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On Classification of Subalgebras of the Poincaré Algebra

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Abstract

The substantiation of the algorithm for classifying subalgebras of the Poincaré algebra $AP(1,n)$ up to $P(1,n)$-conjugacy is completed

1. Introduction. Subalgebras of the Poincaré algebra $AP(1,3)$ have been classified up to conjugacy with respect to the group of inner automorphisms, i.e., up to $P(1,3)$-conjugacy [1–6]. The classification of subalgebras of the algebra $AP(1,4)$ up to $P(1,4)$-conjugacy is done in [7–9]. Subalgebras of the algebra $AP(1,n)$ for an arbitrary n were investigated in [10–12]. In present article we proved a number of Propositions substantiating the procedure of classifying subalgebras of the algebra $AP(1,n)$ up to $P(1,n)$-conjugacy for an arbitrary $n \geq 2$.

We suppose that the Poincaré group $P(1,n)$ is realized as a multiplicative group of matrices of the form

$$\Gamma = \begin{pmatrix} C & Y \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where $C \in O(1,n)$ and Y is a real $(1+n)$-dimensional column. We consider the Poincaré algebra $AP(1,n)$ as the algebra of matrices

$$Z = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where

$$X = \begin{pmatrix} 0 & \beta_{01} & \beta_{02} & \cdots & \beta_{0n} \\ \beta_{01} & 0 & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{02} & -\beta_{12} & 0 & \cdots & \beta_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \beta_{0n} & -\beta_{1n} & -\beta_{2n} & \cdots & 0 \end{pmatrix}.$$

Let $I_{\alpha\beta}$ be a square matrix of order $n+2$, having unity at the intersection of the α-th row and β-th column and zeros elsewhere ($\alpha, \beta = 0, 1, \ldots, n+1$). The basis of the algebra $AP(1,n)$ consists of matrices

$$P_0 = I_{0,n+1}, \quad P_a = I_{a,n+1}, \quad J_{0a} = -I_{0a} - I_{a0},$$

$$J_{ab} = -I_{ab} + I_{ba} \quad (a < b;\ a, b = 1, \ldots, n).$$

These matrices are connected by the following commutation relations:

$$[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta}J_{\beta\gamma} + g_{\beta\gamma}J_{\alpha\delta} - g_{\alpha\gamma}J_{\beta\delta} - g_{\beta\delta}J_{\alpha\gamma},$$

$$[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta, \quad [P_\alpha, P_\beta] = 0,$$

where $\alpha, \beta, \gamma, \delta = 0, 1, \dots, n$ and $(g_{\alpha\beta}) = \mathrm{diag}[1, -1, \dots, -1]$.

It is easy to see that $AP(1,n) = U \supset\!\!\!\!+ AO(1,n)$, where $U = \langle P_0, P_1, \dots, P_n\rangle$ and $AO(1,n) = \langle J_{\alpha\beta} : \alpha, \beta = 0, 1 \dots, n\rangle$. It is convenient to identify elements of the algebra $AO(1,n)$ with matrices X, and elements of the ideal U with real $(1+n)$-dimensional columns Y.

For an arbitrary matrix Γ of the form (1), the mapping $Z \to \Gamma Z \Gamma^{-1}$ is an automorphism of the algebra $AP(1,n)$. Denote this automorphism by φ_Γ and call it $P(1,n)$-*automorphism corresponding to* Γ. If $\Gamma = \mathrm{diag}[C, 1]$, then we shall write φ_C instead of φ_Γ. The automorphism φ_C will be referred to as $O(1,n)$-automorphism of the algebra $AP(1,n)$, corresponding to the matrix C. We shall consider the ideal U as the Minkowski space $\mathbb{R}_{1,n}$ with the orthonormal basis $P_0, P_1, \dots, P_n$. In this case, the restriction of the automorphism φ_C onto U is an isometry of this space, and each isometry of the space U can be obtained in such a manner. Sometimes we shall identify the isometry φ_C with the matrix C.

Subalgebras L_1 and L_2 of the algebra $AP(1,n)$ are called $P(1,n)$-*conjugate*, if $\varphi_\Gamma(L_1) = L_2$ for some matrix $\Gamma \in P(1,n)$. If $\varphi_C(L_1) = L_2$ for $C \in O(1,n)$, then L_1 and L_2 are named *O(1,n)-conjugate*.

We denote the subdirect sum of Lie algebras $A_1, \dots, A_m$ by the symbol $A_1 \dot{+} \cdots \dot{+} A_m$. We shall denote by $\hat{\varepsilon}$ a projection of $AP(1,n)$ onto $AO(1,n)$, and by $\mathrm{Ad}\, L$ the group of inner automorphisms of the algebra L.

Definitions of other notions used in our article are given in [12].

2. Subalgebras of Euclidean algebras. A *Euclidean algebra* $AE(n)$ is a Lie algebra isomorphic to the algebra $Q \supset\!\!\!\!+ AO(n)$, where $Q = \langle P_1, \dots, P_n\rangle$ and $AO(n) = \langle J_{ab} : a, b = 1, \dots, n\rangle$. An *extended Euclidean algebra* $A\tilde{E}(n)$ is a Lie algebra isomorphic to the algebra $AE(n) \supset\!\!\!\!+ \langle D\rangle$, where $D = -I_{11} - I_{22} - \ldots - I_{nn}$. It is easy to make sure that $[D, P_a] = -P_a$, $[D, J_{ab}] = 0$ for all $a, b = 1, 2, \dots, n$.

To simplify the presentation we shall sometimes assume that $AO(n)$ is the algebra of real skew-symmetric matrices of order n and Q is the Euclidean space of n-dimensional columns. Identify a matrix $X \in AO(n)$ with the operator of multiplication of columns of the space Q by X (from the left).

A subalgebra $L \neq 0$ of the algebra $AO(n)$ is called *irreducible* if the space Q has no nonzero L-invariant subspaces different from Q. In the opposite case, L is called the *reducible* subalgebra.

Let L be a reducible subalgebra of the algebra $AO(n)$. Then there is a matrix $C \in O(n)$ such that $C^{-1}LC$ consists of skew-symmetric matrices of the form $X = \mathrm{diag}[X_1, \dots, X_s]$ or the form $X = \mathrm{diag}[X_1, \dots, X_s, 0]$, where for every $j = 1, \dots, s$ the matrix X_j goes through some irreducible subalgebra L_j of the algebra $AO(m_j)$. In what follows, we suppose that $C^{-1}LC = L$. The mapping $\pi_j : L \to AO(m_j)$ defined by the equality $\pi_j(X) = X_j$, is a homomorphism of L onto L_j. The algebra L decomposes into the subdirect product of the algebras $L_1, \dots, L_s$. To denote this we use the notation $L = \pi_1(L) \dot{\times} \cdots \dot{\times} \pi_s(L)$. The algebras $\pi_1(L), \dots, \pi_s(L)$ are called *irreducible parts of the algebra* L.

Proposition 1. *Let*

$$L = \pi_1(L) \dot{\times} \cdots \dot{\times} \pi_s(L), \qquad L' = \pi'_1(L') \dot{\times} \cdots \dot{\times} \pi'_{s'}(L')$$

be decompositions of subalgebras L and L' of the algebra $AO(n)$ into subdirect products of irrreducible parts. The subalgebras L and L' are $O(n)$-conjugate if and only if $s = s'$ and there exists an isomorphism $f : L \to L'$ that, up to the indexing of irreducible parts, $C_j \pi_j(X) C_j^{-1} = \pi'_j(f(X))$ for all $X \in L$ and all $j = 1, \ldots, s$, where C_j is an orthogonal matrix.

Proposition 1 follows from the well-known statements about irreducible subrepresentations of a representation of a Lie algebra by skew-symmetric matrices.

Denote by $O[r, s]$, $r \leq s$ the subgroup of isometries from $O(n)$, that preserve the subspace $\langle P_r, \ldots, P_s\rangle$ and act identically on its orthogonal complement. If $r > s$, then we suppose that $O[r, s]$ consists of an identity isometry. For $r < s$, the Lie algebra $AO[r, s]$ of the group $O[r, s]$ is generated by matrices J_{ab}, where $a, b = r, r + 1, \ldots, s$, and for $r \geq s$ we have $AO[r, s] = 0$.

On the basis of results from [13], each nonzero subalgebra of the algebra $AE(n)$ is conjugate with respect to the group of $E(n)$-automorphisms to a subalgebra of the form

$$V \Supset (F_1 \dot{+} F_2 \dot{+} W), \tag{3}$$

satisfying the following conditions:

1) $V = 0$, $F_1 = 0$ or $V = \langle P_1, \ldots, P_k\rangle$ and F_1 is a subalgebra of the algebra $AO(k)$, not conjugate to a subalgebra of the algebra $AO(k - 1)$;

2) $F_2 = 0$ or F_2 is a subalgebra of the algebra $AO[b+1, b+l]$, $l \geq 2$, not conjugate to a subalgebra of the algebra $AO[b+1, b+l-1]$, where $b = 0$ for $F_1 = 0$ and $b = k$ for $F_1 \neq 0$;

3) $W = 0$ or $W = \langle P_{d+1}, P_{d+2}, \ldots, P_{d+m}\rangle$, where $d = 0$ for $F_1 = F_2 = 0$, $d = k$ for $F_1 \neq 0$, $F_2 = 0$ and $d = b + l$ for $F_2 \neq 0$.

Let us stipulate that $k = 0$ for $F_1 = 0$, $l = 0$ for $F_2 = 0$ and $m = 0$ for $W = 0$. The vector (k, l, m) will be called the *type of subalgebra (3)*.

Each subalgebra of the algebra $A\tilde{E}(n)$ with a nonzero projection onto $\langle D\rangle$ is conjugate with respect to the group of $E(n)$-automorphisms to a subalgebra of the form

$$(V \oplus W) \Supset (F_1 \dot{+} F_2 \dot{+} \langle D\rangle), \tag{4}$$

where V, W, F_1, F_2 are the same as in (3). The type of the subalgebra $(V \oplus W) \Supset (F_1 \dot{+} F_2)$ will be named the *type of subalgebra (4)*.

Obviously, each subalgebra of the form (3) is not conjugate to subalgebras of the form (4).

Theorem 1. *Let L_i $(i = 1, 2)$ be a nonzero subalgebra of the form (3) of the algebra $AE(n)$ and L_i have the type (k_i, l_i, m_i). The subalgebras L_1 and L_2 are $E(n)$-conjugate if and only if $k_1 = k_2 = k$, $l_1 = l_2 = l$, $m_1 = m_2 = m$ and the subalgebras L_1, L_2 are H-conjugate, where*

$$H = O[1, k] \times O[k + 1, k + l] \times O[k + l + 1, k + l + m].$$

Proof. Let $L_i = V_i \Supset (F_{1i} \dotplus F_{2i} \dotplus W_i)$ and $\varphi(L_1) = L_2$ for the $E(n)$-automorphism $\varphi = \varphi_\Gamma$ corresponding to the matrix

$$\Gamma = \begin{pmatrix} \Lambda & Y \\ 0 & 1 \end{pmatrix},$$

where $\Lambda \in O(n)$ and Y is a real n-dimensional column. Then $\varphi_\Lambda(F_{11} \dotplus F_{21}) = F_{12} \dotplus F_{22}$, therefore, on the basis of Proposition 1, $k_1 + l_1 = k_2 + l_2$. Since $[L_1, V_1] = V_1$, $[\varphi(L_1), \varphi(V_1)] = \varphi(V_1)$, whence $\varphi(V_1) \subset V_2$. Reasoning similarly, we obtain $\varphi^{-1}(V_2) \subset V_1$, therefore $\varphi(V_1) = V_2$. It follows that $k_1 = k_2 = k$, $l_1 = l_2 = l$. Since $\varphi_\Lambda(V_1) = V_1$, $\Lambda = \operatorname{diag}[\Lambda_1, \Lambda_1']$, where $\Lambda_1 \in O(k)$. Since $\varphi_\Delta(W_1) = W_2$ for $\Delta = \operatorname{diag}[E, \Lambda_1']$, $m_1 = m_2 = m$ and $\Lambda = \operatorname{diag}[\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4]$, where $\Lambda_2 \in O(l)$, $\Lambda_3 \in O(m)$. One can assume that Λ_4 is the identity matrix and Y is the zero column. Hence, $\Lambda \in H$, which is what had to be proved.

The conjugacy criterion for subalgebras of the form (4) is formulated similarly.

3. On a normalizer of the space $\langle P_0 + P_n \rangle$ in the algebra $AO(1,n)$.

Lemma 1. *If $C \in O(1,n)$ and $C(P_0 + P_n) = \lambda(P_0 + P_n)$, then $\lambda \neq 0$ and*

$$C = \begin{pmatrix} \dfrac{1+\lambda^2(1+\vec{v}^2)}{2\lambda} & \lambda\vec{v}B & \dfrac{-1+\lambda^2(1-\vec{v}^2)}{2\lambda} \\ {}^t\vec{v} & B & -{}^t\vec{v} \\ \dfrac{-1+\lambda^2(1+\vec{v}^2)}{2\lambda} & \lambda\vec{v}B & \dfrac{1+\lambda^2(1-\vec{v}^2)}{2\lambda} \end{pmatrix}, \tag{5}$$

where $B \in O(n-1)$, $\vec{v}$ is an $(n-1)$-dimensional vector line, $\vec{v}^2$ is the scalar square of the vector $\vec{v}$ in the Euclidean space $\mathbb{R}^{n-1}$ and ${}^t\vec{v}$ is the vector-column obtained from $\vec{v}$ as a result of transformation.

Proof. Let $C = (c_{\alpha\beta})$, where $\alpha, \beta = 0, 1, \ldots, n$. Then

$$\begin{cases} c_{00} + c_{0n} = \lambda, \\ c_{10} + c_{1n} = 0, \\ \dots\dots\dots\dots\dots\dots \\ c_{n-1,0} + c_{n-1,n} = 0, \\ c_{n0} + c_{nn} = \lambda. \end{cases} \tag{6}$$

Let $\alpha \neq \beta$ and $\alpha, \beta = 1, \ldots, n-1$. Since rows of the matrix C form an orthonormal system in the Minkowski space $\mathbb{R}_{1,n}$, on the basis of (6) we have

$$-1 = c_{\alpha 0}^2 - c_{\alpha 1}^2 - \cdots - c_{\alpha n}^2 = -c_{\alpha 1}^2 - \cdots - c_{\alpha, n-1}^2,$$

$$0 = c_{\alpha 0}c_{\beta 0} - c_{\alpha 1}c_{\beta 1} - \cdots - c_{\alpha n}c_{\beta n} = -c_{\alpha 1}c_{\beta 1} - \cdots - c_{\alpha, n-1}c_{\beta, n-1}.$$

It follows that

$$C = \begin{pmatrix} c_{00} & \vec{u} & \lambda - c_{00} \\ {}^t\vec{v} & B & -{}^t\vec{v} \\ c_{n0} & \vec{\omega} & \lambda - c_{n0} \end{pmatrix},$$

where $B \in O(n-1)$. It is obvious that

$$C = \begin{pmatrix} 1 & \vec{0} & 0 \\ {}^t\vec{0} & B & {}^t\vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix} \begin{pmatrix} c_{00} & \vec{u} & \lambda - c_{00} \\ {}^t\vec{v}_1 & E_{n-1} & -{}^t\vec{v}_1 \\ c_{n0} & \vec{\omega} & \lambda - c_{n0} \end{pmatrix},$$

where ${}^t\vec{v}_1 = B^{-1}\cdot{}^t\vec{v}$. Since when multiplying vectors by B^{-1}, scalar product is preserved, we have $\vec{v}^2 = \vec{v}_1^2$. It remains to describe matrices of the form

$$\hat{C} = \begin{pmatrix} c_{00} & \vec{u} & \lambda - c_{00} \\ {}^t\vec{v}_1 & E_{n-1} & -{}^t\vec{v}_1 \\ c_{n0} & \vec{\omega} & \lambda - c_{n0} \end{pmatrix}.$$

The matrix $\hat{C}$ is pseudoorthogonal. From the condition $c_{00}^2 - \vec{u}^2 - (\lambda - c_{00})^2 = 1$ we obtain

$$c_{00} = \frac{1 + \lambda^2 + \vec{u}^2}{2\lambda}, \qquad \lambda - c_{00} = \frac{\lambda^2 - 1 - \vec{u}^2}{2\lambda}.$$

It follows from the equality $c_{n0}^2 - \vec{\omega}^2 - (\lambda - c_{n0})^2 = -1$ that

$$c_{n0} = \frac{-1 + \lambda^2 + \vec{\omega}^2}{2\lambda}, \qquad \lambda - c_{n0} = \frac{\lambda^2 + 1 - \vec{\omega}^2}{2\lambda}.$$

Since rows of the matrix $\hat{C}$ are pairwise orthogonal,

$$c_{00}\vec{v}_1 - \vec{u} + (\lambda - c_{00})\vec{v}_1 = \vec{0}, \qquad c_{n0}\vec{v}_1 - \vec{\omega} + (\lambda - c_{n0})\vec{v}_1 = \vec{0},$$

whence $\vec{u} = \lambda\vec{v}_1$, $\vec{\omega} = \lambda\vec{v}_1$. In this case,

$$c_{00} = \frac{1 + \lambda^2(1 + \vec{v}_1^2)}{2\lambda}, \qquad \lambda - c_{00} = \frac{-1 + \lambda^2(1 - \vec{v}_1^2)}{2\lambda},$$

$$c_{n0} = \frac{-1 + \lambda^2(1 + \vec{v}_1^2)}{2\lambda}, \qquad \lambda - c_{n0} = \frac{1 + \lambda^2(1 - \vec{v}_1^2)}{2\lambda}.$$

The orthogonality condition for the first and last rows of the matrix $\hat{C}$ imposes no additional restriction on the elements written above. The Lemma is proved.

Lemma 2. *Let $C \in O(1,n)$ and is of the form (5), where $\lambda > 0$. Then*

$$C = \mathrm{diag}[1, B, 1] \exp[-\ln\lambda\cdot J_{0n}] \cdot \exp(-\beta_1 G_1 - \cdots - \beta_{n-1}G_{n-1}),$$

where $G_a = J_{0a} - J_{an}$ $(a = 1, \ldots, n-1)$, ${}^t(\beta_1, \ldots, \beta_{n-1}) = B^{-1}\cdot{}^t\vec{v}$.

The set H of matrices of the form (5) with the condition $\lambda > 0$ forms a group with respect to ordinary multiplication. The mapping

$$C \to \begin{pmatrix} \lambda B & \lambda\cdot{}^t\vec{v} \\ 0 & 1 \end{pmatrix}$$

is an isomorphism of the group H onto the extended Euclidean group $\tilde{E}(n-1)$. In what follows, we shall imply the group H under the group $\tilde{E}(n-1)$. On the basis of Lemma 2, the Lie algebra AH of the group H is generated by matrices J_{ab}, G_a, J_{0n} $(a < b;\ a, b = 1, \ldots, n-1)$. We have $AH = A\tilde{E}(n-1) = \langle G_1, \ldots, G_{n-1}\rangle \Subset (AO(n-1) \oplus \langle J_{0n}\rangle)$. The matrix J_{0n} generates dilations. The algebra $A\tilde{E}(n-1)$ is the normalizer of the space $\langle P_0 + P_n\rangle$ in the algebra $AO(1,n)$.

Lemma 3. *If $C \in O(1,n)$ and $\pm C \notin \tilde{E}(n-1)$, then $C = \pm A_1 C_1 A_2$, where $A_1, A_2 \in \tilde{E}(n-1)$ and $C_1 = \mathrm{diag}[1,\dots,1,-1]$.*

Proof. For some matrix $\Lambda = \mathrm{diag}[1,\Lambda',1]$, where $\Lambda' \in O(n-1)$, we have $\Lambda C(P_0+P_n) = \alpha P_0 + \beta P_1 + \gamma P_n$ with $\alpha^2 - \beta^2 - \gamma^2 = 0$. If $\beta \neq 0$, then $\alpha^2 - \gamma^2 \neq 0$, therefore $\alpha - \gamma \neq 0$. Let $\theta = \beta(\alpha-\gamma)^{-1}$. Then

$$\exp(\theta G_1)(\alpha P_0 + \beta P_1 + \gamma P_n) = \frac{\alpha-\gamma}{2}(P_0 - P_n).$$

Hence, there is a matrix $\Gamma \in \tilde{E}(n-1)$ such that $\Gamma C(P_0+P_n) = \lambda(P_0 - P_n)$. Then $(C_1\Gamma C)(P_0+P_n) = \lambda(P_0+P_n)$. If $\lambda > 0$, then on the basis of Lemmas 1 and 2 we conclude that $C_1\Gamma C \in \tilde{E}(n-1)$. If $\lambda < 0$, then $-C_1\Gamma C \in \tilde{E}(n-1)$. The Lemma is proved.

Proposition 2. *Let L_1 and L_2 be subalgebras of the algebra $A\tilde{E}(n-1)$, L_1 not being conjugate with respect to the group of inner automorphisms of the algebra $A\tilde{E}(n-1)$ to the subalgebra of $AO(n-1) \oplus \langle J_{0n}\rangle$. The subalgebras L_1 and L_2 are $O(1,n)$-conjugate if and only if they are $\tilde{E}(n-1)$-conjugate.*

Proof. Let subalgebras L_1 and L_2 be $O(1,n)$-conjugate, but not be $\tilde{E}(n-1)$-conjugate. Then $\varphi_{\Lambda}(L_1) = L_2$, where $\Lambda \in O(1,n)$ and $\pm\Lambda \notin \tilde{E}(n-1)$. By virtue of Lemma 3, we have $\Lambda = \pm A_1 C_1 A_2$, therefore $C_1(A_2 L_1 A_2^{-1})C_1 = A_1^{-1} L_2 A_1$. But $C_1 G_a C_1 = C_1(J_{0a} - J_{an})C_1 = J_{0a} + J_{an} \notin A\tilde{E}(n-1)$. The contradiction obtained proves that the Proposition is valid.

Proposition 3. *Let L_1 and L_2 be subalgebras of the algebra $K = AO(n-1) \oplus \langle J_{0n}\rangle$ with nonzero projections onto $\langle J_{0n}\rangle$. They are $O(1,n)$-conjugate if and only if there exists an $O(n-1)$-automorphism φ of the algebra K, such that $\varphi(L_1) = L_2$ or $\varphi(L_1) = C_1 L_2 C_1$, where $C_1 = \mathrm{diag}[1,\dots,1,-1]$.*

Proof. Let $\varphi_{\Lambda}(L_1) = L_2$, where $\Lambda \in O(1,n)$. If $\pm\Lambda \in \tilde{E}(n-1)$ and $\pm\Lambda \notin \tilde{O}(n-1)$, then in view of Lemma 2, the projection of $\varphi_{\Lambda}(L_1)$ onto $\langle G_1,\dots,G_{n-1}\rangle$ is nonzero, therefore the projection of the algebra L_2 onto $\langle G_1,\dots,G_{n-1}\rangle$ differs from zero, contrary to the hypothesis of the Proposition. Hence, if $\pm\Lambda \in \tilde{E}(n-1)$, then the subalgebras L_1, L_2 are conjugate with respect to the group of $O(n-1)$-automorphisms. If $\pm\Lambda \notin \tilde{E}(n-1)$, then, reasoning similarly, we get $\Lambda = \pm A_1 C_1 A_2$, where $A_1, A_2 \in \tilde{O}(n-1)$. Therefore $\Lambda = \pm C_1 A$, where $A \in \tilde{O}(n-1)$. It follows from this that $\varphi(L_1) = C_1 L_2 C_1$ for some $O(n-1)$-automorphism φ. The Proposition is proved.

4. Subalgebras of the algebra $AG_1(n-1) \niplus \langle J_{0n}\rangle$. Let K be a subalgebra of the algebra $AP(1,n)$, such that its projection onto $AO(1,n)$ possesses an invariant isotropic subspace in the $\mathbb{R}_{1,n}$. Up to isometries, one can assume that this subspace is $\langle P_0 + P_n\rangle$. Since the normalizer of $\langle P_0+P_n\rangle$ in $AO(1,n)$ coincides with $A\tilde{E}(n-1)$, the subalgebra K is conjugate to a subalgebra of the algebra $\mathfrak{A} = AG_1(n-1) \niplus \langle J_{0n}\rangle$, where $AG_1(n-1)$ is the classical Galilei algebra [14] with the basis M, T, P_a, G_a, J_{ab} $(a,b = 1,\dots,n-1)$. In this case, $M = P_0 + P_n$, $T = \frac{1}{2}(P_0 - P_n)$. Denote by N the group of automorphisms of the algebra $\mathfrak{A}$, generated by inner automorphisms of this algebra and the $O(1,n)$-automorphism corresponding to the matrix $\mathrm{diag}[1,-1,1,\dots,1] \in O(1,n)$.

Theorem 2. *Let L_1 and L_2 be subalgebras of the algebra $\mathfrak{A}$, L_1 not being conjugate with respect to* $\mathrm{Ad}\,\mathfrak{A}$ *to a subalgebra having a zero projection onto $\langle G_1, \ldots, G_{n-1}\rangle$. If $\varphi_\Lambda(L_1) = L_2$ for $\Lambda \in P(1,n)$, then there is an automorphism $\psi \in N$ such that $\psi(L_1) = L_2$ or $\psi(L_1) = \varphi_\Gamma(L_2)$, where $\Gamma = \mathrm{diag}[-1,1,\ldots,1,-1] \in O(1,n)$.*

Proof. Since N contains inner automorphisms corresponding to elements of the form

$$\exp\left(\sum_{\gamma=0}^{n} a_\gamma P_\gamma\right), \tag{7}$$

and $P(1,n)$ is a semidirect product of the group of matrices of the form (7) and the group $O(1,n)$, one can assume that $\Lambda \in O(1,n)$. On the basis of Proposition 2, $\pm\Lambda \in \tilde{E}(n-1)$. The Theorem is proved.

Theorem 3. *Let $\mathfrak{Q}$ be a Lie algebra with the basis P_0, P_a, P_n, J_{ab}, J_{0n} $(a,b = 1,\ldots,n-1)$ and L_1, L_2 be subalgebras of the algebra $\mathfrak{Q}$, at least one of them has a nonzero projection onto $\langle J_{0n}\rangle$. Then the subalgebras L_1, L_2 are $P(1,n)$-conjugate if and only if there exists an automorphism $\psi \in \mathrm{Ad}\,\mathfrak{Q}$ such that $\psi(L_1) = L_2$ or $\psi(L_1) = \varphi_C(L_2)$, where C is one of the matrices*

$$C_1 = \mathrm{diag}[1,\ldots,1,-1], \quad C_2 = \mathrm{diag}[1,-1,1\ldots,1], \quad C_3 = \mathrm{diag}[1,-1,1,\ldots,1,-1]$$

of order $n+1$.

Proof. Let a projection of L_1 onto $\langle J_{0n}\rangle$ differ from zero and $\varphi_\Lambda(L_1) = L_2$ for $\Lambda \in P(1,n)$. As in the proof of Theorem 2, one can assume that $\Lambda \in O(1,n)$. If $\pm\Lambda \in \tilde{E}(n-1)$, then $\pm\Lambda \in \tilde{O}(n-1)$, therefore φ_Λ differs from an inner automorphism of the algebra $\mathfrak{Q}$ by the mupltiplier φ_C, where C is one of the matrices C_2, C_4, C_2C_4. In this case, $C_4 = \mathrm{diag}[-1,1,\ldots,1,-1]$. Let $\pm\Lambda \notin \tilde{E}(n-1)$. As in the proof of Proposition 3, we obtain $\Lambda = \pm C_1 A$, where $A \in \tilde{O}(n-1)$. This implies $\varphi_C(\psi(L_1)) = L_2$, where $\psi \in \mathrm{Ad}\,\mathfrak{Q}$ and C is one of the matrices C_j, C_jC_4 $(j = 1,3)$.

Employing Proposition I.2.2 [12], it is not easy to prove that $L_1 = K \ni\!\!\!+ \langle X + J_{0n} + Y\rangle$, where $Y \in U$, $\hat{\varepsilon}(K) \oplus \langle X\rangle$ is a subalgebra of the algebra $AO(n-1)$ and K contains its projection onto $\langle P_0, P_n\rangle$. Therefore, the employment of automorphisms corresponding to matrices $\mathrm{diag}[\alpha,1,\ldots,1,\beta]$, where $\alpha,\beta \in \{1,-1\}$, is equivalent to application of φ_{C_1} and the inner automorphism corresponding to the element $\exp(\beta_0 P_0 + \beta_n P_n)$. Theorem 3 is proved.

Denote by $O'(1,n-m)$ the subgroup of all isometries from $O(1,n)$, that act identically on the subspace $\langle P_1,\ldots,P_m\rangle$.

Proposition 4. *Let $2 \le m_i \le n-1$, F_i be a subalgebra of the algebra $AO(m_i)$, not conjugate to subalgebras of the algebra $AO(m_i - 1)$, and L_i be a subalgebra of the algebra $\langle P_0, P_1, \ldots, P_n\rangle \ni\!\!\!+ F_i$ such that $\hat{\varepsilon}(L_i) = F_i$ $(i = 1,2)$. The subalgebras L_1 and L_2 are $P(1,n)$-conjugate if and only if $m_1 = m_2 = m$ and L_1 is conjugate to L_2 with respect to the group of H-automorphisms, where $H = E(m) \times O'(1,n-m)$.*

Proposition 5. *Each subalgebra of the algebra $\mathfrak{A}$, having a nonzero projection onto $\langle J_{0n}\rangle$, is conjugate with respect to* $\mathrm{Ad}\,\mathfrak{A}$ *to a subalgebra of the algebra $\mathfrak{A}$, that decomposes as a vector space into the sum of its projections onto the spaces $\langle M\rangle$, $\langle T\rangle$, $\langle G_1,\ldots,G_{n-1}\rangle$ and $\langle P_1,\ldots,P_{n-1}\rangle \ni\!\!\!+ (AO(n-1) \oplus \langle J_{0n}\rangle)$.*

Proposition 5 follows from Theorem IV.3.2 [12] and Proposition IV.3.2 [12].

5. Subalgebras of the algebra $AP(1,n)$, not conjugate to subalgebras of the algebra $AG_1(n-1)\Supset\langle J_{0n}\rangle$. Let K be a subalgebra of the algebra $AP(1,n)$. The subalgebra K is $P(1,n)$-conjugate to none of subalgebras of the algebra $AG_1(n-1)\Supset\langle J_{0n}\rangle$ if and only if $\hat{\varepsilon}(K)$ has no invariant isotropic subspaces in the space U. This will be if and only if, up to $O(1,n)$-conjugacy, $\hat{\varepsilon}(K)$ is a subalgebra of the algebra $AO(n)$, conjugate to none of subalgebras of the algebra $AO(n-1)$, or $\hat{\varepsilon}(K) = AO(1,k)\oplus F$, where $k \geq 2$ and $F \subset AO[k+1,n]$.

Proposition 6. *Let B be a subalgebra of the algebra $AO(n)$, not conjugate to subalgebras of the algebra $AO(n-1)$. If L is a subalgebra of the algebra $AP(1,n)$ and $\hat{\varepsilon}(L) = B$, then L is conjugate with respect to the group of $P(1,n)$-automorphisms to the algebra $W\Supset C$, where $W \subset \langle P_1,\ldots,P_n\rangle$ and C is a subalgebra of the algebra $B\oplus\langle P_0\rangle$. Two subalgebras $W_1\Supset C_1$ and $W_2\Supset C_2$ of the type under consideration are $P(1,n)$-conjugate if and only if they are conjugate with respect to the group of $O(1)\times O(n)$-automorphisms.*

Proposition 7. *Let $B = AO(1,k)\oplus C$, where $k\geq 2$ and $C\subset AO[k+1,n]$. If L is a subalgebra of the algebra $AP(1,n)$ and $\hat{\varepsilon}(L) = B$, then L is $P(1,n)$-conjugate to $L_1\oplus L_2$, where $L_1 = AO(1,k)$ or $L_1 = AP(1,k)$ and L_2 is a subalgebra of the Euclidean algebra $AE[k+1,n]$ with the basis P_a, J_{ab} $(a,b = k+1,\ldots,n)$. Two algebras $L_1\oplus L_2$ and $L_1'\oplus L_2'$ of the type under consideration are $P(1,n)$-conjugate if and only if $k = k'$, $L_1 = L_1'$ and L_2 is conjugate to L_2' with respect to the group of $E[k+1,n]$-automorphisms.*

The proofs of Propositions 6 and 7 are similar to one of Theorem 1.

References

[1] Belko I.V. and Fedenko A.S., Subgroups of the Lorentz group, *Dokl. AN BSSR*, 1970, V.14, N 5, 393–395 (in Russian).

[2] Belko I.V., Subgroups of the Lorentz–Poincaré group, *Izvestiya AN BSSR*, 1971, N 1, 5–13 (in Russian).

[3] Lassner W., Realizations of the Poincaré group on homogeneous spaces, *Acta phys. slov.*, 1973, V.23, N 4, 193–202.

[4] Bacry H., Combe Ph. and Sorba P., Connected subgroups of the Poincaré group. I, *Rep. Math. Phys.*, 1974, V.5, N 2, 145–186.

[5] Bacry H., Combe Ph. and Sorba P., Connected subgroups of the Poincaré group. II, *Rep. Math. Phys.*, 1974, V.5, N 3, 361–392.

[6] Patera J., Winternitz P. and Zassenhaus H., Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group, *J. Math. Phys.*, 1975, V.16, N 8, 1597–1624.

[7] Fedorchuk V.M., Splitting subalgebras of the Lie algebra of the generalized Poincaré group $P(1,4)$, *Ukrain. Mat. Zh.*, 1979, V.31, N 6, 717–722 (in Russian).

[8] Fedorchuk V.M., Nonsplitting Lie subalgebras of the generalized Poincaré group $P(1,4)$, *Ukrain. Mat. Zh.*, 1981, V.33, N 5, 696–700 (in Russian).

[9] Fushchych W.I., Barannyk A.F., Barannyk L.F. and Fedorchuk V.M., Continuous subgroups of the Poincaré group $P(1,4)$, *J. Phys. A: Math. Gen.*, 1985, V.18, N 5, 2893–2899.

[10] Barannyk L.F., Barannyk A.F. and Fushchych W.I., On the continuous subgroups of the generalized Poincaré group $P(1,n)$, In: Group theoretical methods in physics: Proceedings of the third seminar, Yurmala, 1985, Moscow, Nauka, 1986, V.2, 169–176 (in Russian).

[11] Barannyk L.F. and Fushchych W.I., On subalgebras of the Lie algebra of the extended Poincaré group $\tilde{P}(1,n)$, *J. Math. Phys.*, 1987, V.28, N 7, 1445–1458.

[12] Fushchych W.I., Barannyk L.F. and Barannyk A.F., Subgroup analysis of the Galilei, Poincaré groups and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991 (in Russian).

[13] Fushchych W.I., Barannyk A.F. and Barannyk L.F., Continuous subgroups of the generalized Euclidean group, *Ukrain. Mat. Zh.*, 1986, V.38, N 1, 67–72 (in Russian).

[14] Barannyk L., On the classification of subalgebras of the Galilei algebras, *J. Nonlin. Math. Phys.*, 1995, V.2, N 3–4, 263–268.