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Transformation Properties of $\ddot{x}+f_1(t)\dot{x}+f_2(t)x+f_3(t)x^n=0$

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Abstract

In this paper, we consider a general anharmonic oscillator of the form $\ddot{x}+f_1(t)\dot{x}+f_2(t)x+f_3(t)x^n=0$, with $n\in\mathcal{Q}$. We seek the most general conditions on the functions f_1, f_2 and f_3, by which the equation may be integrable, as well as conditions for the existence of Lie point symmetries. Time-dependent first integrals are constructed. A nonpoint transformation is introduced by which the equation is linearized.

1 Introduction

Recently we have reported some results on the integrability of the nonlinear anharmonic oscillator

$$\ddot{x}+f_1(t)\dot{x}+f_2(t)x+f_3(t)x^n=0. \tag{1}$$

Here $\dot{x}\equiv dx/dt$, $\ddot{x}\equiv d^2x/dt^2$ and $n\in\mathcal{Q}$. Conditions on the functions f_1, f_2, and f_3 as well as the constant n were derived for which the equation admits point transformations in integrable equations. The Lie point symmetries were obtained only for the case where f_1, f_2 and f_3 are constants. The Painlevé analysis for special cases of n was performed. For more details, we refer to the papers of Euler *et al* [6], Duarte *et al* [2] and Duarte *et al* [3]. In the present paper, we generalize those results, introduce a nonpoint transformation which linearizes (1), and do a Lie point symmetry classification of (1), whereby conditions for the existence of Lie point symmetries are given on f_1, f_2 and f_3. Before doing so, we would like to make some literatorical remarks on point transformations, nonpoint transformations, and integrability of ordinary differential equations (ODEs), relevant in the present considerations.

In being faced with a nonlinear ordinary differential equation (NODE), one unsually wants to construct its general solution. If the general solution can be obtained, the equation is said to be integrable. Constructing such solutions for NODEs is in general difficult. In fact, in most cases the general solution of NODEs cannot be obtained in closed form, so that one has to be satisfied by solving the equation numerically or by constructing some special exact solutions. Much attention has been focused on the classification of NODEs as integrable and nonintegrable ones. In the case of second order ODEs, the construction of a first integral is of fundamental importance. It is desirable to have a simple approach to obtaining time-dependent first integrals of NODEs.

Several methods for the identification of integrable ODEs have been proposed. A method dating back to the beginning of the development of differential calculus, is to find a coordinate transformation which transforms a particular differential equation in a differential equation with a known general solution. To find a transformation which transforms a NODE in a linear ODE would certainly be a way in which to solve the NODE in general. In particular, the problem of linearizing second-order ODEs has been of great interest. The utilization of point transformations for the linearization is the usual procedure (see, for example Duarte *et al* [1], Sarlet *et al* [13], and Moreira [11]). Since the time of Tresse [17], it is known that the most general second-order ODE which may be linearized by a point transformation, is of the form

$$\ddot{x} + \Lambda_3(x,t)\dot{x}^3 + \Lambda_2(x,t)\dot{x}^2 + \Lambda_1(x,t)\dot{x} + \Lambda_0(x,t) = 0, \tag{2}$$

whereby the functions Λ_j must satisfy the following conditions:

$$\begin{aligned}
&\Lambda_{1xx} - 2\Lambda_{2xt} + 3\Lambda_{3tt} + 6\Lambda_3\Lambda_{0x} + 3\Lambda_0\Lambda_{3x} - 3\Lambda_3\Lambda_{1t} - 3\Lambda_1\Lambda_{3t} - \Lambda_2\Lambda_{1x} + 2\Lambda_2\Lambda_{2t} = 0,\\
&\Lambda_{2tt} - 2\Lambda_{1xt} + 3\Lambda_{0xx} - 6\Lambda_0\Lambda_{3t} - 3\Lambda_3\Lambda_{0t} + 3\Lambda_0\Lambda_{2x} + 3\Lambda_2\Lambda_{0x} + \Lambda_1\Lambda_{2t} - 2\Lambda_1\Lambda_{1x} = 0.
\end{aligned} \tag{3}$$

(We use the notation $\Lambda_{1x} \equiv \partial\Lambda_1/\partial x$, $\Lambda_{1xx} \equiv \partial^2\Lambda_1/\partial x^2$, etc.) In fact, (2) is the most general second-order ODE which may be point transformed by the invertible point transformation

$$X(T) = F(x,t), \qquad T(x,t) = G(x,t), \qquad \frac{\partial(T,X)}{\partial(t,x)} \neq 0, \tag{4}$$

in the free particle equation

$$\frac{d^2X}{dT^2} = 0. \tag{5}$$

Transformation (4) is obtained by solving F and G from

$$\begin{aligned}
\Lambda_3 &= (G_xF_{xx} - G_{xx}F_x)\,\Delta^{-1},\\
\Lambda_2 &= (G_tF_{xx} + 2G_xF_{tx} - 2F_xG_{tx} - F_tG_{xx})\,\Delta^{-1},\\
\Lambda_1 &= (G_xF_{tt} + 2G_tF_{tx} - 2F_tG_{tx} - F_xG_{tt})\,\Delta^{-1}\\
\Lambda_0 &= (G_tF_{tt} - G_{tt}F_t)\,\Delta^{-1}.
\end{aligned} \tag{6}$$

Here $\Delta \equiv G_tF_x - G_xF_t \neq 0$.

The compatibility condition of system (6) is given by (3). If the point transformation (4) is known, the first integrals, Lie point symmetries, and general solution of (5) may be used to obtain the corresponding ones for (2). We use this result in Section 4 in the classification of Lie point symmetries for (1). In particular, the first integral of (5) is

$$I\left(\frac{dX}{dT}\right) = \frac{dX}{dT},$$

so that the first integral of (2) takes the form

$$I(t,x,\dot{x}) = \frac{F_t + F_x\dot{x}}{G_t + G_x\dot{x}};$$

which is, in general, a time-dependent first integral. The free particle equation (5) admits eight Lie point symmetry generators forming the $sl(3,\mathcal{R})$ Lie algebra under the Lie bracket. Those Lie point symmetry generators are

$$\mathcal{G}_1 = \frac{\partial}{\partial T}, \qquad \mathcal{G}_2 = \frac{\partial}{\partial X}, \qquad \mathcal{G}_3 = T\frac{\partial}{\partial T}, \qquad \mathcal{G}_4 = X\frac{\partial}{\partial X}, \qquad \mathcal{G}_5 = X\frac{\partial}{\partial T},$$
$$\mathcal{G}_6 = T\frac{\partial}{\partial X}, \qquad \mathcal{G}_7 = T\left(T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}\right), \qquad \mathcal{G}_8 = X\left(T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}\right).$$

This is the maximum number of Lie point symmetries which any second-order ODE might admit. In fact, any nonlinear second order ODE may admit the $sl(3,\mathcal{R})$ Lie point symmetry algebra provided it admits eight Lie point symmetries. In such a case a point transformation can be found which would linearize the equation, i.e., transform the equation in the free particle equation (5). This leads to the statement:

A necessary and sufficient condition for a second-order ODE to be linearizable by a point transformation, is that the equation admits the $sl(3,\mathcal{R})$ Lie point symmetry algebra.

Linearization by point transformations was studied in some detail by several authors (see for example the works of Leach [9], Sarlet *et al* [13], and Duarte *et al* [1]). An example of a nonlinear second-order ODE that admits the $sl(3,\mathcal{R})$ Lie point symmetry algebra is the equation (Leach [9])

$$\ddot{x} + \alpha x\dot{x} + \frac{\alpha^2}{9}x^3 = 0, \tag{7}$$

where α is an arbitrary real constant. This equation plays an important role in our Lie point symmetry classification of (1) (see Section 4). The general solution of (7) was obtained by Duarte *et al* [1] by the invertible point transformation

$$X(T) = \frac{t}{x} - \frac{1}{6}\alpha t^2, \qquad T(x,t) = \frac{1}{x} - \frac{1}{3}\alpha t,$$

which transforms (7) in the free particle equation (5). The general solution of the free particle equation is $X(T) = k_1 T + k_2$, so that the general solution of (7) follows:

$$x(t) = \frac{t - k_1}{\frac{1}{6}\alpha t^2 - k_1\frac{1}{3}\alpha t + k_2}\,. \tag{8}$$

Here k_1 and k_2 are integrating constants. This result is used to solve some of the equations in Table 1 and Table 2 of Section 4.

It is clear that if one is able to find the invertible point transfromation by which a NODE may be linearized, the general solution of the NODE is easily obtained. We refer to the book of Steeb [14]. Since (1) is not linearizable by a point transformation, we aim to find point transformations in other integrable equations (Section 2 and Section 5), and to linearize (1) by a nonpoint transformation (Section 3).

If a NODE admits a Lie point symmetry, the symmetry may be used to calculate point transformations which transform the NODE either in an autonomous ODE or an ODE with lower order. A Lie point symmetry classification of (1) is performed in Section 4. For more details on Lie point symmetries, we refer to the books of Olver [12], Fushchych *et al* [8] and Steeb [15].

The problem of classifying second order ODEs with respect to the singularity structure of their soluitions, was considered by a school of French Mathematicians under the leadership of P. Painlevé in the period from 1893 till 1902. They classified the equation

$$A_1(x,t)\ddot{x} + A_2(x,t)\dot{x}^2 + A_3(x,t)\dot{x} + A_4(x,t) = 0, \qquad \frac{\partial^{m_j} A_j}{\partial x^{m_j}} = 0, \quad j = 1, \ldots, 4 \quad (9)$$

($m_1, \ldots, m_4$ may be different integers) with respect to the following classification criterion:

The critical points of solutions of (9), that are branch points and essential singularities, should be fixed points.

Any function which is a solution of an equation of this class of ODEs would, therefore, have only poles as movable singularities. They obtained fifty second-order ODEs. The equations satisfying the above criterion are said to have the Painlevé property. Fourty-four of these fifty equations can be solved by standard functions. The remaining six are known as the Painlevé transcendents; they define transcendental functions. It is important to note that the Painlevé transcendents admit no Lie point symmetry transformations. The classification of (9) was done under the Mobius group of transformations

$$X(T) = \frac{\psi_1(t)x + \psi_2(t)}{\psi_3(t)x + \psi_4(t)}, \qquad T = \phi(t),$$

where ψ_j and ϕ are analytic functions of t. Given a particular nonlinear second-order ODE, one could ask the question:

Does there exist an invertible point transformation which may transform a given nonlinear ODE in one of the integrable second-order ODEs classified by Painlevé?

This is generally a difficult question to answer. In our paper, Euler *et al* [7], an invertible point transformation was obtained for an anharmonic oscillator of the form (1) by which the equation may be transformed in the second Painlevé transcendent. We discuss this result in Section 5 of the present paper in detail.

It is clear that the point transformation (4) preserves the Lie point symmetry stucture as well as the integrability structure of a given ODE. By introducing a nonpoint transformation of the form

$$X(T) = F(x,t), \qquad dT(x,t) = G(x,t)dt, \tag{10}$$

one preserves only the integrability structure and not the symmetry structure of the equation. A transformation of this type was considered by Euler *et al* (1994) in their calculations of approximate solutions of nonlinear multidimensional heat equations. Duarte *et al* (1994) made use of transformation (10) and obtained equations which may be nonpoint transformed in the free particle equation (5). They showed, by way of examples, transformation (10) may lead to the linearization of NODEs not linearizable by a point transformation. In Section 3 of the present paper, we utilize this transformation for the linearization of (1).

2 First integrals by point transformations

In this section we consider the problem of constructing invertible point transformations of the form (4), i.e.,

$$X(T) = F(x,t), \qquad T(x,t) = G(x,t),$$

for equation (1), i.e.,

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^n = 0.$$

Note that (1) is a special case of (2). That is, for $\Lambda_3 = \Lambda_2 = 0$, $\Lambda_1 = \alpha(t)$ equation (2) takes the form

$$\ddot{x} + \alpha_1(t)\dot{x} + \Lambda_0(x,t) = 0. \tag{11}$$

By condition, (3) it follows that (11) may be linearized by a point transformation of the form (4) if and only if Λ_0 is a linear function of x, where α is an arbitrary function of t. This leads to the following result:

Equation (1), with $n \notin \{0,1\}$, cannot be linearized by a point transformation.

We now consider the integrable equation

$$\frac{d^2X}{dT^2} + X^n = 0, \tag{12}$$

which admits the first integral

$$I\left(X, \frac{dX}{dT}\right) = \frac{1}{2}\left(\frac{dX}{dT}\right)^2 + \frac{X^{n+1}}{n+1}.$$

By the point transformation (4) equation (12) takes the form

$$\ddot{x} + A_3\dot{x}^3 + A_2\dot{x}^2 + A_1\dot{x} + A_0 = 0, \tag{13}$$

where

$$\begin{aligned}
A_3 &= \left(F_{xx}G_x - G_{xx}F_x + G_x^3F^n\right)\Delta^{-1},\\
A_2 &= \left(G_tF_{xx} + 2G_xF_{xt} - 2F_xG_{xt} - F_tG_{xx} + 3G_tG_x^2F^n\right)\Delta^{-1},\\
A_1 &= \left(G_xF_{xt} + 2G_tF_{xt} - 2F_tG_{xt} - F_xG_{tt} + 3G_t^2G_xF^n\right)\Delta^{-1},\\
A_0 &= \left(G_tF_{tt} - F_tG_{tt} + G_t^3F^n\right)\Delta^{-1}
\end{aligned} \tag{14}$$

and $\Delta \equiv F_xG_t - F_tG_x \neq 0$. In order to obtain an equation of the form (1), we set

$$F(x,t) = f(t)x, \qquad G(x,t) = g(t), \tag{15}$$

where f, g are smooth functions, to be determined in terms of the coefficient functions of (1), namely f_1, f_2 and f_3. System (14) leads to

$$A_3 = A_2 = 0, \qquad A_1 = \frac{2\dot{f}\dot{g} - f\ddot{g}}{f\dot{g}}, \qquad A_0 = \frac{\dot{g}\ddot{f} - \dot{f}\ddot{g}}{f\dot{g}}\, x + \dot{g}f^{n-1}x^n.$$

The functions f_1, f_2 and f_3 then take the form

$$f_1(t) = \frac{2\dot{f}}{f} - \frac{\ddot{g}}{\dot{g}}, \qquad f_2(t) = \frac{\ddot{f}}{f} - \frac{\dot{f}}{f}\frac{\ddot{g}}{\dot{g}}, \qquad f_3(t) = \dot{g}^2 f^{n-1}. \tag{16}$$

We can state the following

Theorem 1: *Equation*

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^n = 0$$

may be point transformed in the equation

$$\frac{d^2X}{dT^2} + X^n = 0,$$

by the transformation

$$X(T) = f(t)x, \qquad T(x,t) = g(t)$$

in the following cases:

a) *For $n \notin \{-3, 0, 1\}$ the transformation coefficients are*

$$f(t) = C f_3^{1/(n+3)}(t) \exp\left(\int^t \frac{2f_1(\zeta)}{n+3}\, d\zeta\right), \tag{17}$$

$$g(t) = \int^t \frac{f_3^{1/2}(\zeta)}{f^{(n-1)/2}(\zeta)}\, d\zeta \tag{18}$$

with the following conditions on the equation coefficients

$$f_2 = \frac{1}{n+3}\frac{\ddot{f}_3}{f_3} - \frac{n+4}{(n+3)^2}\left(\frac{\dot{f}_3}{f_3}\right)^2 + \frac{n-1}{(n+3)^2}\left(\frac{\dot{f}_3}{f_3}\right) f_1 + 2\frac{1}{n+3}\dot{f}_1 + 2\frac{n+1}{(n+3)^2} f_1^2. \tag{19}$$

b) *For $n = -3$ the transformation coefficients are*

$$g(t) = \int^t \sqrt{f_3(\rho)} \exp\left(2\int^\rho \phi(\zeta)\, d\zeta\right) d\rho, \tag{20}$$

$$f(t) = \exp\left(\int^t \phi(\zeta)\, d\zeta\right), \tag{21}$$

where ϕ is the solution of the Riccati equation

$$\dot{\phi} = \phi^2 - f_1(t)\phi + f_2(t). \tag{22}$$

The condition on the equation coefficients is

$$f_1(t) = -\frac{1}{2}\frac{\dot{f}_3}{f_3}. \tag{23}$$

To prove Theorem 1 one needs to invert system (16) and integrate to obtain f and g. The compatibility condition of (16) results in the differential relations (19) and (23), which provides the condition of existence of an invertible point transformation of (1) in the integrable equation (12).

By the point transformation (4) with (15), the first integral of (1) is

$$I(t,x,\dot{x}) = \frac{1}{2}\left(\frac{\dot{f}}{\dot{g}}x + \frac{f}{\dot{g}}\dot{x}\right)^2 + \frac{1}{n+1}f^{n+1}x^{n+1},$$

($n \neq -1$), where f and g as well as the corresponding conditions on f_1, f_2, and f_3, are given in Theorem 1.

3 Linearization by nonpoint transformation

In this section, we make use of the nonpoint transformation (10), i.e.,

$$X(T) = F(x,t), \qquad dT(x,t) = G(x,t)dt.$$

Let us pose the following problem: Find functions F and G in transformation (10), by which the general anharmonic oscillator (1) transforms in

$$\frac{d^2X}{dT^2} + k_1\frac{dX}{dT} + k_2X^p = 0. \tag{24}$$

Here k_1, k_2 are real constants, and $p \in \mathcal{Q}$. Applying transformation (10) to (24), we obtain

$$\ddot{x} + A_2(x,t)\dot{x}^2 + A_1(x,t)\dot{x} + A_0(x,t) = 0 \tag{25}$$

where

$$A_2(x,t) = \frac{F_{xx}}{F_x} - \frac{G_x}{G}, \qquad A_1(x,t) = 2\frac{F_{xt}}{F_x} - \frac{G_t}{G} - \frac{G_x}{G}\frac{F_t}{F_x} + k_1,$$
$$A_0(x,t) = \frac{F_{tt}}{F_x} - \frac{F_t}{F_x}\frac{G_t}{G} + k_1\frac{F_t}{F_x} - k_2G^2\frac{F^p}{F_x}.$$

In order to obtain an equation of the form (1), we set

$$A_2 = 0, \qquad A_1 = f_1(t), \qquad A_0 = f_2(t)x + f_3(t)x^n.$$

The condition $A_2 = 0$ leads to the following special form for (10):

$$X(T) = f(t)x^m, \qquad dT(x,t) = g(t)x^{m-1}dt, \tag{26}$$

so that

$$f_1(t) = \frac{m+1}{m}\frac{\dot{f}}{f} - \frac{\dot{g}}{g} + k_1, \quad f_2(t) = \frac{1}{m}\left(\frac{\ddot{f}}{f} - \frac{\dot{f}}{f}\frac{\dot{g}}{g} + k_1\frac{\dot{f}}{f}\right), \quad f_3(t) = \frac{k_2}{m}g^2f^{p-1}.$$

We can now state

Theorem 2: *Equation*

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^n = 0$$

may be nonpoint transformed in the equation

$$\frac{d^2X}{dT^2} + k_1\frac{dX}{dT} + k_2X^p = 0, \qquad k_1, k_2 \in \mathcal{R}, \quad p \in \mathcal{Q},$$

by transformation (26), with

$$\begin{aligned} f(t) &= f_3^{m/(n+3)} \exp\left\{\frac{2m}{n+3}\int^t f_1(\rho)d\rho - 2k_1\frac{m}{n+3}t\right\}, \\ g(t) &= \left(\frac{m}{k_2}\right)^{1/2} f^{1-(n+1)/(2m)} \end{aligned} \tag{27}$$

and

$$p = \frac{n+1}{m} - 1 \qquad n \notin \{-3, 1\}, \qquad m \notin \{0, 1\}, \qquad p \neq 1, \qquad m(p+1) \neq -2,$$

if and only if

$$\begin{aligned} f_2 = &\frac{1}{n+3}\frac{\ddot{f}_3}{f_3} - \frac{n+4}{(n+3)^2}\left(\frac{\dot{f}_3}{f_3}\right)^2 + \frac{n-1}{(n+3)^2}\left(\frac{\dot{f}_3}{f_3}\right) f_1 + 2\frac{1}{n+3}\dot{f}_1 \\ &+2\frac{n+1}{(n+3)^2}f_1^2 + \frac{k_1}{(n+3)^2}\left\{4\frac{\dot{f}_3}{f_3} - 2(n-1)f_1 - 4k_1\right\}. \end{aligned} \tag{28}$$

Remark: Conditions (19) and (28) are identical if $k_1 = 0$. The nonpoint transformation does, therefore, not identify a wider class of integrable equations of the form (1).

Let us now find a nonpoint transformation which linearizes (1). Note that the constant m, in the nonpoint transformation (26), may be chosen arbitrary (except for 0 and 1). With the choice

$$m = n + 1,$$

equation (1), for $n \in \mathcal{Q}\backslash\{-3, -1, 1\}$, is linearized in

$$\frac{d^2X}{dT^2} + k_1\frac{dX}{dT} + k_2 = 0, \qquad k_2 \neq 0. \tag{29}$$

With this value for m, transformation (26) reduces to

$$X(T) = f(t)x^{n+1}, \qquad dT = \sqrt{\frac{n+1}{k_2}f_3(t)f(t)}\; x^n\, dt, \tag{30}$$

where

$$f(t) = f_3^{(n+1)/(n+3)} \exp\left\{2\left(\frac{n+1}{n+3}\right)\int^t f_1(\rho)d\rho - 2k_1\left(\frac{n+1}{n+3}\right)t\right\}. \tag{31}$$

Thus, if condition (28) holds, (1) may be linearized by transformation (30). Note also that (29) may be point transformed in the free particle equation. For $k_1 = 0$, a first integral of (1) takes the form

$$I(t, x, \dot{x}) = \frac{1}{2}\left(\frac{F_t + F_x\dot{x}}{G}\right)^2 + F, \tag{32}$$

with

$$F(x, t) = f(t)x^{n+1}, \qquad G(x, t) = \sqrt{\frac{n+1}{k_2}f_3(t)f(t)}\, x^n,$$

and f given by (31) if condition (28) (with $k_1 = 0$) is satisfied.

4 Lie point symmetry transformations

4.1 Introduction

In this section, we obtain continuous transfromations which leave equation (1) invariant, and therefore transform solutions of (1) to solutions of (1). This type of transformations forms a group, namely, the Lie point transfromation group. Let a Lie point transformation be given in the following form:

$$\tilde{t} = \varphi(x, t, \varepsilon), \qquad \tilde{x} = \psi(x, t, \varepsilon). \tag{33}$$

Here ε is the group parameter, the group identity is the identity transformation at $\varepsilon = 0$, and the group inverse is the inverse transformation. One can define an infinitesimal generator Z for the Lie point transformation group by

$$Z = \xi(x,t)\frac{\partial}{\partial t} + \eta(x,t)\frac{\partial}{\partial x} \tag{34}$$

so that

$$\tilde{t}(x,t,\varepsilon) = t + \varepsilon Z t + O(\varepsilon^2), \qquad \tilde{x}(x,t,\varepsilon) = x + \varepsilon Z x + O(\varepsilon^2).$$

Integral curves of the generator Z are group orbits of the transformation group; that is by integrating the autonomous system

$$\frac{d\tilde{t}}{d\varepsilon} = \xi(\tilde{x}, \tilde{t}), \qquad \frac{d\tilde{x}}{d\varepsilon} = \eta(\tilde{x}, \tilde{t}) \tag{35}$$

with the initial conditions $\tilde{x}(\varepsilon = 0) = x$, $\tilde{t}(\varepsilon = 0) = t$, we arrive at the finite transformation (33). A function $J(x,t)$ is an invariant of the Lie point transformation group (invariant under the action of the transformation group) if and only if

$$ZJ(x,t) = 0. \tag{36}$$

This is known as the invariance condition. Clearly, the invariant functions of a Lie point transformation group are the first integrals of the corresponding autonomous system (35). In order to find a Lie point transformation group which leaves a second order ODE

$$F(t, x, \dot{x}, \ddot{x}) = 0 \tag{37}$$

invariant, we need to prolong the infinitesimal generator Z to

$$Z^{(2)} = Z + \eta^{(1)}\frac{\partial}{\partial \dot{x}} + \eta^{(2)}\frac{\partial}{\partial \ddot{x}}$$

and apply the invariance condition to the ODE at $F = 0$, i.e.,

$$Z^{(2)}F\Big|_{F=0} = 0. \tag{38}$$

The prolongation coefficients of Z are

$$\eta^{(r)} = \frac{d^r}{dt^r}\left[\eta(x,t) - \dot{x}\xi(x,t)\right] + x^{(r+1)}\xi(x,t).$$

A generator which satisfies condition (38) for a particular ODE is known as a Lie point symmetry generator for that ODE. The corresponding Lie point transformation is known as a Lie point symmetry transformation for the particular ODE. For some ODE, the invariance condition may lead to several Lie point symmetry generators. This set of Lie point symmetry generators form an algebra under the Lie bracket, known as a Lie point symmetry algebra for the equation.

It is clear that an invertible point transformation which transforms one ODE in another, will also transform Lie point symmetry generators of one equation in Lie point symmetry generators of other equation. In particular, the $sl(3, \mathcal{R})$ Lie point symmetry algebra of (2) is spanned by the following Lie point symmetry generators

$$\begin{aligned}
&\mathcal{G}_1 = Q_T\frac{\partial}{\partial t} + P_T\frac{\partial}{\partial x}, \quad \mathcal{G}_2 = Q_X\frac{\partial}{\partial t} + P_X\frac{\partial}{\partial x}, \quad \mathcal{G}_3 = G\left(Q_T\frac{\partial}{\partial t} + P_T\frac{\partial}{\partial x}\right),\\
&\mathcal{G}_4 = F\left(Q_X\frac{\partial}{\partial t} + P_X\frac{\partial}{\partial x}\right), \quad \mathcal{G}_5 = F\left(Q_T\frac{\partial}{\partial t} + P_T\frac{\partial}{\partial x}\right), \quad \mathcal{G}_6 = G\left(Q_X\frac{\partial}{\partial t} + P_X\frac{\partial}{\partial x}\right),\\
&\mathcal{G}_7 = G\left(GQ_T + FQ_X\right)\frac{\partial}{\partial t} + G\left(GP_T + FP_X\right)\frac{\partial}{\partial x},\\
&\mathcal{G}_8 = F\left(FQ_X + GQ_T\right)\frac{\partial}{\partial t} + F\left(FP_X + GP_T\right)\frac{\partial}{\partial x}.
\end{aligned}$$

This is obtained by applying the point transformation (4) and transforming the Lie point symmetry generators of the free particle equation (5). We denote the inverse transformation by $x(X,T) = P(X,T),\ t(X,T) = Q(X,T))$. The integrable equation (12) admits the following Lie point symmetry generators

$$\mathcal{G}_1 = \frac{\partial}{\partial T}, \qquad \mathcal{G}_2 = T\frac{\partial}{\partial T} - \left(\frac{2}{n-1}\right)X\frac{\partial}{\partial X},$$

so that Lie point symmetry generators of (1) can be obtained by the point transformations derived in Section 2 if the appropriate conditions are satisfied. The Lie point symmetry generators for (1), obtained by the point transformation of the form (4) with F and G given by (15), are of the form

$$\begin{aligned}
&\mathcal{G}_1 = Q_T\frac{\partial}{\partial t} + P_T\frac{\partial}{\partial x},\\
&\mathcal{G}_2 = \left\{g(t)Q_T - \left(\frac{2}{n-1}\right)xQ_X\right\}\frac{\partial}{\partial t} - \left\{g(t)P_T - \left(\frac{2}{n-1}\right)f(t)xP_X\right\}\frac{\partial}{\partial x},
\end{aligned} \tag{39}$$

whereby the conditions given in Theorem 1 have to be satisfied. This result is contained in our Lie point symmetry classification of (1) (see subsection 4.2).

Lie point symmetries of an ODE may be used to find invertible point transfromations for ODEs. Let (37) admit the Lie point symmetry generator (34). An invertible point transformation of the form (4), which transforms (37) in an ODE of the autonomous form

$$G(X, \dot{X}, \ddot{X}) = 0,$$

is obtained by solving the system of first-order PDEs

$$ZT = 1, \qquad ZX = 0,$$

whereas the solution of

$$ZT = 0, \qquad ZX = 1$$

provides the point transformation in an equation of the form

$$H(T, Y, \dot{Y}) = 0,$$

where $\dot{X}(T) = Y(T)$. Thus, if an ODE admits Lie point symmetries, it may be used to find first integrals of the ODE. This procedure was followed by Leach and Maharaj [10] for an anharmonic oscillator with multiple anharmonicities.

4.2 Lie point symmetry classification of (1)

Our aim in this section is to do a general Lie point symmetry classification of (1), that is, we give the most general conditions on f_1, f_2 and f_3 for which Lie point symmetries of (1) exist. The aim is not to find all possible functions by which (1) admits Lie point symmetries but merely to give theorems of existence. We consider this form of classification useful since particular equations of the form (1) can easily be tested for the existence of Lie point symmetries.

On applying the invariance condition (38) on (1), we obtain the following restrictions on the infinitesimal functions ξ and η for generator (34):

$$\xi(x,t) = h_1(t)x + h_2(t), \qquad \eta(x,t) = (\dot{h}_1 - f_1 h_1)x^2 + g_2(t)x + g_1(t).$$

Here h_j, g_j are smooth functions to be determined by the conditions

$$\begin{aligned} &x^{n+1}A_1 + x^n A_2 + x^{n-1}A_3 + x^2 A_4 + xA_5 + A_6 = 0, \\ &x^n B_1 + xB_2 + B_3 = 0. \end{aligned} \tag{40}$$

The A's and B's are functions of f_1, f_2, f_3, h_1, h_2, g_1 and g_2. In particular,

$$A_1 = (2-n)f_1 f_3 + h_1\dot{f}_3 + nf_3\dot{h}_1, \quad A_2 = (n-1)f_3 g_2 + h_2\dot{f}_3 + 2f_3\dot{h}_2, \quad A_3 = nf_3 g_1,$$

$$A_4 = f_1 f_2 h_1 - f_1 h_1 \dot{f}_1 + \frac{d}{dt}(f_2 h_1) - f_1^2\dot{h}_1 - 2\dot{f}_1\dot{h}_1 - h_1\ddot{f}_1 + h_1^{(3)},$$

$$A_5 = f_1\dot{g}_2 + h_2\dot{f}_2 + 2f_2\dot{h}_2 + \ddot{g}_2, \quad A_6 = f_2 g_1 + f_1\dot{g}_1 + \ddot{g}_1,$$

$$B_1 = 3f_3 h_1, \quad B_2 = 3f_2 h_1 - 3\frac{d}{dt}(h_1 f_1) + 3\ddot{h}_1, \quad B_3 = 2\dot{g}_2 + \frac{d}{dt}(f_1 h_2) - \ddot{h}_2.$$

In general, one has to consider three cases depending on the nonlinearity: The linear case $n \in \{0, 1\}$, the case $n = 2$ as well as the case $n \in \mathcal{Q}\backslash\{0, 1, 2\}$.

Case 1: The linear case, i.e., $n = 0$ and $n = 1$. The equation can be point transformed in the free particle equation. The Lie point symmetry algebra is $sl(3, \mathcal{R})$, as discussed in the introduction. The Lie point symmetry generators are of the form

$$Z = (h_1(t)x + h_2(t))\frac{\partial}{\partial t} + \left\{\left(\dot{h}_1(t) - f_1(t)h_1(t)\right)x^2 + g_1(t)x + g_2(t)\right\}\frac{\partial}{\partial x},$$

where h_1, h_2, g_1, and g_2 take on particular functional forms, in terms of f_1, f_2 and f_3. This case was discussed in detail by Duarte *et al* [1].

Case 2: $n = 2$. System (40) reduces to the system

$$A_1 = 0, \quad A_2 + A_4 = 0, \quad A_3 + A_5 = 0, \quad A_6 = 0,$$
$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0.$$

From the equation $B_1 = 0$, it follows that $h_1 = 0$ so that the Lie point symmetry generator takes on the form

$$Z = h_2(t)\frac{\partial}{\partial t} + (g_2(t)x + g_1(t))\frac{\partial}{\partial x}, \tag{41}$$

where the remaining conditions on g_1, g_2 and h_2 are

$$f_3 g_2 + h_2 \dot{f}_3 + 2 f_3 \dot{h}_2 = 0, \tag{42}$$

$$2 f_3 g_1 + f_1 \dot{g}_2 + h_2 \dot{f}_2 + 2 f_2 \dot{h}_2 + \ddot{g}_2 = 0, \tag{43}$$

$$f_2 g_1 + f_1 \dot{g}_1 + \ddot{g}_1 = 0, \tag{44}$$

$$2\dot{g}_2 + \frac{d}{dt}(f_1 h_2) - \ddot{h}_2 = 0. \tag{45}$$

Note that $h_1 = 0$ for all $n \geq 2$. In solving conditions (42)–(45) we have to consider two subcases, namely $g_1 = 0$ and $g_1 \neq 0$.

Subcase 2.1: $g_1 = 0$. By (42) and (45), we obtain

$$g_2(t) = -\frac{d}{dt}[\ln f_3]\, h_2 - 2\dot{h}_2 \equiv \frac{1}{2}\dot{h}_2 - \frac{1}{2} f_1 h_2, \tag{46}$$

$$\dot{h}_2 = c_1 - \frac{1}{5}\left(2\frac{d}{dt}[\ln f_3] - f_1\right) h_2. \tag{47}$$

Inserting (46) and (47) into (43) leads to an expression of the form

$$F_1(f_1, f_2, f_3)\, h_2 + c_1 F_2(f_1, f_2, f_3) = 0, \tag{48}$$

where

$$\begin{aligned} F_1 = \Big(&-12 f_1^3 f_3^3 + 50 f_1 f_2 f_3^3 - 80 f_1 f_3^3 \dot{f}_1 + 125 f_3^3 \dot{f}_2 + 22 f_1^2 f_3^2 \dot{f}_3 \\ &-100 f_2 f_3 \dot{f}_3 + 35 f_3^2 \dot{f}_1 \dot{f}_3 + 21 f_1 f_3 \dot{f}_3^2 - 84 \dot{f}_3^3 - 50 f_3^3 \ddot{f}_1 \\ &-15 f_1 f_3^2 \ddot{f}_3 + 105 f_3 \dot{f}_3 \ddot{f}_3 - 25 f_3^2 f_3^{(3)}\Big) / \left(125 f_3^3\right), \end{aligned} \tag{49}$$

$$F_2 = 2\left(-6 f_1^2 f_3^2 + 25 f_2 f_3^2 - 10 f_3^2 \dot{f}_1 - f_1 f_3 \dot{f}_3 + 6 \dot{f}_3^2 - 5 f_3 \ddot{f}_3\right) / \left(25 f_3^2\right). \tag{50}$$

This leads to the following

Theorem 3: *The most general Lie point symmetry generator (34), which the equation*

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^2 = 0$$

may admit, is of the form

$$Z = h_2(t)\frac{\partial}{\partial t} + g_2(t)x\frac{\partial}{\partial x}$$

if and only if f_1, f_2 and f_3 satisfy one of the following conditions:

a) $F_2 = 0$, then h_2 is given by

$$h_2(t) = f_3^{-2/5} \exp\left(\frac{1}{5}\int^t f_1(\zeta)d\zeta\right)\left[c_1\int^t f_3^{2/5}(\rho)\exp\left(-\frac{1}{5}\int^\rho f_1(\zeta)d\zeta\right)d\rho + c_2\right].$$

and g_2 is given by (46).

b) $F_1 = 0$ with $c_1 = 0$, then h_2 is given by

$$h_2(t) = c_2 f_3^{-2/5}\exp\left(\frac{1}{5}\int^t f_1(\zeta)d\zeta\right)$$

and g_2 is given by (46).

c) $F_1 \neq 0$ and $F_2 \neq 0$ with $c_1 \neq 0$, then

$$h_2(t) = -\frac{c_1 F_2}{F_1}$$

and g_2 is given by (46), whereby the condition on f_1, f_2 and f_3 is

$$\dot F_1 F_2 - \dot F_2 F_1 - F_1^2 - \left(\frac{2}{5}\dot f_3 f_3^{-1} - \frac{1}{5}f_1\right)F_1F_2 = 0.$$

Here F_1 and F_2 are given by (49) and (50), respectively.

A note on the proof of Theorem 3: If $F_2 = 0$, it follows that $F_1 \equiv 0$. The infinitesimal function h_2 in the symmetry generator (41) is then obtained by integrating (47), whereby g_2 is given by (46). If $F_2 \neq 0$ and $c_1 \neq 0$, then F_1 must be nonzero, so that $h_2 = -c_1F_1/F_2$ has to satisfy (47).

Remark: Condition $F_2 = 0$ is identical to the condition by which (1) is point transformable in the integrable equation (12) and linearizable by the nonpoint transformation (30) (with $n = 2$). The Lie point symmetries (39) (with $n = 2$) obtained by the point transformations of Section 2, are those corresponding to Theorem 3a. The most general Lie point symmetry generator which follows from the conditions of Theorem 3b, and Theorem 3c cannot be obtained from the symmetries of the integrable equation (12).

Subcase 2.2: $g_1 \neq 0$. Equations (42) and (45) remain the same, therefore, relations (46) and (47) hold also for this subcase. By (43), g_1 is given by

$$g_1(t) = -\frac{1}{2f_3}\left(f_1\dot g_2 + h_2\dot f_2 + 2f_2\dot h_2 + \ddot g_2\right), \tag{51}$$

so that (44) leads to the expression

$$F_1(f_1, f_2, f_3)\, h_2 + c_1F_2(f_1, f_2, f_3) = 0$$

where

$$
\begin{aligned}
F_1 = \Big(&72 f_1^5 f_3^5 - 1250 f_1 f_2^2 f_3^5 + 1800 f_1^3 f_3^5 \dot{f}_1 + 5000 f_1 f_3^5 \dot{f}_1^2 - 2500 f_1^2 f_3^5 \dot{f}_2 - 3125 f_2 f_3^5 \dot{f}_2 \\
&-3125 f_3^5 \dot{f}_1 \dot{f}_2 - 720 f_1^4 f_3^4 \dot{f}_3 + 2500 f_1^2 f_2 f_3^4 \dot{f}_3 + 2500 f_2^2 f_3^4 \dot{f}_3 - 8300 f_1^2 f_3^4 \dot{f}_1 \dot{f}_3 \\
&+3125 f_2 f_3^4 \dot{f}_1 \dot{f}_3 - 6875 f_3^4 \dot{f}_1^2 \dot{f}_3 + 13125 f_1 f_3^4 \dot{f}_2 \dot{f}_3 + 2730 f_1^3 f_3^3 \dot{f}_3^2 - 13125 f_1 f_2 f_3^3 \dot{f}_3^2 \\
&+14100 f_1 f_3^3 \dot{f}_1 \dot{f}_3^2 - 22500 f_3^3 \dot{f}_2 \dot{f}_3^2 - 1485 f_1^2 f_3^2 \dot{f}_3^2 + 22500 f_2 f_3^2 \dot{f}_3^3 - 3150 f_3^2 \dot{f}_1 \dot{f}_3^3 \\
&-20790 f_1 f_3 \dot{f}_3^4 + 49896 \dot{f}_3^5 + 4000 f_1^2 f_3^5 \ddot{f}_1 + 6250 f_3^5 \dot{f}_1 \ddot{f}_1 - 10375 f_1 f_3^4 \dot{f}_3 \ddot{f}_1 \\
&+7875 f_3^3 \dot{f}_3^2 \ddot{f}_1 - 5625 f_1 f_3^5 \ddot{f}_2 + 11250 f_3^4 \dot{f}_3 \ddot{f}_2 - 1100 f_1^3 f_3^4 \ddot{f}_3 + 5625 f_1 f_2 f_3^4 \ddot{f}_3 \\
&-5625 f_1 f_3^4 \dot{f}_1 \ddot{f}_3 + 9375 f_3^4 \dot{f}_2 \ddot{f}_3 + 600 f_1^2 f_3^3 \dot{f}_3 \ddot{f}_3 - 20625 f_2 f_3^3 \dot{f}_3 \ddot{f}_3 + 1875 f_3^3 \dot{f}_1 \dot{f}_3 \ddot{f}_3 \\
&+33300 f_1 f_3^2 \dot{f}_3^2 \ddot{f}_3 - 103950 f_3 \dot{f}_3^3 \ddot{f}_3 - 3125 f_3^4 \ddot{f}_1 \ddot{f}_3 - 5625 f_1 f_3^3 \ddot{f}_3^2 + 39375 f_3^2 \dot{f}_3 \ddot{f}_3^2 \\
&+3750 f_1 f_3^5 f_1^{(3)} - 4375 f_3^4 \dot{f}_3 f_1^{(3)} - 3125 f_3^5 f_2^{(3)} + 125 f_1^2 f_3^4 f_3^{(3)} + 3125 f_2 f_3^4 f_3^{(3)} \\
&-8625 f_3^3 \dot{f}_3 f_3^{(3)} + 29250 f_3^2 \dot{f}_3^2 f_3^{(3)} - 9375 f_3^3 \ddot{f}_3 f_3^{(3)} + 1250 f_3^5 f_1^{(4)} + 1250 f_1 f_3^4 f_3^{(4)} \\
&-5625 f_3^3 \dot{f}_3 f_3^{(4)} + 625 f_3^4 f_3^{(5)}\Big) \,/\, (6250 f_3^6)\,,
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
F_2 = \Big(&36 f_1^4 f_3^4 - 625 f_2^2 f_3^4 + 720 f_1^2 f_3^4 \dot{f}_1 + 700 f_3^4 \dot{f}_1^2 - 1250 f_1 f_3^4 \dot{f}_2 - 288 f_1^3 f_3^3 \dot{f}_3 \\
&+1250 f_1 f_2 f_3^3 \dot{f}_3 - 1630 f_1 f_3^3 \dot{f}_1 \dot{f}_3 + 2500 f_3^3 \dot{f}_2 \dot{f}_3 + 429 f_1^2 f_3^2 \dot{f}_3^2 - 2500 f_2 f_3^2 \dot{f}_3^2 \\
&+680 f_3^2 \dot{f}_1 \dot{f}_3 + 1188 f_1 f_3 \dot{f}_3^3 - 3564 \dot{f}_3^4 + 1100 f_1 f_3^4 \ddot{f}_1 - 950 f_3^3 \dot{f}_3 \ddot{f}_1 \\
&-1250 f_3^4 \ddot{f}_2 - 190 f_1^2 f_3^3 \ddot{f}_3 + 1250 f_2 f_3^3 \ddot{f}_3 - 300 f_3^3 \dot{f}_1 \ddot{f}_3 - 1390 f_1 f_3^2 \dot{f}_3 \ddot{f}_3 \\
&+5940 f_3 \dot{f}_3^2 \ddot{f}_3 - 1075 f_3^2 \ddot{f}_3^2 + 500 f_3^4 f_1^{(3)} + 300 f_1 f_3^3 f_3^{(3)} \\
&-1600 f_3^2 \dot{f}_3 f_3^{(3)} + 250 f_3^3 f_3^{(4)}\Big) \,/\, (625 f_3^5)\,.
\end{aligned}
\tag{53}
$$

This leads to the following

Theorem 4: *The most general Lie point symmetry generator (34), which the equation*

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^2 = 0$$

may admit, is of the form

$$Z = h_2(t)\frac{\partial}{\partial t} + \{g_2(t)x + g_1(t)\}\,\frac{\partial}{\partial x}$$

if and only if f_1, f_2 and f_3 satisfy one of the following conditions:

a) $F_2 = 0$, then h_2 is given by

$$h_2(t) = f_3^{-2/5} \exp\left(\frac{1}{5}\int^t f_1(\zeta)d\zeta\right)\left[c_1 \int^t f_3^{2/5}(\rho)\exp\left(-\frac{1}{5}\int^\rho f_1(\zeta)d\zeta\right)d\rho + c_2\right],$$

g_1 by (51), and g_2 is given by (46).

b) $F_1 = 0$ with $c_1 = 0$, then h_2 is given by

$$h_2(t) = c_2 f_3^{-2/5} \exp\left(\frac{1}{5}\int^t f_1(\zeta)d\zeta\right),$$

g_1 by (51), and g_2 is given by (46).

c) $F_1 \neq 0$ and $F_2 \neq 0$ with $c_1 \neq 0$, then

$$h_2(t) = -\frac{c_1 F_2}{F_1},$$

g_1 by (51), and g_2 is given by (46), whereby the condition on f_1, f_2 and f_3 is

$$\dot{F}_1 F_2 - \dot{F}_2 F_1 - F_1^2 - \left(\frac{2}{5}\dot{f}_3 f_3^{-1} - \frac{1}{5} f_1\right) F_1 F_2 = 0$$

Here F_1 and F_2 are given by (52) and (53), respectively.

Case 3: $n \in \mathcal{Q}\backslash\{0,1,2\}$. This leads to the system

$$A_2 = 0 \qquad A_3 = 0, \qquad B_3 = 0, \qquad A_5 = 0.$$

From the equation $A_2 = 0$, it follows that $g_1 = 0$, so that the remaining conditions on h_2 and g_2 are

$$(n-1) f_3 g_2 + h_2 \dot{f}_3 + 2 f_3 \dot{h}_2 = 0, \tag{54}$$

$$f_1 \dot{g}_2 + h_2 \dot{f}_2 + 2 f_2 \dot{h}_2 + \ddot{g}_2 = 0, \tag{55}$$

$$2\dot{g}_2 + \frac{d}{dt}(f_1 h_2) - \ddot{h}_2 = 0. \tag{56}$$

To solve this system of equations, we need to consider two subcases:

Subcase 3.1: $n = -3$. This leads to

Theorem 5: *The most general Lie point symmetry generator (34), which the equation*

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^{-3} = 0$$

may admit, is of the form

$$Z = h_2(t)\frac{\partial}{\partial t} + g_2(t)x\frac{\partial}{\partial x}$$

if and only if

$$f_1 = -\frac{1}{2}\frac{\dot{f}_3}{f_3}, \tag{57}$$

where h_2 is a solution of

$$h_2^{(3)} + 4\Gamma(t)\dot{h}_2 + 2\dot{\Gamma}(t)h_2 = 0 \tag{58}$$

with

$$\Gamma(t) = -\frac{1}{16}\left[\frac{d}{dt}(\ln f_3)\right]^2 + \frac{1}{4}\frac{d^2}{dt^2}(\ln f_3) + f_2$$

and

$$g_2(t) = \frac{1}{4}\frac{d}{dt}(\ln f_3) + \frac{1}{2}\dot{h}_2.$$

Note that condition (57) is identical to the condition derived in Section 2, for an invertible point transformation of (1) with $n = -3$. Therefore, all Lie point symmetries of (1) with $n = -3$, which follow from Theorem 5, may also be obtained by applying the point transformation, in Section 2, in the Lie point symmetries of (12) (with $n = -3$).

Note that (58) may be transformed in the free particle equation (5): Let

$$A(t) = \frac{\dot{h}_2}{h_2}.$$

In this case, (58) reduces to

$$\ddot{A} + 3A\dot{A} + 4\Gamma(t)A + A^3 + 2\dot{\Gamma}(t) = 0.$$

Comparing this equation with (2), we find that condition (3) is satisfied.

Subcase 3.2: $n \neq -3$. By (54) and (56), we obtain

$$g_2(t) = -\left(\frac{1}{n-1}\right) h_2 \frac{d}{dt}(\ln f_3) - 2\left(\frac{1}{n-1}\right)\dot{h}_2, \tag{59}$$

$$\dot{h}_2 = c_1 - \left(\frac{n-1}{n+3}\right)\left\{\frac{2}{n-1}\frac{d}{dt}(\ln f_3) - f_1\right\} h_2. \tag{60}$$

Inserting (59) and (60) into (55) leads to

$$F_1(f_1, f_2, f_3)\, h_2 + c_1 F_2(f_1, f_2, f_3) = 0,$$

where

$$\begin{aligned} F_1 \;=\; & \Big[-4(n^2-1)f_1^3 f_3^3 + 2(n^3+5n^2+3n-9)f_1 f_2 f_3^3 - 8n(n+3)f_1 f_3^3 \dot{f}_1 \\ & +(n^3+9n^2+27n+27)f_3^3\dot{f}_2 - 2(n^2-6n-3)f_1^2 f_3^2 \dot{f}_3 - 4(n+3)^2 f_2 f_3^2 \dot{f}_3 \\ & -(n+3)(n-9)f_3^2\dot{f}_1\dot{f}_3 + 3(n-1)(n+5)f_1 f_3 \dot{f}_3^2 - 2(n+4)(n+5)\dot{f}_3^3 \\ & -2(n+3)^2 f_3^3 \ddot{f}_1 - 3(n-1)(n+3)f_1 f_3^2 \ddot{f}_3 + 3(n+3)(n+5)f_3\dot{f}_3\ddot{f}_3 \\ & -(n+3)^2 f_3^2 f_3^{(3)}\Big] \,/\, \left[(n+3)^3 f_3^3\right], \end{aligned} \tag{61}$$

$$\begin{aligned} F_2 \;=\; & 2\Big[-2(n+1)f_1^2 f_3^2 + (n+3)^2 f_2 f_3^2 - 2(n+3)f_3^2\dot{f}_1 \\ & -(n-1)f_1 f_3 \dot{f}_3 + (n+4)\dot{f}_3^2 - (n+3)f_3\ddot{f}_3\Big] \,/\, \left[(n+3)^2 f_3^2\right]. \end{aligned} \tag{62}$$

This leads to

Theorem 6: *The most general Lie point symmetry generator (34), which the equation*

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^n = 0,$$

with $n \in \mathcal{Q}\backslash\{-3, 0, 1, 2\}$*, may admit, is of the form*

$$Z = h_2(t)\frac{\partial}{\partial t} + g_2(t)x\frac{\partial}{\partial x}$$

if and only if f_1*,* f_2 *and* f_3 *satisfy one of the following conditions:*

a) $F_2 = 0$, *then* h_2 *is given by*

$$h_2(t) = f_3^{-2/(n+3)} \exp\left(\frac{n-1}{n+3}\int^t f_1(\zeta)d\zeta\right) \times \\ \times \left[c_1 \int^t f_3^{2/(n+3)}(\rho) \exp\left(-\frac{n-1}{n+3}\int^\rho f_1(\zeta)d\zeta\right) d\rho + c_2\right],$$

and g_2 *is given by (59).*

b) $F_1 = 0$ *with* $c_1 = 0$, *then* h_2 *is given by*

$$h_2(t) = c_2 f_3^{-2/n+3} \exp\left(\frac{n-1}{n+3}\int^t f_1(\zeta)d\zeta\right),$$

and g_2 *is given by (59).*

c) $F_1 \neq 0$ *and* $F_2 \neq 0$ *with* $c_1 \neq 0$, *then*

$$h_2(t) = -\frac{c_1 F_2}{F_1},$$

and g_2 *is given by (46), whereby the condition on* f_1, f_2 *and* f_3 *is*

$$\dot{F}_1 F_2 - \dot{F}_2 F_1 - F_1^2 - \frac{n-1}{n+3}\left(\frac{2}{n-1}\dot{f}_3 f_3^{-1} - f_1\right) F_1 F_2 = 0.$$

Here F_1 *and* F_2 *are given by (52) and (53), respectively.*

Remark: The condition $F_2 = 0$ is identical to the condition for transforming (1) in the integrable equation (12) and linearizing (1) by the non-oint transformation (30).

An important special case of equation (1) is the case where $f_1 = f_2 = 0$. By applying Theorem 3 to Theorem 6 we calculate the conditions on f_3 for which there exist Lie point symmetries of (1) (see Table 1 and Table 2). The general form of f_3 can be given in all cases, except where $g_1 \neq 0$, i.e., Theorem 4. For this case, we only list the conditions on f_3 (Table 1). Let us view the original determining equations for this case:

$$\ddot{g}_1 = 0, \qquad g_2(t) = \frac{1}{2}\dot{h}_2 - \frac{1}{2}c_1, \qquad f_3(t) = -\frac{1}{4}\frac{h_2^{(3)}}{g_1}.$$

This set of equations may be converted into a condition on h_2, namely

$$h_2 h_2^{(4)} - \left(-\frac{5}{2}\dot{h}_2 + \frac{\dot{g}_1}{g_1}h_2 - \frac{1}{2}c_1\right) h_2^{(3)} = 0,$$

where $h_2^{3)} \neq 0$. We did study solutions of this equation.

Table 1: $\ddot{x} + f_3(t)\, x^2 = 0$

Theorem 3: Lie Symmetry Generator $Z = h_2(t)\dfrac{\partial}{\partial t} + g_2(t)x\dfrac{\partial}{\partial x}$
Theorem 3a
$h_2(t) = -\dfrac{c_1}{k_1}(k_1 t + k_2) + c_2(k_1 t + k_2)^2$ $g_2(t) = c_2 k_1(k_1 t + k_2) - 3c_1$ $Z_1 = (k_1 t + k_2)^2\dfrac{\partial}{\partial t} + k_1(k_1 t + k_2)x\dfrac{\partial}{\partial x}, \quad Z_2 = \dfrac{1}{k_1}(k_1 t + k_2)\dfrac{\partial}{\partial t} + \dfrac{n+1}{n-1}x\dfrac{\partial}{\partial x}$ $[Z_1, Z_2] = -Z_1$
Condition on f_3
$F_2 \equiv \dfrac{2}{25 f_3^2}\left\{6\dot{f}_3^2 - 5 f_3 \ddot{f}_3\right\} = 0$
General solution for f_3
$f_3(t) = (k_1 t + k_2)^{-5}, \qquad k_1, k_2 \in \mathcal{R}$
Theorem 3b
$h_2(t) = c_2 k_3^{-2/5}\left(-\dfrac{1}{5}t^2 + \dfrac{2}{5}k_1 t + k_2\right), \qquad g_2(t) = -\dfrac{c_2}{5}k_3^{-2/5}(t - k_1)$
Condition on f_3
$F_1 \equiv \dfrac{1}{125 f_3^3}\left\{-84\dot{f}_3^3 + 105 f_3 \dot{f}_3 \ddot{f}_3 - 25 f_3^2 f_3^{(3)}\right\} = 0$
General solution for f_3
$f_3(t) = k_3\left(-\dfrac{1}{5}t^2 + \dfrac{2}{5}k_1 t + k_2\right), \qquad k_1, k_2, k_3 \in \mathcal{R}$

Table 1 (continued)

Thoerem 3c
$h_2(t)=k_1t^2+k_2t+k_3 \qquad g_2(t)=k_1t+\frac{1}{2}(k_2-c_1)$
Condition on f_3
$21f_3^2\dot{f}_3^2-42f_3\ddot{f}_3^3-24\dot{f}_3^3f_3^{(3)}+62f_3\dot{f}_3\ddot{f}_3f_3^{(3)}-15f_3^2\left(f_3^{(3)}\right)^2-12f_3\dot{f}_3^2f_3^{(4)}+10f_3^2\ddot{f}_3f_3^{(4)}=0$
Condition on f_3 with the subsititions: $A(t)=\dot{f}_3f_3^{-1}$ and $B(A)=\dot{A}(t)$
$\left(10B^3-2A^2B^2\right)B''-\left(5B^2+2A^2B\right)(B')^2+12AB^2B'-12B^3=0$ Note: The equation for B can be linearized by a point transformation.
General solution for f_3
$f_3(t)=k_4(k_1t^2+k_2t+k_3)^{-5/2}\exp\left\{\frac{c_1}{2}\int^t\frac{d\rho}{k_1\rho^2+k_2\rho+k_3}\right\}, \qquad k_j\in\mathcal{R}$

Theorem 4: Lie Symmetry Generator $Z=h_2(t)\frac{\partial}{\partial t}+\{g_1(t)+g_2(t)x\}\frac{\partial}{\partial x}$
Theorem 4a
$h_2(t)=f_3^{-2/5}\left\{c_1\int^t f_3^{2/5}(\rho)d\rho+c_2\right\}, \qquad g_2(t)=-\frac{1}{5}\frac{\dot{f}_3}{f_3}h_2-2c_1$ $g_1(t)=k_1t+k_2, \qquad k_1,k_2\in\mathcal{R}$
Condition on f_3
$F_2\equiv\frac{1}{625f_3^5}\left\{-3564\dot{f}_3^4+5940f_3\dot{f}_3^2\ddot{f}_3-1075f_3^2\ddot{f}_3^2-1600f_3^2\dot{f}_3f_3^{(3)}+250f_3^3f_3^{(4)}\right\}=0$

Table 1 (continued)

Condition on f_3 with the subsititions: $A(t) = \dot{f}_3 f_3^{-1}$ and $B(A) = \dot{A}(t)$

$$250B^2B'' + 250B(B')^2 - 600ABB' - 325B^2 + 490A^2B - 49A^4 = 0$$

Note: The equation for B cannot be linearized by a point transformation.

Theorem 4b

$$h_2(t) = c_2 f_3^{-2/5}, \qquad g_2(t) = -\frac{c_2}{5}\dot{f}_3 f_3^{-7/5}$$

$$g_1(t) = k_1 t + k_2, \qquad k_1, k_2 \in \mathcal{R}$$

Condition on f_3

$$F_1 \equiv \frac{1}{6250 f_3^6}\Big\{49896\dot{f}_3^5 - 103950 f_3\dot{f}_3^3\ddot{f}_3 + 39375 f_3^2\dot{f}_3\ddot{f}_3^2 + 29250 f_3^2\dot{f}_3^2 f_3^{(3)}$$

$$-9375 f_3^3\ddot{f}_3 f_3^{(3)} - 5625 f_3^3\dot{f}_3 f_3^{(4)} + 625 f_3^4 f_3^{(5)}\Big\} = 0$$

Condition on f_3 with the subsititions: $A(t) = \dot{f}_3 f_3^{-1}$ and $B(A) = \dot{A}(t)$

$$625B^3B''' + 2500B^2(B' - A)B'' + 625B(B')^3 - 2500BA(B')^2 + 125B(29A^2 - 25B)B'$$

$$+3750B^2A - 2450BA^3 + 196A^5 = 0$$

Theorem 4c

$$h_2(t) = -\frac{F_2}{F_1}, \qquad g_2(t) = -\frac{1}{5}\frac{\dot{f}_3}{f_3}h_2 - 2c_1$$

$$g_1(t) = k_1 t + k_2, \qquad k_1, k_2 \in \mathcal{R}$$

Condition on f_3

$$24948\dot{f}_3^6\ddot{f}_3^2 + \cdots \text{ 31 terms } \cdots + 500 f_3^6 f_3^{(4)} f_3^{(6)} = 0$$

Table 2: $\ddot{x} + f_3(t)\, x^n = 0$

Theorem 5: $n = -3$. Lie Symmetry Generator $Z = h_2(t)\dfrac{\partial}{\partial t} + g_2(t)x\dfrac{\partial}{\partial x}$

$$h_2(t) = c_1 t^2 + c_2 t + c_3, \qquad g_2(t) = c_1 t + \frac{1}{2}c_2, \qquad f_3 = \text{ constant}$$

$$Z_1 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x}, \qquad Z_2 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x}, \qquad Z_3 = \frac{\partial}{\partial t}$$

$$[Z_1, Z_2] = -Z_1, \qquad [Z_1, Z_3] = -2Z_2, \qquad [Z_2, Z_3] = -Z_3$$

Theorem 6: $n \in \mathcal{Q}\backslash\{-3, 0, 1, 2\}$. Lie Symmetry Generators $Z = h_2(t)\dfrac{\partial}{\partial t} + g_2(t)x\dfrac{\partial}{\partial x}$

Theorem 6a

$$h_2(t) = -\frac{c_1}{k_1}(k_1 t + k_2) + c_2(k_1 t + k_2)^2, \qquad g_2(t) = c_2 k_1(k_1 t + k_2) - c_1\left(\frac{n+1}{n-1}\right)$$

$$Z_1 = (k_1 t + k_2)^2\frac{\partial}{\partial t} + k_1(k_1 t + k_2)x\frac{\partial}{\partial x}, \qquad Z_2 = \frac{1}{k_1}(k_1 t + k_2)\frac{\partial}{\partial t} + \left(\frac{n+1}{n-1}\right)x\frac{\partial}{\partial x}$$

$$[Z_1, Z_2] = -Z_1$$

Condition on f_3

$$F_2 \equiv \frac{2}{(n+3)^2 f_3^2}\left\{(n+4)\dot{f}_3^2 - (n+3)f_3\ddot{f}_3\right\} = 0$$

General solution for f_3

$$f_3(t) = (k_1 t + k_2)^{-(n+3)}, \qquad k_1, k_2 \in \mathcal{R}$$

Theorem 6b

$$h_2(t) = c_2 k_3^{-2/(n+3)}\left\{-\frac{1}{n+3}t^2 + k_1\frac{2}{n+3}t + k_2\right\}, \qquad g_2(t) = -\frac{c_2}{n+3}k_3^{-2/(n+3)}(t - k_1)$$

Table 2 (continued)

Condition on f_3
$F_1 \equiv -\frac{1}{(n+3)^3 f_3^3}\left\{-2(n^2+9n+20)\dot{f}_3^3 + 3(n^2+8n+15) f_3 \dot{f}_3 \ddot{f}_3 - (n+3)^2 f_3^2 f_3^{(3)}\right\} = 0$
General solution for f_3
$f_3(t) = k_3\left\{-\frac{1}{n+3}t^2 + k_1\frac{2}{n+3}t + k_2\right\}^{-(n+3)/2}$
Theorem 6c:
$h_2(t) = k_1 t^2 + k_2 t + k_3, \qquad g_2 = k_1 t + \frac{1}{2}(k_2 - c_1)$
Condition on f_3
$3(n+5)\dot{f}_3^2 \ddot{f}_3^2 - 6(n+5) f_3 \ddot{f}_3^3 - 4(n+4)\dot{f}_3^3 f_3^{(3)} + (42+10n) f_3 \dot{f}_3 \ddot{f}_3 f_3^{(3)}$ $-3(n+3) f_3^2 \left(f_3^{(3)}\right)^2 - 2(n+4) f_3 \dot{f}_3^2 f_3^{(4)} + 2(n+3) f_3^2 \ddot{f}_3 f_3^{(4)} = 0$
Condition on f_3 with the subsititions: $A(t) = \dot{f}_3 f_3^{-1}$ and $B(A) = \dot{A}(t)$
$2\left\{(n+3)(B^3) - A^2 B^2\right\} B'' - \left\{(n+3)B^2 + 2A^2 B\right\}(B')^2 + 12AB^2B' - 12B^3 = 0$ Note: The equation for B can be linearized by a point transformation.
General solution for f_3
$f_3(t) = k_4(k_1 t^2 + k_2 t + k_3)^{-(n+3)/2} \exp\left\{c_1\left(\frac{n-1}{2}\right)\int^t \frac{d\rho}{k_1\rho^2 + k_2\rho + k_3}\right\}, \qquad k_j \in \mathcal{R}$

4.3 Invertible point transformations by Lie point symmetries

One can now use the Lie point symmetry generators obtained above to construct time dependent first integrals of (1). We give the point transformation in general form and do an example to illustrate the procedure.

In Table 1 and Table 2 we make use of the substitution

$$\frac{\dot f_3}{f_3} = A(t), \qquad B(A) = \dot A(t),$$

by which we are able to reduce the order of the differential conditions on f_3 by two. It follows that

$$\begin{aligned}
&\frac{\ddot f_3}{f_3} = B + A^2, \qquad \frac{f^{(3)}}{f_3} = B'B + 3BA + A^3,\\
&\frac{f_3^{(4)}}{f_3} = B''B^2 + (B')^2B + 4B'BA + 3B^2 + 6BA^2 + A^4,\\
&\frac{f_3^{(5)}}{f_3} = B'''B^3 + 4B''B'B^2 + (B')^3B + 5B''B^2A + 5(B')^2BA + 10B'B^2\\
&\qquad +10B'BA^2 + 15B^2A + 10BA^3 + A^5,
\end{aligned}$$

where $B' \equiv dB/dA$, etc.

Let us consider the transformation of (1) in an autonomous form by the use of the most general Lie point symmetry generator for the nonlinear equation (1), namely

$$Z = h_2(t)\frac{\partial}{\partial t} + \{g_1(t) + g_2(t)x\}\frac{\partial}{\partial x}.$$

The defining equations for the point transformations

$$X(T) = F(x,t), \qquad T(x,t) = G(x,t)$$

are

$$\begin{aligned}
&h_2(t)\frac{\partial T}{\partial t} + \{g_1(t) + g_2(t)x\}\frac{\partial T}{\partial x} = 1,\\
&h_2(t)\frac{\partial X}{\partial t} + \{g_1(t) + g_2(t)x\}\frac{\partial X}{\partial x} = 0.
\end{aligned}$$

The general solution of this system is

$$\begin{aligned}
&X(T) = \varphi_1(\omega), \qquad T(x,t) = \int^t \frac{1}{h_2(\rho)}\,d\rho + \varphi_2(\omega),\\
&\omega = x\,\exp\left\{-\int^t \frac{g_2(\rho)}{h_2(\rho)}\,d\rho\right\} - \int^t \frac{g_1(\rho)}{h_2(\rho)}\exp\left\{-\int^\rho \frac{g_2(\zeta)}{h_2(\zeta)}d\zeta\right\}d\rho,
\end{aligned} \tag{63}$$

where φ_1 and φ_2 are arbitrary functions of ω and may be chosen in a convenient form.

To construct a point transformation which may transfrom (1) in a first-order ODE, one needs to solve the system

$$\begin{aligned}
&h_2(t)\frac{\partial T}{\partial t} + \{g_1(t) + g_2(t)x\}\frac{\partial T}{\partial x} = 0,\\
&h_2(t)\frac{\partial X}{\partial t} + \{g_1(t) + g_2(t)x\}\frac{\partial X}{\partial x} = 1.
\end{aligned}$$

The general solution of this system is

$$X(T) = \int^t \frac{1}{h_2(\rho)}\, d\rho + \psi_1(\omega), \qquad T(x,t) = \psi_2(\omega),$$
$$\omega = x \exp\left\{-\int^t \frac{g_2(\rho)}{h_2(\rho)}\, d\rho\right\} - \int^t \frac{g_1(\rho)}{h_2(\rho)} \exp\left\{-\int^\rho \frac{g_2(\zeta)}{h_2(\zeta)} d\zeta\right\} d\rho,$$

where ψ_1 and ψ_2 may be chosen arbitrary. It can be shown that this point transformation reduces (1) in an Abel equation of the first kind, i.e.,

$$\frac{dY}{dT} + \Gamma_3(T)Y^3 + \Gamma_2(T)Y^2 + \Gamma_1(T)Y + \Gamma_0(T) = 0$$

with $Y = dX/dT$. Since the known solutions of the Abel equation are restricted to special forms of the functions Γ, it is usually a useless exercise. However, if the equation admits a two-dimensional Lie point symmetry algebra, it is well known that the reduction may be performed such that the reduced equation (in this case, the Abel equation) admits a Lie point symmetry. In particular, if Z_1 and Z_2 are symmetry generators of the equation to be reduced, and

$$[Z_1, Z_2] = \lambda Z_1,$$

then the generator Z_1 should be used to perform the reduction. This will ensure that the symmetry generator Z_2 is 'preserved' by the reduction, i.e., the reduced equation will admit the generator Z_2 in transformed form. This is important to note, since the integrating factor μ of any first-order ODE of the form $\dot{x} = -M(x,t)/N(x,t)$, which admits the Lie point symmetry generator $Z = \xi\partial/\partial t + \eta\partial/\partial x$, is given by $\mu = (N\eta + M\xi)^{-1}$. Note also that it is not a simple task to find Lie point symmetry generators for a first order ODE. In particular, the determining equation of a Lie point symmetry generator for the ODE $\dot{x} = f(x,t)$ is

$$-\xi\frac{\partial f}{\partial t} - \eta\frac{\partial f}{\partial x} + \frac{\partial \eta}{\partial t} + f\frac{\partial \eta}{\partial x} - f\frac{\partial \xi}{\partial t} - f^2\frac{\partial \xi}{\partial x} = 0.$$

As an example, we transform

$$\ddot{x} + k_3\left(-\frac{1}{n+3}\, t^2 + \frac{2k_1}{n+3}\, t + k_2\right)^{-(n+3)/2} x^n = 0, \qquad k_1, k_2 \in \mathcal{R}, \tag{64}$$

for $n \in \mathcal{Q}\backslash\{-3, 0, 1\}$, in an autonomous form. This case corresponds to Theorem 6b given in Table 2. Note that

$$g_2 = \frac{1}{2}\dot{h}_2, \qquad h_2(t) = -\frac{1}{n+3}\, t^2 + \frac{2k_1}{n+3}\, t + k_2.$$

The general form of the point transformation is given by (63). We let $\varphi_1 = \omega$, $\varphi_2 = 0$. It follows that

$$X(T) = h_2^{-1/2}\, x, \qquad T(x,t) = \int^t \frac{d\rho}{h_2(\rho)}.$$

This transformation leads to the autonomous equation

$$\frac{d^2X}{dT^2} - \left\{\left(\frac{k_1}{n+3}\right)^2 + \frac{k_2}{n+3}\right\} X + c_2^{(n+3)/2} X^n = 0$$

which has the first integral

$$T\left(X, \frac{dX}{dT}\right) = \left(\frac{dX}{dT}\right)^2 - \left\{\left(\frac{k_1}{n+3}\right)^2 + \frac{k_2}{n+3}\right\} X^2 + \left(\frac{2}{n+1}\right) c_2^{(n+3)/2} X^{n+1} = 0.$$

A first integral for (64) is then

$$\begin{aligned} I(t, x, \dot{x}) &= \frac{1}{h_2}\left(-\frac{1}{2}\dot{h}_2 x + h_2 \dot{x}\right)^2 - \frac{1}{h_2}\left\{\left(\frac{k_1}{n+3}\right)^2 + \frac{k_2}{n+3}\right\} x^2 \\ &\quad + \frac{2}{n+1} c_2^{(n+3)/2} h_2^{-(n+1)/2} x^{n+1}. \end{aligned}$$

5 Transforming in the second Painlevé transcendent

By using the Lie point symmetry generators classified in the previous section, we are able to construct point transformations by which (1) may be transformed in a second equation. However, this method does not allow for a transformation in an equation without a Lie point symmetry. This can easily be shown: Consider a second equation in the variables (X, T) with no symmetry. To construct a point transformation in this equation by a Lie point symmetry generator $Z = \xi\partial/\partial t + \eta\partial/\partial t$ of the first equation with variables (x, t), we need to solve the system

$$\xi \frac{\partial X}{\partial t} + \eta \frac{\partial X}{\partial x} = 0, \qquad \xi \frac{\partial T}{\partial t} + \eta \frac{\partial T}{\partial x} = 0,$$

which implies that the Jacobian is zero.

There is an important class of equations which do not admit Lie point symmetries, but are integrable. These are the six Painlevé transcendents, as discussed in the introduction. We consider the following problem (Euler *et al* 1991): Find the condition on f_1, f_2, and f_3 for which (1), with $n = 3$, i.e.,

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^3 = 0,$$

may be point transformed in the second Painlevé transcendent

$$\frac{d^2X}{dT^2} - TX - 2X^3 - a = 0, \qquad a \in \mathcal{R}. \tag{65}$$

By the point transformation

$$X(T) = f(t)x, \qquad T(x, t) = g(t) \tag{66}$$

we obtain the following expressions:

$$f_1(t) = 2\frac{\dot{f}}{f} - \frac{\ddot{g}}{\dot{g}}, \qquad f_2(t) = \frac{\ddot{f}}{f} - \frac{\dot{f}}{f}\frac{\ddot{g}}{\dot{g}} - g\dot{g}^2, \qquad f_3(t) = -2(f\dot{g})^2.$$

Inverting this system leads to

Theorem 7: *Equation*

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^3 = 0$$

may be point transformed in

$$\frac{d^2X}{dT^2} - TX - 2X^3 = 0$$

by the invertible point transformation

$$X(T) = f(t)x, \qquad T(x,t) = g(t),$$

where

$$f(t) = kf_3^{1/6} \exp\left\{\int^t \frac{1}{3} f_1(\rho)d\rho\right\},$$
$$g(t) = \frac{k^2}{18 f_3^{8/3}} \left(-6f_3\ddot{f}_3 + 7\dot{f}_3^2 - 2f_1f_3\dot{f}_3 - 12\dot{f}_1f_3^2 + 36f_2f_3^2 - 8f_1^2f_3^2\right) \exp\left\{\frac{2}{3}\int^t f_1(\rho)d\rho\right\},$$

under the following conditions:

$$\begin{aligned} &9f_3^{(4)}f_3^3 - 54f_3^{(3)}\dot{f}_3f_3^2 + 18f_3^{(3)}f_3^3f_1 - 36\ddot{f}_3^2f_3^2 + 192\ddot{f}_3\dot{f}_3^2f_3 - 78\ddot{f}_3\dot{f}_3f_3^2f_1 + 36\ddot{f}_3f_3^3f_2 \\ &+3\ddot{f}_3f_3^3f_1^2 - 112\dot{f}_3^4 + 64\dot{f}_3^3f_3f_1 + 6\dot{f}_3^2\dot{f}_1f_3^2 - 72\dot{f}_3^2f_3^2f_2 + 90\dot{f}_3\dot{f}_2f_3^3 - 27\dot{f}_3\ddot{f}_1f_3^3 \\ &-57\dot{f}_3\dot{f}_1f_3^3f_1 + 72\dot{f}_3f_3^3f_2f_1 - 14\dot{f}_3f_3^3f_1^3 - 54\ddot{f}_2f_3^4 - 90\dot{f}_2f_3^4f_1 + 18f_1^{(3)}f_3^4 \\ &+54\ddot{f}_1f_3^4f_1 + 36\dot{f}_1^2f_3^4 - 36\dot{f}_1f_3^4f_2 + 60\dot{f}_1f_3^4f_1^2 - 36f_3^4f_2f_1^2 + 8f_3^4f_1^4 = 0, \end{aligned} \tag{67}$$

with

$$-6f_3\ddot{f}_3 + 7\dot{f}_3^2 - 2f_1f_3\dot{f}_3 - 12\dot{f}_1f_3^2 + 36f_2f_3^2 - 8f_1^2f_3^2 \neq 0. \tag{68}$$

Remark: The l.h.s. of (68) equal to zero is identical to the condition obtained in Section 2 by which (1) may be point transformed in $d^2X/dT^2 + X^3 = 0$, and which linearizes (1) by a nonpoint transformation (with $n = 3$).

It is easy to show that if f_1, f_2 and f_3 are such that the l.h.s. of (68) is equal to zero, then condition (67) is satisfied identically for those functional forms. In Euler *et al* [6], we showed that (1) passes the Painlevé test if and only if condition (67) is satisfied. (We refer to the book of Steeb and Euler [16] for more details on the Painlevé test of nonlinear evolution equations.) Thus, if conditions (67) and (68) are satisfied, equation (1) has the Painlevé property and is therefore integrable; this is true since there exists an invertible point transformation in the second Painlevé transcendent. Moreover, equation (1) with f_1, f_2 and f_3, which make the l.h.s. of (68) equal to zero, also has the Painlevé property, since the Painlevé test is passed and the equation can invertibly be point transformed in the integrable equation $d^2X/dT^2 + X^3 = 0$.

Let us finally consider the special case where $f_1 = f_2 = 0$. By (67), we obtain the condition

$$9A^{(3)} - 18\ddot{A}A + 12\dot{A}A^2 - 9\dot{A}^2 - A^4 = 0, \tag{69}$$

where $A(t) = \dot{f}_3/f_3$. This equation admits three Lie point symmetry generators

$$Z_1 = -\frac{t^2}{6}\frac{\partial}{\partial t} + \left(\frac{1}{3}At + 1\right)\frac{\partial}{\partial A}, \qquad Z_2 = t\frac{\partial}{\partial t} - A\frac{\partial}{\partial A}, \qquad Z_3 = \frac{\partial}{\partial t}.$$

By using these symmetry properties, (69) may be transformed in the following Abel equation

$$\frac{dU}{dT} = g_1(T)U + g_2(T)U^2 + g_3(T)U^3,$$

where

$$U(T) = \left(\frac{dX}{dT}\right)^{-1}, \quad T(u, A) = u \exp\{-2X\}, \quad X(T) = \ln A, \qquad u(A) = \frac{dA}{dt},$$
$$g_1(T) = \frac{1}{T^2}\left(\frac{1}{9} - \frac{4}{3}T + 5T^2 - 6T^3\right), \quad g_2(T) = \frac{1}{T}(2 - 7T), \quad g_3(T) = -\frac{1}{T}.$$

6 Conclusion

We have seen that if condition (19) (with $n \notin \{-3, -1, 0, 1\}$) is satisfied, equation (1) has the following properties:

a) Equation (1) may be point transformed in

$$\frac{d^2X}{dT^2} + X^n = 0.$$

b) Equation (1) may be linearized in

$$\frac{d^2X}{dT^2} + k_2 = 0, \qquad k_2 \in \mathcal{R}\backslash\{0\}$$

by a nonpoint transformation.

c) Equation (1) admits a two-dimensional Lie point symmetry algebra.

By the Lie point symmetry classification, we have observed that (1) admits a Lie symmetry generator with $g_1 \neq 0$ only in the case $n = 2$. This is a very complicated case to solve in general. We have given the general conditions on f_1, f_2 and f_3 for the existence of Lie symmetries in this case as well as all other cases of n. Leach and Maharaj [10] calculated some special cases where the Lie point symmetries may be given explicitly.

For the case $n = 3$, we have given the necessary and sufficient conditions on f_1, f_2 and f_3 by which (1) has the Painlevé property. This includes also condition (19) with $n = 3$. A detailed Painlevé analysis of (1) will be the subject of a future paper.

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