States of a Charged Particle with a Tensor-Like Mass in External Constant Magnetic Field

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Abstract

Solutions of the Schrödinger equation for a particle with a tensor-like mass are considered. It is shown that the problem of determination of the coherent states in this case is reduced to integration of the nonlinear system of ordinary differential equations.

In many cases a charged particle in solid state can have a tensor-like mass [1]. For this reason it is interesting to investigate behavior of such a particle in the constant magnetic field $H \parallel z$. This problem is more difficult for investigation than the problem of usual axial symmetric oscillator [2], but some results can be obtained.

Let us consider the Schrödinger equation

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t} + \mathbf{H}\right)\psi = 0, \qquad \mathbf{H} = -\frac{\hbar^2}{2}m_{mn}^{-1}D_nD_m, \tag{1}$$

where m_{ik}^{-1} is an inverse mass tensor and

$$D_i = \frac{\partial}{\partial x_i} + \frac{ie}{\hbar c}A_i. \tag{2}$$

Electromagnetic potential for a constant magnetic field is

$$A_i = \frac{H}{2}\left(-x_2, x_1, 0, 0\right). \tag{3}$$

It is convenient to take the transversal variables x_1, x_2 in the axially-symmetric form

$$x = x_1 + ix_2, \qquad x^+ = x_1 - ix_2. \tag{4}$$

Then we obtain the Hamiltonian $\left(\Omega = \dfrac{eH}{2c}\right)$

$$
\begin{aligned}
\mathbf{H} = -\frac{\hbar^2}{2}\Bigg[&\mu\frac{\partial^2}{\partial x^2} + \mu^+\frac{\partial^2}{\partial x^{+2}} + 4\chi\frac{\partial^2}{\partial x\partial x^+} + \frac{2\Omega}{\hbar}\chi\left(x^+\frac{\partial}{\partial x^+} - x\frac{\partial}{\partial x}\right) - \\
&\frac{\Omega}{\hbar}\left(\mu^+x\frac{\partial}{\partial x^+} - \mu x^+\frac{\partial}{\partial x}\right) - \left(\frac{\Omega}{2\hbar}\right)^2\left(\mu^+x^2 + \mu x^{+2} - 4\chi x^+ x\right)\Bigg] -
\end{aligned}
\tag{5}
$$

$$\frac{i\hbar}{2}p\left[\nu\frac{\partial}{\partial x} - \nu^+\frac{\partial}{\partial x^+} + \frac{\Omega}{2\hbar}\left(\nu x^+ - \nu^+ x\right)\right] + \frac{1}{2}m_{33}^{-1}p^2,$$

where p is the impulse of the particle along the z-axis. There are the following abbreviations in Eq. (5)

$$\mu = m_{11}^{-1} - m_{22}^{-1} + 2im_{12}^{-1}, \qquad \mu^+ = m_{11}^{-1} - m_{22}^{-1} - 2im_{12}^{-1}, \tag{6}$$

$$\nu = 2\left(m_{31}^{-1} + im_{23}^{-1}\right), \qquad \nu^+ = 2\left(m_{13}^{-1} - im_{23}^{-1}\right), \tag{7}$$

$$\chi = \frac{1}{2}\left(m_{11}^{-1} + m_{22}^{-1}\right). \tag{8}$$

Following the general principles of quantum mechanics we look for a solution of Eq. (5) in the quasiclassical form

$$\psi(\vec{x}, t) = \phi(t)\exp\left(\frac{i}{\hbar}s(x, x^+, t)\right), \tag{9}$$

where ϕ and s are complex functions. Substituting (9) into Eq.(1) we obtain the equation for the amplitude ϕ and Hamilton-Jacobi equation for s

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\mu\frac{\partial^2 s}{\partial x^2} + \mu^+\frac{\partial^2 s}{\partial x^{+2}} + 4\chi\frac{\partial^2 s}{\partial x\partial x^+}\right]\phi = 0, \tag{10}$$

$$\frac{\partial s}{\partial t} + \frac{1}{2}\left[\mu\left(\frac{\partial s}{\partial x}\right)^2 + \mu^+\left(\frac{\partial s}{\partial x^+}\right)^2 + 4\chi\frac{\partial s}{\partial x}\frac{\partial s}{\partial x^+} - 2i\chi\Omega\left(x^+\frac{\partial s}{\partial x^+} - x\frac{\partial s}{\partial x}\right) +\right.$$

$$\left. i\Omega\left(\mu x\frac{\partial s}{\partial x^+} - \mu^+ x^+\frac{\partial s}{\partial x}\right) - \left(\frac{\Omega}{2\hbar}\right)^2\left(\mu^+ x^2 + \mu x^{+2} - 4\chi x^+ x\right)\right] + \tag{11}$$

$$\frac{1}{2}p\left[\nu\frac{\partial s}{\partial x} - \nu^+\frac{\partial s}{\partial x^+} + i\frac{\Omega}{2}\left(\nu^+ x - \nu^+ x^+\right)\right] + \frac{1}{2}m_{33}^{-1}p^2 = 0.$$

Since H is a quadratic Hamiltonian, we may assume that s is also quadratic [2]

$$s = A(t) + B(t)x + B^+(t)x^+ + C(t)x^+ x + F(t)x^2 + F^+(t)x^{+2}; \tag{12}$$

and we obtain the amplitude equation

$$\frac{\partial\phi}{\partial t} + \left(\mu F + \mu^+ F^+ + 2\chi C\right)\phi = 0 \tag{13}$$

and a system of equations for the other unknown functions A, B, B^+, C, F, F^+

$$\frac{dA}{dt} + \frac{1}{2}(\mu B^2 + \mu^+ B^{+2} + 4\chi B^+ B) = 0, \tag{14}$$

$$\frac{dB}{dt} + \left(C + \frac{i\Omega}{2}\right)\left(2\chi B + \mu^+ B^+ + \frac{1}{2}\nu^+ p\right) + \left(2\chi B^+ + 2\mu B + \nu p\right)F = 0, \tag{15}$$

$$\frac{dB^+}{dt} + \left(C - \frac{i\Omega}{2}\right)\left(2\chi B^+ + \mu B + \frac{1}{2}\nu p\right) + \left(2\chi B + 2\mu^+ B^+ + \nu^+ p\right)F^+ = 0, \tag{16}$$

$$\frac{dC}{dt} + 2\chi\left(C^2 + 4F^+F + \frac{\Omega^2}{4}\right) + 2\mu\left(C - \frac{i\Omega}{2}\right)F + 2\mu^+\left(C + \frac{i\Omega}{2}\right)F^+ = 0, \qquad (17)$$

$$\frac{dF}{dt} + 4\chi\left(C + \frac{i\Omega}{2}\right)F + 2\mu F^2 + \frac{1}{2}\mu^+\left(C + \frac{i\Omega}{2}\right)^2 = 0, \qquad (18)$$

$$\frac{dF^+}{dt} + 4\chi\left(C - \frac{i\Omega}{2}\right)F^+ + 2\mu^+F^{+2} + \frac{1}{2}\mu\left(C - \frac{i\Omega}{2}\right)^2 = 0. \qquad (19)$$

We see that when the coordinates are adopted on the axes of the tensor m_{ik}^{-1}, the whole problem is simplifield and

$$\mu = \mu^+, \quad \nu = \nu^+ = 0 \qquad (20)$$

and only two parameters μ and χ, remain. The solution of the initial problem is reduced to solution of the system of ordinary differential equations (14)–(19) but only the nonlinear equations (17)–(19) are essential. When they are solved, integration of Eqs. (14)–(16) becomes trivial.

There are coherent states in the system of solutions of Eqs. (18)–(19) if $C = C_0 = \frac{i\Omega}{2}$, $F^+ = 0$. Then the function F has the form

$$F = u(t) + a, \qquad (21)$$

where

$$u(t) = \frac{u_0}{-is\alpha + (1 + is\alpha)\exp(-4is\chi g\Omega t)}, \quad a = -\frac{i\chi\Omega}{\mu}\left(1 + sg\right), \qquad (22)$$

where the abbreviations are introduced

$$\alpha = \frac{\mu u_0}{2\chi g\Omega}, \quad g = \sqrt{1 - \frac{\mu^+\mu}{4\chi^2}}, \quad s = \pm 1. \qquad (23)$$

The stability of solutions is guaranteed by the conditions

$$|ReF| < \frac{\Omega}{2}, \qquad |ImF| < \frac{\Omega}{2}. \qquad (24)$$

Solutions with constants C, F, F^+ are also new. There are solutions of the system (17)–(19) when $s^+ = \pm 1$ but they do not depend on s

$$\begin{aligned}
F &= -\frac{\chi}{\mu}\left(c + \frac{i\Omega}{2}\right)\left(1 + sg\right), \\
F^+ &= -\frac{\chi}{\mu}\left(c - \frac{i\Omega}{2}\right)\left(1 + s^+g\right).
\end{aligned} \qquad (25)$$

In this case Eq. (17) for the function C may be satisfied if $s^+ = -s$. But it is satisfied identically and the constant function C may be arbitrary but $C^2 = -\frac{\Omega^2}{4}$.

It is important to have solutions of the system of equations with a real constant g. It is guaranteed if the next conditions are satisfied:

$$4\chi^2 \geq \mu^+\mu \qquad (26)$$

and

$$2m_{11}^{-1}m_{22}^{-1} - (m_{12}^{-1})^2 \geq 0. \tag{27}$$

When $m_{12}^{-1} = 0$ (it exists ever) then $\mu = \mu^+$. Selecting g=0, $\mu^+\mu = 4\chi^2$ we have a simpler solution for the function $u(t)$

$$u = \frac{u_0[1 + 2Re(\mu u_0)t - 2iIm(\mu u_0)t]}{1 + 4Re(\mu u_0)t + 4\rho^2 t^2}, \tag{28}$$

where ρ is the modulus of the expression μu_0. It is sufficient to set

$$\left| Re\left(u_0 - \frac{i\chi\Omega}{\mu} \right) \right| \leq \frac{\Omega}{2},$$

$$\left| Im\left(u_0 - \frac{i\chi\Omega}{\mu} \right) \right| \leq \frac{\Omega}{2}$$

in order that the solution (28) have the stability. This depends on the choice of u_0.

The calculation of averages in such a problem in L^2-space is of great interest. It is associated with calculation of the norm N of a wave function

$$N = \int (d\vec{x})\psi^+(\vec{x}, t)\psi(\vec{x}, t) = C_\infty | \phi |^2 \int (d\vec{x}) \exp\left(-\frac{2Ims}{\hbar} \right). \tag{29}$$

From the Eq. (12) we find that

$$Ims = ImA, \tag{30}$$

where the following notations are introduced

$$\begin{aligned} b_1 &= ImB + ImB^+, \\ b_2 &= ReB - ReB^+, \\ a_1 &= ImC + ImF + ImF^+, \\ a_2 &= ImC - ImF - ImF^+, \\ q &= ReF - ReF^+. \end{aligned} \tag{31}$$

The norm of Exp. (29) is of the form

$$N = \frac{\pi\hbar}{2} \frac{1}{\sqrt{a_1 a_2 - q^2}} \exp\left(-\frac{2ImA}{\hbar} \right) \exp\left(\frac{b_1^2}{2\hbar a_1} + \frac{(b_2 - \frac{qb_1}{a_1})^2}{2\hbar(a_2 - \frac{q^2}{a_1})} \right) \tag{32}$$

It follows from above that average coordinates can be calculated

$$\langle \Delta x_i \rangle = -\frac{\hbar}{2N} \frac{\partial N}{\partial b_i}, \quad \langle x_i^2 \rangle = -\frac{\hbar}{2N} \frac{\partial N}{\partial a_i}, \quad i = 1, 2 \tag{33}$$

and indetermination, e.g., of the x_2-coordinate is equal to

$$\Delta x_2^2 = \langle x_2^2 \rangle - \langle x_2 \rangle^2 = \frac{\hbar a_1}{4(a_1 a_2 - q^2)}. \tag{34}$$

We can see that the presence of q increases indetermination.

The entropy $\mathbf{S}$ of the states with the given $\quad a_1, a, q \quad$ may be calculated as

$$\mathbf{S} = -\int (d\vec{x}_T)\psi^+\psi \ln \psi^+\psi = 1 + \ln \frac{\pi\hbar}{2} - \ln \sqrt{a_1 a_2 - q^2}. \tag{35}$$

And we assume coherent states have minimal entropy.

More common classification of the solutions of the system (14)–(19) requires employment of computer analysis.

References

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