! \cdots j_m!} (\mathrm{ad}_{X_{i_1}})^{j_1} (\mathrm{ad}_{X_{i_2}})^{j_2} \cdots (\mathrm{ad}_{X_{i_m}})^{j_m} \rho_i, \quad \rho_i \in E(R_i), \end{aligned} \tag{4.33}$$

where $\frac{\partial}{\partial t} \beta_j(t) = \alpha_j(t),\ 1 \leq j \leq m$. Moreover, this map is a Lie algebra homomorphism between the graded vector field algebra defined in (4.29) and the graded symmetry algebra defined by (4.32). In other words, we have

$$\left\{ \begin{array}{l} [\sigma_{i_1 i_2 \cdots i_m}(X_{m_1}), \sigma_{i_1 i_2 \cdots i_m}(X_{m_2})] = 0, \\ [\sigma_{i_1 i_2 \cdots i_m}(X_{m_1}), \sigma_{i_1 i_2 \cdots i_m}(Y_{j_1 n_1})] = \sigma_{i_1 i_2 \cdots i_m}([X_{m_1}, Y_{j_1 n_1}]), \\ [\sigma_{i_1 i_2 \cdots i_m}(Y_{j_1 n_1}), \sigma_{i_1 i_2 \cdots i_m}(Y_{j_2 n_2})] = \sigma_{i_1 i_2 \cdots i_m}([Y_{j_1 n_1}, Y_{j_2 n_2}]). \end{array} \right. \tag{4.34}$$

The symmetries $\sigma_{i_1 i_2 \cdots i_m}(Y_{in})$ contain m given arbitrary functions of time and thus the polynomial-in-time symmetries (4.30) generated by the master symmetries Y_{kn} are no more than special cases amongst these.

Some concrete examples of the Lax operators A, B and the corresponding eigenvector fields X, Y are listed in the following:-

(i) The Lax operators and vector fields of the modified KP equations:

$$\begin{aligned} A_0 &= P(1) = \partial_x,\ X_0 = u_x; \\ A_1 &= P(0,1) = \partial_x^2 + u\partial_x,\ X_1 = -u_y; \\ A_2 &= P(0,0,1) = \sum_{k=1}^{3} a_k^{\{2\}} \partial_x^k = \partial_x^3 + \frac{3}{2} u \partial_x^2 + \left(\frac{3}{8} u^2 + \frac{3}{4} u_x - \frac{3}{4} \partial_x^{-1} u_y \right) \partial_x, \\ X_2 &= \sum_{k=1}^{3} a_k^{\{2\}} (\partial_x^k u) - (a_{1xx}^{\{2\}} + u a_{1x}^{\{2\}} + a_{1y}^{\{2\}}) \\ &= \frac{1}{4} u_{xxx} - \frac{3}{8} u^2 u_x - \frac{3}{4} u_x \partial_x^{-1} u_y + \frac{3}{4} \partial_x^{-1} u_{yy}. \end{aligned} \tag{4.35}$$

(ii) The Lax operators and i-th master symmetries of the modified KP equations:

$$\begin{aligned} B_{i0} &= P(y^i) = y^i\partial_x = y^iA_0,\ Y_{i0} = y^iu_x - iy^{i-1} = y^iX_0 - iy^{i-1}; \\ B_{i1} &= \sum_{l=1}^{2} b_l^{\{i1\}}\partial_x^l = y^i\partial_x^2 + \left(-\frac{1}{2}ixy^{i-1} + y^iu\right)\partial_x = y^iA_1 - \frac{1}{2}ixy^{i-1}A_0, \\ Y_{i1} &= \sum_{l=1}^{2} b_l^{\{i1\}}(\partial_x^l u) - (b_{1xx}^{\{i1\}} + ub_{1x}^{\{i1\}} + b_{1y}^{\{i1\}}) \\ &= -y^iu_y - \frac{1}{2}ixy^{i-1}u_x - \frac{1}{2}iy^{i-1}u + \frac{1}{2}i(i-1)xy^{i-2} \\ &= y^iX_1 - \frac{1}{2}ixy^{i-1}X_0 - \frac{1}{2}iy^{i-1}u + \frac{1}{2}i(i-1)xy^{i-2}. \end{aligned}$$

From these, we see that $u_t = X_2$ is, indeed, the normal modified KP equation introduced in [23]. It is also a two-dimensional generalization of the modified KdV equation and may be connected with the KP equation by a Miura transformation [23].

5 Concluding remarks

A main result is the construction of graded symmetry algebras for a broad class of variable-coefficient evolution equations with given arbitrary time-dependent functions as coefficients, starting from known graded Lie algebras, in particular, centerless Virasoro algebras. A Lie homomorphism $\exp(\mathrm{ad}_T)$ of the vector field Lie algebra plays a central role in the construction of time-dependent symmetries of these equations. On the other hand, a graded Lax operator algebra and a graded symmetry algebra are presented for the modified KP hierarchy, in the style of the Lax operator method of [7]. This gives a concrete example of graded Lie algebras. Furthermore, the time-dependent symmetries are obtained for variable-coefficient modified KP equations as an application of our main result.

We point out that coupled systems of soliton equations may be constructed by perturbation [25], [26] and, thus, various new realizations of graded symmetry algebras may be presented for integrable coupled systems, based upon the theory presented in this paper. Moreover by applying the perturbation iteratively, we may obtain infinitely many new realizations of graded Lie algebras, by starting from a known one.

There are also some other symmetries which may be constructed. From the scale transformations, for example, we may take $T_i = x_iu_{x_i},\ x = (x_i)$, as a key vector field. The application of $\exp(\mathrm{ad}_{T_i})$ to the known particular equations may lead to some important equations with specific space-dependent coefficients. An interesting remaining problem is whether or not we can construct integrable evolution equations with *given arbitrary space-dependent functions* as coefficients and in what fashion we can construct their symmetries with space-dependent functions as coefficients if they exist. Of course, we may also ask whether or not there is any important application of these various equations with variable-coefficients to physical problems - such as, for example, to applicable conformal field theory. All of these ideas need further investigation.

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