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Invariant Solutions of a Nonlinear System of Differential Equations for Electromagnetic Field

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Abstract

Solutions invariant under subalgebras of the affine algebra $AIGL(3,\mathbb{R})$ are found.

1 Introduction

The system of nonlinear differential equations

$$\frac{\partial E_k}{\partial t} + H_l \frac{\partial E_k}{\partial x_l} = 0, \qquad \frac{\partial H_k}{\partial t} + E_l \frac{\partial H_k}{\partial x_l} = 0 \qquad (k, l = 1, 2, 3) \tag{1}$$

was proposed in [1] to study electromagnetic fields. For $E_k = H_k$ $(k = 1, 2, 3)$ system (1) becomes the Euler system of equations for ideal fluid that was studied in [2–6]. Symmetry properties of system (1) were investigated in [7]. It was stated that the maximal invariance algebra of the system under consideration is the affine algebra $AIGL(4,\mathbb{R})$, three basis elements of which are nonlinear differential operators.

The aim of our research is to construct solutions of system (1) with the help of symmetry reduction of this system to systems of ordinary differential equations. In this paper, we restrict ourselves to the case where E_k, H_k don't depend on the variable x_3.

Let us give a more detailed characteristic of each Section of the paper.

In Section 2, we get a number of statements concerning systems of linear invariants of subalgebras of the algebra $AIGL(4,\mathbb{R})$, on which we construct ansatzes [8, 9]. Note that overwhelming majority of classes of conjugate subalgebras contains subalgebras for which there exist linear ansatzes.

In Section 3, we obtain solutions to system (1) that are functions only of t and x_1. In this case, we can consider functions E_1, H_1 as components of a solution to the system of equations

$$\frac{\partial E_1}{\partial t} + H_1 \frac{\partial E_1}{\partial x_1} = 0, \qquad \frac{\partial H_1}{\partial t} + E_1 \frac{\partial H_1}{\partial x_1} = 0, \tag{2}$$

functions E_2, E_3 as solutions of the homogeneous equation

$$\frac{\partial E_k}{\partial t} + H_1 \frac{\partial E_k}{\partial x_1} = 0, \tag{3}$$

and functions H_2, H_3 as solutions of the homogeneous equation

$$\frac{\partial H_k}{\partial t} + E_1 \frac{\partial H_k}{\partial x_1} = 0. \tag{4}$$

System of equations (2) is invariant under the affine algebra $AIGL(2,\mathbb{R})$. For this reason, with the help of one-dimensional subalgebras of the algebra $AIGL(2,\mathbb{R})$, one can reduce system (2) to systems of ODEs. If the function $E_1(t,x_1)$ is not constant, then each solution of equation (3) can be presented in the form $\psi(E_1(t,x_1))$, where ψ is some differentiable function. Moreover, for each differentiable function ψ, the function $\psi(E_1(t,x_1))$ is a solution of equation (3). One can say the same about equation (4).

In Section 4, we construct solutions to system (1), that are functions of t, x_1, x_2. Functions E_1, E_2, H_1, H_2 are components of a solution to the system of equations

$$\frac{\partial E_k}{\partial t} + H_l \frac{\partial E_k}{\partial x_l} = 0, \qquad \frac{\partial H_k}{\partial t} + E_l \frac{\partial H_k}{\partial x_l} = 0 \qquad (k,l = 1,2), \tag{5}$$

function E_3 can be regarded as a solution of the homogeneous equation

$$\frac{\partial E_3}{\partial t} + H_1 \frac{\partial E_3}{\partial x_1} + H_2 \frac{\partial E_3}{\partial x_2} = 0, \tag{6}$$

and H_3 as a solution of the homogeneous equation

$$\frac{\partial H_3}{\partial t} + E_1 \frac{\partial H_3}{\partial x_1} + E_2 \frac{\partial H_3}{\partial x_2} = 0.$$

If the functions $E_1(t,x_1,x_2)$, $E_2(t,x_1,x_2)$ are functionally independent in some domain Γ, then each solution to equation (6) can be presented in this domain in the form $\psi(E_1(t,x_1,x_2),E_2(t,x_1,x_2))$, where ψ is some differentiable function. In addition, for an arbitrary differentiable function, ψ the function $\psi(E_1(t,x_1,x_2),E_2(t,x_1,x_2))$ of the variables t, x_1, x_2 is a solution of equation (6).

System of equations (5) is invariant under the affine algebra $AIGL(3,\mathbb{R})$. To perform reduction of system (5) to systems of ODEs, we need two-dimensional subalgebras of the algebra $AIGL(3,\mathbb{R})$, that have only one main invariant of the variables t, x_1, x_2. We obtain the list of such subalgebras up to affine conjugation from results of the classification performed in [10].

2 Linear ansatzes

To unify systems (1), (2) and (5), we consider the system of equations

$$\frac{\partial E_k}{\partial x_0} + H_l \frac{\partial E_k}{\partial x_l} = 0, \qquad \frac{\partial H_k}{\partial x_0} + E_l \frac{\partial H_k}{\partial x_l} = 0 \qquad (k,l = 1,2,\dots,n). \tag{7}$$

Here $x_0 = t$, n is an arbitrary natural number. By the same reasoning as in [7], we deduce that the invariance algebra of system (7) is the affine algebra $AIGL(n+1,\mathbb{R})$, the basis of which is generated by the vector fields:

$$\begin{aligned}
&P_\mu = \frac{\partial}{\partial x_\mu} \quad (\mu = 0,1,\dots,n), \quad \Gamma_{ab} = x_a \frac{\partial}{\partial x_b} + E_a \frac{\partial}{\partial E_b} + H_a \frac{\partial}{\partial H_b},\\
&G_a = x_0 \frac{\partial}{\partial x_a} + \frac{\partial}{\partial E_a} + \frac{\partial}{\partial H_a}, \quad G'_a = x_a \frac{\partial}{\partial x_0} - E_a E_l \frac{\partial}{\partial E_l} - H_a H_l \frac{\partial}{\partial H_l},\\
&\Lambda = x_0 \frac{\partial}{\partial x_0} + x_l \frac{\partial}{\partial x_l} \quad (a,b = 1,2,\dots,n).
\end{aligned} \tag{8}$$

Mark that summation over the repeated index l is from 1 to n.

The linear span of the system of operators, obtained from basis (8) as a result of excluding operators G'_a $(a = 1, 2, \ldots, n)$, forms a Lie subalgebra of the algebra $AIGL(n+1, \mathbb{R})$. Denote this subalgebra by Q and call it the linear part of the invariance algebra of system (7).

Each operator $Y \in Q$ can be presented in the form

$$Y = a_\alpha(x)\frac{\partial}{\partial x_\alpha} + b_{ij}\left(E_j\frac{\partial}{\partial E_i} + H_j\frac{\partial}{\partial H_i}\right) + c_i\left(\frac{\partial}{\partial E_i} + \frac{\partial}{\partial H_i}\right), \tag{9}$$

where $x = (x_0, x_1, \ldots, x_n)$; b_{ij}, c_i are real numbers; $\alpha = 0, 1, 2, \ldots, n;\ i, j = 1, 2, \ldots, n.$

Definition. An invariant of the subalgebra Q that is a linear function of variables E_a, H_a $(a = 1, \ldots, n)$ is called *linear*.

Let

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

$$U = \begin{pmatrix} u_{11}(x) & u_{12}(x) & \ldots & u_{1n}(x) \\ u_{21}(x) & u_{22}(x) & \ldots & u_{2n}(x) \\ \ldots & \ldots & \ldots & \ldots \\ u_{n1}(x) & u_{n2}(x) & \ldots & u_{nn}(x) \end{pmatrix}, \qquad V = \begin{pmatrix} v_1(x) \\ v_2(x) \\ \vdots \\ v_n(x) \end{pmatrix}.$$

Theorem. *A system of functions $f_q = u_{qi}(x)E_i + v_q(x)$, $q = 1, 2, \ldots, n$, is a system of linear invariants of an operator Y, functionally independent of variables $E_1, E_2, \ldots, E_n$ if and only if*

$$a_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB = 0, \qquad a_\alpha(x)\frac{\partial V}{\partial x_\alpha} + UC = 0 \tag{10}$$

and $\det U \neq 0$ *in some domain of the point x space.*

Proof. Obviously,

$$Y(f_q) = a_\alpha(x)\frac{\partial u_{qi}(x)}{\partial x_\alpha}E_i + a_\alpha(x)\frac{\partial v_q(x)}{\partial x_\alpha} + b_{ij}u_{qi}E_j + c_i u_{qi}.$$

Therefore, $Y(f_q) = 0$ if and only if

$$a_\alpha(x)\frac{\partial u_{qi}}{\partial x_\alpha} + u_{ql}b_{li} = 0, \qquad a_\alpha(x)\frac{\partial v_q}{\partial x_\alpha} + u_{ql}c_l = 0$$

for all values of $i, q = 1, 2, \ldots, n$. This system of equations can be rewritten in the matrix form of matrix equation (10).

The matrix U is a Jacobi matrix for functions $f_1, f_2, \ldots, f_n$ with respect to variables $E_1, E_2, \ldots, E_n$. Hence, the system of these functions is functionally independent if and only if $\det U \neq 0$ in some domain of the point x space. The theorem is proved.

Proposition 1. *Let $B \neq 0$. The matrix $U = \exp(f(x)B)$ satisfies the first equation of system (10) if and only if*

$$a_\alpha(x)\frac{\partial f}{\partial x_\alpha} = -1.$$

Proof. Since

$$\frac{\partial U}{\partial x_\alpha} = \frac{\partial f}{\partial x_\alpha} UB,$$

U satisfies the first equation of system (10) if and only if

$$\left(a_\alpha(x)\frac{\partial f}{\partial x_\alpha} + 1\right) UB = 0.$$

The last equality holds if and only if

$$a_\alpha(x)\frac{\partial f}{\partial x_\alpha} + 1 = 0.$$

The proposition is proved.

Proposition 2. *Let*

$$Y_1 = a_\alpha^{(1)}(x)\frac{\partial}{\partial x_\alpha} + \dots, \qquad Y_2 = a_\alpha^{(2)}(x)\frac{\partial}{\partial x_\alpha} + \dots$$

be operators of the form (9) and their corresponding matrices B_1 and B_2 mutually commute and are also linearly independent. The matrix $U = \exp(f(x)B_1) \times \exp(g(x)B_2)$ satisfies the system of equations

$$a_\alpha^{(1)}(x)\frac{\partial U}{\partial x_\alpha} + UB_1 = 0, \qquad a_\alpha^{(2)}(x)\frac{\partial U}{\partial x_\alpha} + UB_2 = 0 \tag{11}$$

if and only if

$$\begin{aligned} &a_\alpha^{(1)}(x)\frac{\partial f(x)}{\partial x_\alpha} + 1 = 0, \qquad a_\alpha^{(1)}(x)\frac{\partial g(x)}{\partial x_\alpha} = 0, \\ &a_\alpha^{(2)}(x)\frac{\partial f(x)}{\partial x_\alpha} = 0, \qquad a_\alpha^{(2)}(x)\frac{\partial g(x)}{\partial x_\alpha} + 1 = 0. \end{aligned} \tag{12}$$

Proof. The matrix U satisfies system (11) if and only if

$$\begin{cases} a_\alpha^{(1)}(x)\dfrac{\partial f(x)}{\partial x_\alpha}UB_1 + a_\alpha^{(1)}(x)\dfrac{\partial g(x)}{\partial x_\alpha}UB_2 + UB_1 = 0, \\ a_\alpha^{(2)}(x)\dfrac{\partial f(x)}{\partial x_\alpha}UB_1 + a_\alpha^{(2)}(x)\dfrac{\partial g(x)}{\partial x_\alpha}UB_2 + UB_2 = 0. \end{cases}$$

Since the matrices B_1 and B_2 are linearly independent, the last system holds if and only if equations (12) hold. The proposition is proved.

Proposition 3. *Let*

$$Y_1 = a_\alpha^{(1)}(x)\frac{\partial}{\partial x_\alpha} + \dots, \qquad Y_2 = a_\alpha^{(2)}(x)\frac{\partial}{\partial x_\alpha} + \dots$$

be operators of the form (9) and their corresponding matrices B_1 and B_2 be connected by the relation $[B_2, B_1] = B_2B_1 - B_1B_2 = \lambda B_1$, $\lambda \neq 0$. The matrix $U = \exp(f(x)B_1)\exp(g(x)B_2)$ satisfies system of equations (11) if and only if

$$\begin{aligned} &a_\alpha^{(1)}(x)\,\mathrm{e}^{-\lambda g(x)}\,\frac{\partial f(x)}{\partial x_\alpha} + 1 = 0, \qquad a_\alpha^{(1)}(x)\frac{\partial g(x)}{\partial x_\alpha} = 0,\\ &a_\alpha^{(2)}(x)\frac{\partial f(x)}{\partial x_\alpha} = 0, \qquad a_\alpha^{(2)}(x)\frac{\partial g(x)}{\partial x_\alpha} + 1 = 0. \end{aligned} \tag{13}$$

Proof. Since, by Campbell-Hausdorff's formula,

$$\exp(\theta B_2)\cdot B_1\cdot\exp(-\theta B_2) = B_1 + \frac{\lambda\theta}{1!}B_1 + \frac{(\lambda\theta)^2}{2!}B_1 + \dots = \mathrm{e}^{\lambda\theta}\,B_1,$$

we have

$$B_1\exp(g(x)B_2) = \mathrm{e}^{-\lambda g(x)}\exp(g(x)B_2)B_1.$$

The matrix U satisfies system (11) if and only if

$$\begin{cases} a_\alpha^{(1)}(x)\dfrac{\partial f(x)}{\partial x_\alpha}\,\mathrm{e}^{-\lambda g(x)}\,UB_1 + a_\alpha^{(1)}(x)\dfrac{\partial g(x)}{\partial x_\alpha}UB_2 + UB_1 = 0,\\ a_\alpha^{(2)}(x)\dfrac{\partial f(x)}{\partial x_\alpha}\,\mathrm{e}^{-\lambda g(x)}\,UB_1 + a_\alpha^{(2)}(x)\dfrac{\partial g(x)}{\partial x_\alpha}UB_2 + UB_2 = 0. \end{cases}$$

By the hypothesis, $[B_2, B_1] \neq 0$, therefore, the matrices B_1 and B_2 are linearly independent. Having equated to zero the expressions at B_1 and B_2 on the left-hand sides of the equalities written down, we obtain equalities (13). The proposition is proved.

Let

$$\vec{E} = \begin{pmatrix} E_1\\ E_2\\ \vdots\\ E_n \end{pmatrix}, \qquad \vec{H} = \begin{pmatrix} H_1\\ H_2\\ \vdots\\ H_n \end{pmatrix}, \qquad \vec{V} = \begin{pmatrix} v_1(x)\\ v_2(x)\\ \vdots\\ v_n(x) \end{pmatrix}.$$

It is easy to see that, if components of the vector-function $U\vec{E} + \vec{V}$ are linear invariants of a subalgebra $F \subset Q$ for some $n \times n$-matrix $U = U(x)$, components of the vector-function $U\vec{H} + \vec{V}$ are also linear invariants of this subalgebra F.

To perform symmetry reduction of system (7) to systems of ODEs we need n-dimensional subalgebras of the algebra F that have one main invariant ω that depends only on the variables $x_0, x_1, \dots, x_n$. On such subalgebras, we construct ansatzes of the form

$$U\vec{E} + \vec{V} = \vec{M}(\omega), \qquad U\vec{H} + \vec{V} = \vec{N}(\omega), \tag{14}$$

where $\vec{M}(\omega)$, $\vec{N}(\omega)$ are unknown n-component functions, matrices U, $\vec{V}$ are known, and $\det U \neq 0$ in some domain of the point x space. Ansatz (14) can be presented in the form

$$\vec{E} = U^{-1}\vec{M}(\omega) - U^{-1}\vec{V}, \qquad \vec{H} = U^{-1}\vec{N}(\omega) - U^{-1}\vec{V}. \tag{15}$$

We call ansatzes of the form (14) or (15) *linear*. In their searching for, we use the statements proved.

3 Solutions of system (1) that are functions of t and x_1

To obtain such solutions, one should solve system (2). As was noticed in Section 2, the invariance algebra of system (2) is the affine algebra $AIGL(2,\mathbb{R})$ with the generators

$$D = t\frac{\partial}{\partial t} - x_1\frac{\partial}{\partial x_1} - 2E_1\frac{\partial}{\partial E_1} - 2H_1\frac{\partial}{\partial H_1}, \quad S = x_1\frac{\partial}{\partial t} - E_1^2\frac{\partial}{\partial E_1} - H_1^2\frac{\partial}{\partial H_1},$$
$$T = t\frac{\partial}{\partial x_1} + \frac{\partial}{\partial E_1} + \frac{\partial}{\partial H_1}, \quad P_0 = \frac{\partial}{\partial t}, \quad P_1 = \frac{\partial}{\partial x_1}, \quad Z = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1}.$$

Up to conjugacy with respect to the group of inner automorphisms the algebra $AIGL(2,\mathbb{R})$ has the following one-dimensional subalgebras:

$$\langle D - \alpha Z\rangle\ (\alpha \geq 0,\ \alpha \neq 1),\ \langle D - Z - 2\alpha P_0\rangle\ (\alpha \in \mathbb{R}),\ \langle Z\rangle,\ \langle P_1\rangle,$$
$$\langle T + \alpha P_0\rangle,\ \langle T + \alpha Z\rangle\ (\alpha \neq 0),\ \langle T - S + \alpha Z\rangle\ (\alpha \in \mathbb{R}).$$

Employing some of these subalgebras, let us perform reduction of system (2) to systems of ODEs, on solutions of which we shall construct the corresponding solutions of system (2) and then ones of systems (1).

3.1. $\langle D - Z\rangle:\ E_1 = x_1 M_1(\omega),\ H_1 = x_1 N_1(\omega),\ \omega = t,$

$$\dot{M}_1 + N_1 M_1 = 0, \qquad \dot{N}_1 + M_1 N_1 = 0.$$

If $M_1 = N_1$, then $E_1 = \dfrac{x_1}{t + C'}$, $H_1 = \dfrac{x_1}{t + C'}$ and E_2, E_3, H_2, H_3 are arbitrary functions of $\dfrac{x_1}{t + C'}$. If $M_1 = N_1 + C$, $C \neq 0$, then

$$E_1 = \frac{Cx_1}{1 - \tilde{C}\,\mathrm{e}^{-Ct}}, \qquad H_1 = \frac{C\tilde{C}x_1\,\mathrm{e}^{-Ct}}{1 - \tilde{C}\,\mathrm{e}^{-Ct}}.$$

Components E_2, E_3 may be arbitrary functions of $\dfrac{Cx_1}{1 - \tilde{C}\,\mathrm{e}^{-Ct}}$ and components H_2, H_3 may be arbitrary functions of $\dfrac{C\tilde{C}x_1\,\mathrm{e}^{-Ct}}{1 - \tilde{C}\,\mathrm{e}^{-Ct}}$.

3.2. $\langle T + \alpha P_0\rangle\ (\alpha \neq 0):\ E_1 = \dfrac{1}{\alpha}t + M_1(\omega),\ H_1 = \dfrac{1}{\alpha}t + N_1(\omega),\ \omega = x_1 - \dfrac{t^2}{2\alpha},$

$$\frac{1}{\alpha} + N_1\dot{M}_1 = 0, \qquad \frac{1}{\alpha} + M_1\dot{N}_1 = 0.$$

The pair of functions

$$M_1 = (C\omega + C')^{\frac{1}{2}}, \qquad N_1 = -\frac{2}{\alpha C}(C\omega + C')^{\frac{1}{2}}$$

is a solution of the reduced system and the corresponding solution of system (2) is of the form

$$E_1 = \frac{t}{\alpha} + \left[C\left(x_1 - \frac{t^2}{2\alpha}\right) + C'\right]^{\frac{1}{2}}, \quad H_1 = \frac{t}{\alpha} - \frac{2}{\alpha C}\left[C\left(x_1 - \frac{t^2}{2\alpha}\right) + C'\right]^{\frac{1}{2}}.$$

Components E_2, E_3 are arbitrary differentiable functions of E_1 and components H_2, H_3 are arbitrary differentiable functions of H_1.

$$\mathbf{3.3}.\ \langle T\rangle:\ E_1=\frac{x_1}{t}+M_1(\omega),\ H_1=\frac{x_1}{t}+N_1(\omega),\ \omega=t,$$

$$\dot{M}_1+\frac{1}{\omega}N_1=0,\qquad \dot{N}_1+\frac{1}{\omega}M_1=0.$$

The reduced system has the general solution $M_1=C\omega+\dfrac{\tilde{C}}{\omega}$, $N_1=-C\omega+\dfrac{\tilde{C}}{\omega}$ and the corresponding solution of system (2) is

$$E_1=\frac{x_1}{t}+Ct+\frac{\tilde{C}}{t},\qquad H_1=\frac{x_1}{t}-Ct+\frac{\tilde{C}}{t}.$$

Components E_2, E_3 are arbitrary differentiable functions of E_1 and components H_2, H_3 are arbitrary differentiable functions of H_1.

$$\mathbf{3.4}.\ \langle T-S\rangle:\ E_1=\frac{\frac{x_1}{t}+M_1(\omega)}{1-\frac{x_1}{t}M_1(\omega)},\ H_1=\frac{\frac{x_1}{t}+N_1(\omega)}{1-\frac{x_1}{t}N_1(\omega)},\ \omega=t^2+x_1^2,$$

$$2\omega\dot{M}_1+(1+M_1^2)N_1=0,\qquad 2\omega\dot{N}_1+(1+N_1^2)M_1=0.$$

To the solution $M_1=N_1=(C\omega-1)^{-\frac{1}{2}}$ of the reduced system, there corresponds the following solution of system (2):

$$E_1=H_1=\frac{1+\frac{x_1}{t}\left[C(t^2+x_1^2)-1\right]^{\frac{1}{2}}}{\left[C(t^2+x_1^2)-1\right]^{\frac{1}{2}}-\frac{x_1}{t}}.$$

Components E_2, E_3, H_2, H_3 are arbitrary differentiable functions of E_1.

4 Solutions of system (1) that are functions of t, x_1, x_2

The problem is to constract solutions that are not equivalent to solutions depending only on t and x_1. To obtain them for system (5), one should restrict oneself in reducing to employing only two-dimensional subalgebras of the algebra $AIGL(3,\mathbb{R})$ that have a zero intersection with the translation space $\langle P_0,P_1,P_2\rangle$. Let us give the list of such subalgebras basing on results of the classification of all subalgebras of the algebra $AIGL(3,\mathbb{R})$ with respect to affine conjugation performed in [10].

Let

$$B=-\Gamma_{11}+\Gamma_{22},\quad C=\Gamma_{21},\quad F=-\Gamma_{12},\quad Z=\Gamma_{11}+\Gamma_{22},$$
$$D=2\Lambda-\Gamma_{11}-\Gamma_{22}\quad\text{(see notations (8))}.$$

Up to affine conjugation, the algebra $AIGL(3,\mathbb{R})$ contains only the following two-dimensional subalgebras that have zero intersection with the space $\langle P_0,P_1,P_2\rangle$:

$\langle G_1,G_2\rangle$, $\langle G_1+P_2,G_2+\varepsilon P_1\rangle$ $(\varepsilon=0,1)$, $\langle G_1,G_2+P_2\rangle$, $\langle C,G_1\rangle$,
$\langle C+P_2,G_1+\varepsilon P_0\rangle$ $(\varepsilon=\pm1)$, $\langle C+P_2,G_1\rangle$, $\langle C+G_2,G_1\rangle$,

$\langle C+G_2, G_1+P_1\rangle,\ \langle D+Z, C\rangle,\ \langle D+Z, C+G_2\rangle,\ \langle D+Z+2C, G_1\rangle,$

$\langle D+Z+2C+2G_2, G_1\rangle,\ \langle D+Z+2G_1, G_2\rangle,\ \langle D+Z+2G_1, C+G_2\rangle,$

$\langle D+\alpha Z, G_1\rangle,\ \langle Z, G_1\rangle,\ \langle Z-D+2P_1, G_1\rangle,\ \langle Z-D+2P_2, G_1\rangle,$

$\langle D+3Z, G_1+P_0\rangle,\ \langle Z, G_1+P_2\rangle,\ \langle D+\alpha Z, C\rangle,\ \langle Z, C\rangle,$

$\langle Z-D+2P_1, C\rangle,\ \langle Z-D+2P_1, C+P_2\rangle,\ \langle Z-D, C+P_2\rangle,$

$\langle Z+\alpha P_0, C+\beta P_0\rangle\ (\alpha,\beta=0,1;\ \alpha+\beta=1),\ \langle Z+C, D+Z\rangle,$

$\langle D+\alpha Z+\beta C, G_1\rangle,\ \langle Z+\alpha C, G_1\rangle,\ \langle Z-D+2C+2P_2, G_1\rangle,$

$\langle Z+C+P_0, G_1+P_2\rangle,\ \langle D+3Z+2C, G_1+P_0\rangle,\ \langle Z+\alpha C, D+Z+2C\rangle,$

$\langle B+\alpha D+(1+\alpha)Z, G_2\rangle,\ \langle B+D+2Z, G_2+P_1\rangle,\ \langle B+D+2Z, G_2+P_0\rangle,$

$\langle B-D+P_2, G_2\rangle,\ \langle B+Z+P_1, G_2\rangle,\ \langle B+\alpha D+(1+\alpha)Z+G_1, G_2\rangle,$

$\langle B-D+G_1+P_2, G_2\rangle,\ \langle B+Z+G_1, G_2+P_2\rangle,$

$\langle B+D+2Z+G_1, G_2+P_1\rangle,\ \langle B+\alpha D+\beta Z, C\rangle,$

$\langle B+2Z-D+P_1, C+P_0\rangle,\ \langle B+\alpha D+(1-\alpha)Z+P_1, C\rangle\ (\alpha\neq 0,1),$

$\langle B-3Z+P_0, C+P_2\rangle,\ \langle B+\alpha Z+P_0, C\rangle\ (\alpha\neq\pm1),$

$\langle B+\alpha D-(\alpha+3)Z, C+P_2\rangle\ (\alpha\neq -1,-2),$

$\langle B+(\alpha-1)Z-D, C+P_0\rangle\ (\alpha\neq\pm1),$

$\langle B+\alpha D+(\alpha-3)Z, C+G_2\rangle,\ \langle B+2D-Z+P_1, C+G_2\rangle,$

$\langle B-D-4Z, C+G_2+P_0\rangle,\ \langle F+C+\alpha Z, D+Z\rangle\ (\alpha\geq 0),$

$\langle F+C+\alpha(D+Z), Z+\beta(D+Z)\rangle\ (\alpha\geq 0),\ \langle D, Z\rangle,$

$\langle B+\alpha Z, D+Z\rangle\ (0\leq\alpha<1),$

$\langle B+\alpha(D+Z), Z+\beta(D+Z)\rangle\ (\alpha\geq 3\beta+1),$

$\langle B+D, Z\rangle,\ \langle 2B+D+Z, D-Z\rangle,\ \langle 2B+D+Z+2P_1, D-Z+\alpha P_1\rangle,$

$\langle 2B+D+Z, Z-D+2P_1\rangle,\ \langle F+C+P_0, Z+\alpha P_0\rangle\ (\alpha\geq 0),\ \langle F+C, Z+P_0\rangle.$

From this list, we exclude the subalgebra $\langle C, G_1\rangle$, the rank of which is equal to 1 with respect to the variables t, x_1, x_2.

Let us present some examples of symmetry reduction of system (5), performed with the help of given subalgebras.

4.1. $\langle G_1, G_2+P_2\rangle:\ E_1=\dfrac{x_1}{t}+M_1(\omega),\ E_2=\dfrac{x_2}{t+1}+M_2(\omega),$

$$H_1=\frac{x_1}{t}+N_1(\omega),\ H_2=\frac{x_2}{t+1}+N_2(\omega),\ \omega=t,$$

$$\begin{cases} N_1+\omega\dot{M}_1=0, \quad N_2+(\omega+1)\dot{M}_2=0,\\ M_1+\omega\dot{N}_1=0, \quad M_2+(\omega+1)\dot{N}_2=0.\end{cases}$$

The general solution of the reduced system

$$M_1 = C_1\omega + \frac{C_2}{\omega}, \qquad M_2 = C_3(\omega+1) + \frac{C_4}{\omega+1},$$
$$N_1 = -C_1\omega + \frac{C_2}{\omega}, \qquad N_2 = -C_3(\omega+1) + \frac{C_4}{\omega+1},$$

is associated with the following solution of system (5):

$$E_1 = \frac{x_1}{t} + C_1 t + \frac{C_2}{t}, \qquad E_2 = \frac{x_2}{t+1} + C_3(t+1) + \frac{C_4}{t+1},$$
$$H_1 = \frac{x_1}{t} - C_1 t + \frac{C_2}{t}, \qquad H_2 = \frac{x_2}{t+1} - C_3(t+1) + \frac{C_4}{t+1}.$$

In this case, E_3 is an arbitrary differentiable function of E_1 and E_2 and H_3 is an arbitrary differentiable function of H_1 and H_2.

4.2. $\langle C + P_2, G_1 + \varepsilon P_0\rangle$ $(\varepsilon = \pm 1)$: $E_1 = \varepsilon t + M_1(\omega) + x_2 M_2(\omega)$, $E_2 = M_2(\omega)$,

$$H_1 = \varepsilon t + N_1(\omega) + x_2 N_2(\omega), \quad H_2 = N_2(\omega), \ \omega = t^2 - 2\varepsilon x_1 + \varepsilon x_2^2,$$

$$\begin{cases} \varepsilon - 2\varepsilon N_1 \dot{M}_1 + N_2 M_2 = 0, \quad N_1 \dot{M}_2 = 0, \\ \varepsilon - 2\varepsilon M_1 \dot{N}_1 + M_2 N_2 = 0, \quad M_1 \dot{N}_2 = 0. \end{cases}$$

The solution of the reduced system $M_1 = N_1 = 0$, $M_2 = M(\omega)$, $N_2 = -\dfrac{\varepsilon}{M(\omega)}$, where $M(\omega)$ is an arbitrary nonzero function, is associated with the solution of system (1):

$$E_1 = \varepsilon t + x_2 M(\omega), \quad H_1 = \varepsilon t - \varepsilon x_2 M^{-1}(\omega),$$
$$E_2 = M(\omega), \quad H_2 = -\varepsilon M^{-1}(\omega), \quad \omega = t^2 - 2\varepsilon x_1 + \varepsilon x_2^2,$$
$$E_3 = g(\varepsilon t + x_2 M(\omega), M(\omega)), \quad H_3 = h(\varepsilon t - \varepsilon x_2 M^{-1}(\omega), -\varepsilon M^{-1}(\omega)),$$

where g, h are arbitrary differentiable functions.

The solution $M_1 = 0$, $M_2 = C_1$, $N_1 = C_2$, $N_2 = -\dfrac{\varepsilon}{C_1}$, is associated with the solution of system (5)

$$E_1 = \varepsilon t + C_1 x_2, \quad E_2 = C_1, \quad H_1 = \varepsilon t - \frac{\varepsilon}{C_1} x_2 + C_2, \quad H_2 = -\frac{\varepsilon}{C_1}.$$

For it,

$$E_3 = g\left(x_2 + \frac{\varepsilon}{C_1} t, x_1 - \frac{\varepsilon}{2} t^2 + \frac{\varepsilon}{C_1} t x_2 + \frac{t^2}{2C_1^2} - C_2 t\right),$$
$$H_3 = h\left(x_2 - C_1 t, x_1 - C_1 t x_2 + \frac{C_1^2 - \varepsilon}{2} t^2\right),$$

where g, h are arbitrary differentiable functions.

4.3. $\langle C + P_2, G_1\rangle$: $E_1 = \dfrac{x_1}{t} - \dfrac{x_2^2}{2t} + M_1(\omega) + x_2 M_2(\omega)$, $E_2 = M_2(\omega)$,

$$H_1 = \frac{x_1}{t} - \frac{x_2^2}{2t} + N_1(\omega) + x_2 N_2(\omega), \ H_2 = N_2(\omega), \ \omega = t,$$

$$\begin{cases} \omega \dot{M}_1 + N_1 + \dot{M}_2 \dot{N}_2 \omega = 0, & \dot{M}_2 = 0, \\ \omega \dot{N}_1 + M_1 + \dot{M}_2 \dot{N}_2 \omega = 0, & \dot{N}_2 = 0. \end{cases}$$

The reduced system has the general solution $M_1 = C_3\omega + \dfrac{C_4}{\omega}$, $N_1 = -(C_1C_2 + C_3)\omega + \dfrac{C_4}{\omega}$, $M_2 = C_1$, $N_2 = C_2$. It is associated with the following solution of system (5):

$$E_1 = \frac{x_1}{t} - \frac{x_2^2}{2t} + C_1x_2 + C_3t + \frac{C_4}{t}, \quad E_2 = C_1,$$
$$H_1 = \frac{x_1}{t} - \frac{x_2^2}{2t} + C_2x_2 - (C_1C_2 + C_3)t + \frac{C_4}{t}, \quad H_2 = C_2.$$

Components E_3, H_3 are given by the formulas:

$$E_3 = g\left(x_2 - C_2t, x_1 - \left(\frac{C_2^2}{2} - C_1C_2 - C_3\right)t^2 + C_4 - \frac{(x_2 - C_2t)^2}{2}\right),$$
$$H_3 = h\left(x_2 - C_1t, x_1 - \left(\frac{C_1^2}{2} + C_3\right)t^2 + C_4 - \frac{(x_2 - C_1t)^2}{2}\right),$$

where g, h are arbitrary differentiable functions.

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