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Bonferroni and Gini Indices and Recurrence Relations for Moments of Progressive Type-II Right Censored Order Statistics from Marshall-Olkin Exponential Distribution

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In this paper, we derive explicit expressions for Bonferroni Curve (BC), Bonferroni index (BI), Lorenz Curve (LC) and Gini index (GI) for the Marshall-Olikn Exponential (MOE) distribution, which have mainly concern with some aspects like poverty, welfare, decomposability, reliability, sampling and inference. We also establish several recurrence relations satisfied by the single and the product moments of progressive Type-II right censored order statistics from MOE distribution, to enable one to evaluate the single and product moments of all order in a simple recursive way.

Keywords: Bonferroni Curve; Bonferroni index; Lorenz Curve; Gini index; Marshall-Olkin exponential distribution; Order statistics; Progressive Type-II censored sampling.

2010 Mathematics Subject Classification: 62G30, 62E10

1. Introduction

Marshall and Olkin (1997) proposed a new method for adding a parameter to a family of distributions. In particular, they consider a two-parameter generalization of the one-parameter exponential distribution as one parameter exponential family of distributions is not broad enough to model data from various real contexts, which plays a central role in reliability and life time data analysis. Gaver and Lewis (1980) developed a first order autoregressive time series model with exponential stationary marginal distribution. This type of process is found to be useful for modeling time series data from various contexts such as hydrology, wind velocity, life time, etc. In industry the breakdown times of dual generators in a power plant or failure time of twin engines in a two engine airplane

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are examples which could be modeled by bivariate survival variable.

In this paper, we shall derive explicit expressions for Bonferroni Curve and Bonferroni index for Marshall-Olkin exponential distribution. We shall also establish several recurrence relations satisfied by single and the product moments of Progressive Type-II right censored order statistics from Marshall-Olkin exponential distribution, which will enable one to compute the single and the product moments of all order in a simple recursive way. It may be mentioned that Bonferroni Curve and Bonferroni index are well-known measures of income inequality. These measures have some relationship with Lorenz Curve, Gini index and certain concepts used in reliability, life testing and renewal theory [cf. Pundir et al. (2005) and Giorgi and Nadarajah (2010)].

The cdf of Marshall-Olkin extended distribution is defined by

$$F(x) = \frac{H(x)}{\alpha + (1-\alpha)H(x)}, \quad -\infty < x < \infty, \;\; \alpha > 0, \tag{1.1}$$

where $H(x)$ is the cdf of an arbitrary family of distributions (see Marshall and Olkin(1997)).
Or, equivalently, if we have a survival function $\bar{H}(x)$, then the Marshall-Olkin extended survival distribution is given by

$$\bar{F}(x) = \frac{\alpha \bar{H}(x)}{1-(1-\alpha)\bar{H}(x)}, \quad -\infty < x < \infty, \;\; \alpha > 0. \tag{1.2}$$

In particular, for exponential distribution with cdf

$$H(x) = 1 - e^{-x}, \;\; 0 \leq x < \infty, \tag{1.3}$$

if we substitute equation (1.3) into equation (1.1), we get the so called Marshall-Olkin exponential (MOE) distribution (cf. Salah et al., 2009). This distribution has been used by Ghitany et al. (2005) for analyzing censored samples. Thus the cdf of MOE distribution is given by

$$F(x) = 1 - \frac{\alpha e^{-x}}{1-(1-\alpha)e^{-x}}, \;\; 0 \leq x < \infty, \;\; \alpha > 0 \tag{1.4}$$

and the pdf is given by

$$f(x) = \frac{\alpha e^{-x}}{(1-(1-\alpha)e^{-x})^2}, \;\; 0 \leq x < \infty, \;\; \alpha > 0, \tag{1.5}$$

(see Fig.1). From (1.4) and (1.5), one can observe that the characterizing differential equation for MOE distribution is

$$f(x) = [1-F(x)] + \frac{1-\alpha}{\alpha}[1-F(x)]^2. \tag{1.6}$$

In particular, for $\alpha = 1$, the MOE distribution reduces to the standard exponential distribution.

2. Bonferroni Curve and Bonferroni index for MOE distribution

Let X be a non-negative random variable with cumulative distribution function (c.d.f.) $F(x) = \int_0^x f(t)dt$, which is absolutely continuous and at least twice differentiable.

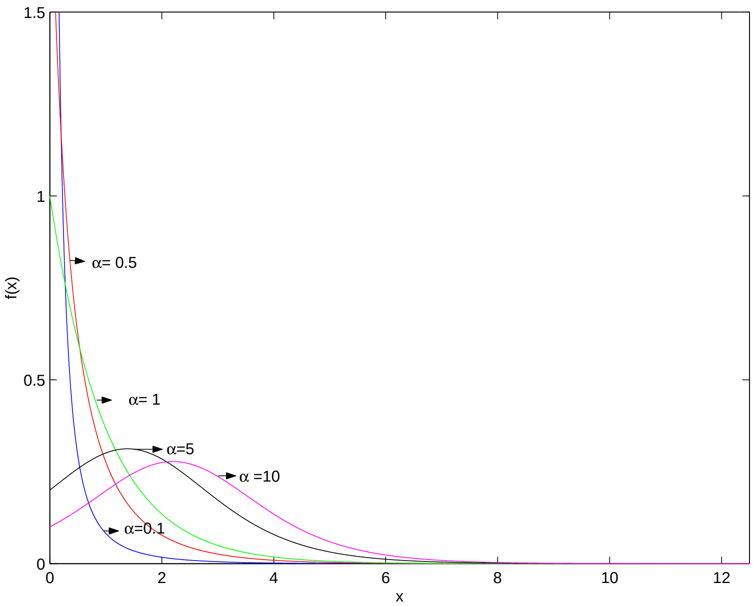

Fig. 1. pdf of Marshall-Olkin exponential distribution

The Bonferroni Curve of the random variable X is defined in the orthogonal plane $[F(x), B_F(F(x))]$ (Fig. 2) within a unit square (cf. Giorgi and Crescenzi, 2001).

Writing $p = F(x)$, the parametric expression of the Bonferroni Curve is given as

$$B_F(p) = \frac{1}{p\mu}\int_0^p F^{-1}(t)dt, \quad p \in (0,1], \tag{2.1}$$

where $F^{-1}(t) = inf\{x : F(x) \geq t\}$.
As $p \to 0, B_F(p)$ takes the form 0/0, hence the Bonferroni Curve does not always start from the origin of the orthogonal plane, and since $\frac{\partial B}{\partial p} > 0$, the graph of $B_F(p)$ is strictly increasing but nothing can be said about the sign of its second derivative. Hence, Bonferroni Curve could be convex in some parts and concave in some others (Giorgi and Crescenzi, 2001).
The Bonferroni index, B, is defined as the area enclosed by the ordinate axis, the Egalitarian Line (a line connecting point (0,1) to (1,1)) and the Bonferroni Curve and is given by

$$B = 1 - \int_0^1 B_F(p)dp. \tag{2.2}$$

The relation between the Bonferroni Curve $B_F(p)$ and mean residual life time $\varepsilon_F(x)$ (a common reliability measure) is given by

$$B_F(F(x)) = \frac{1}{F(x)} - \frac{1}{\mu}\frac{\bar{F}(x)}{F(x)}(\varepsilon_F(x) + x), \tag{2.3}$$

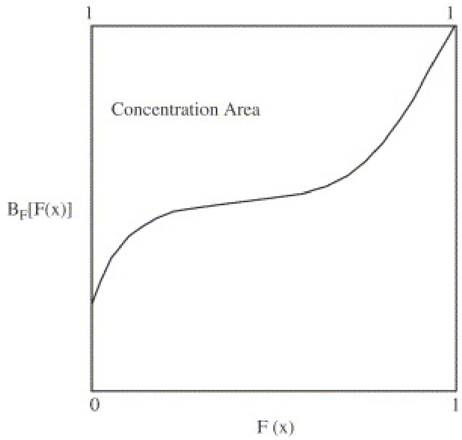

Fig. 2. Bonferroni Curve

and

$$\varepsilon_F(x) = \frac{\mu[1 - F(x)B_F(F(x))]}{\bar{F}(x)} - x \tag{2.4}$$

(cf. Pundir et al., 2005, p. 142).
From (1.4), the quantile function of the Marshall-Olkin exponential distribution is given by

$$F^{-1}(t) = \log\left\{(1-\alpha) + \frac{\alpha}{1-t}\right\}. \tag{2.5}$$

Thus from (2.1), the Bonferroni Curve (BC) of the random variable X, following MOE distribution (1.5), is defined by (Fig. 3)

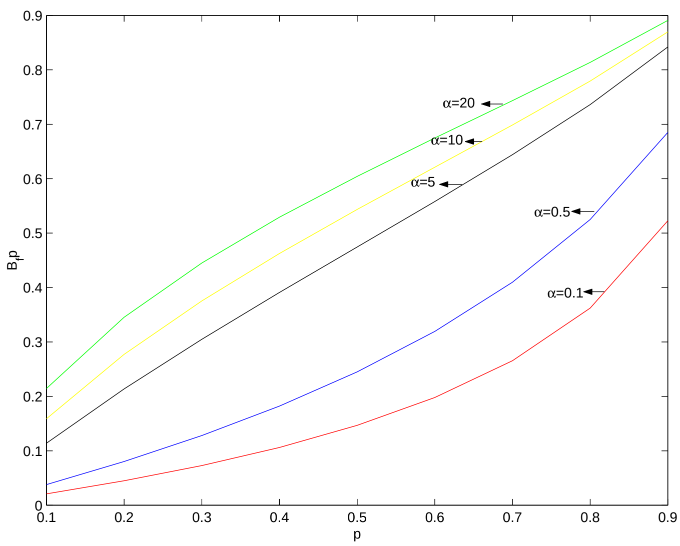

Fig. 3. Bonferroni Curve for MOE distribution

$$B_F(p) = \Big[\alpha\log\Big(-\frac{-p+1+\alpha p}{p-1}\Big)p - \log\Big(-\frac{-p+1+\alpha p}{p-1}\Big)p + \log\Big(-\frac{-p+1+\alpha p}{p-1}\Big) - \alpha\log\Big(-\frac{\alpha}{p-1}\Big) + \alpha\log(\alpha)\Big]\Big/((-1+\alpha)p\mu), \quad (2.6)$$

where $\mu = E(x) = \frac{\alpha\log\alpha}{\alpha-1}$, $\alpha > 0$, $\alpha \neq 1$ (cf. Salah et al., 2009).

From the equation (2.2), the Bonferroni index (BI) for MOE distribution is given by

$$B = 1+\frac{1}{6}\Big[-3\log\Big(\frac{\alpha}{-1+\alpha}\Big)^2 - 2\pi^2 + 6\log(\alpha)\log\Big(\frac{\alpha}{-1+\alpha}\Big) + \alpha\pi^2 - 3\log(\alpha)^2 - 6\alpha\log(\alpha) - 6dilog\Big(\frac{\alpha}{-1+\alpha}\Big)\Big]\,((-1+\alpha)\mu). \quad (2.7)$$

The Lorenz Curve (LC) for MOE distribution is given by (Fig. 4)

$$L_F(p) = \frac{1}{\mu}\int_0^p F^{-1}(t)dt$$

$$= \Big[\alpha\log\Big(-\frac{-p+1+\alpha p}{p-1}\Big)p - \log\Big(-\frac{-p+1+\alpha p}{p-1}\Big)p + \log\Big(-\frac{-p+1+\alpha p}{p-1}\Big) - \alpha\log\Big(-\frac{\alpha}{p-1}\Big) + \alpha\log(\alpha)\Big]\Big/((-1+\alpha)\mu). \quad (2.8)$$

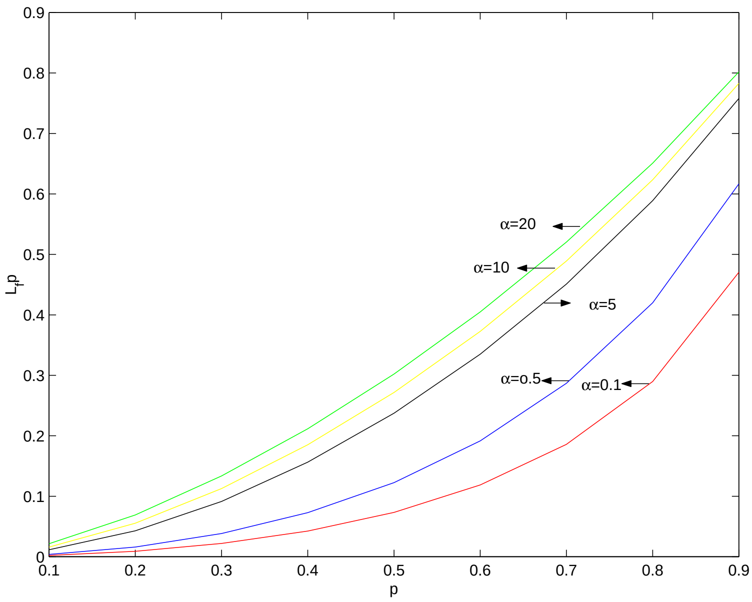

Fig. 4. Lorenz Curve for MOE distribution

And the Gini index (GI) for MOE distribution is given by

$$G = 1 - 2\int_0^1 L_F(p)dp$$

$$= 1 - \frac{\alpha(\alpha\log(\alpha) - \alpha + 1)}{(-1+\alpha)^2\mu}. \tag{2.9}$$

3. Progressive Type-II Right Censored Order Statistics

Let $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)} < X_{2:m:n}^{(R_1,R_2,\ldots,R_m)} < \ldots < X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ be the m ordered observed failure times in a sample of size n from the Marshall-Olkin exponential distribution (1.4) under progressive Type-II right censoring scheme $(R_1, R_2, \ldots, R_m)$. Then the joint pdf of $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ is given by (Balakrishnan and Sandhu, 1995)

$$f_{1,2,\ldots,m:m:n}(x_1,x_2,\ldots,x_m) = A(n,m-1)\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i}, \quad 0 \le x_1 < x_2 < \cdots < x_m < \infty, \tag{3.1}$$

where $A(n,m-1) = \prod_{i=0}^{m-1}(n - S_i - i)$ with $S_0 = 0$ and $S_i = R_1 + R_2 + \cdots + R_i = \sum_{k=1}^{i} R_k$ for $1 \le i \le m-1$.

Here all the factors in $A(n,m-1)$ are positive integers. Similarly, for convenience in notation, let us define for $q = 0,1,\ldots,p-1$,

$$A(p,q) = p(p-R_1-1)(p-R_1-R_2-2)\ldots(p-R_1-R_2-\cdots-R_q-q)$$

with all the factors being positive integers. Thus

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = E\left[X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}\right]^k$$

$$= A(n,m-1)\int\int\cdots\int_{0<x_1<x_2<\ldots<x_m<\infty} x_r^k \prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i} dx_i. \tag{3.2}$$

In Subsections 3.1 and 3.2, we will derive recurrence relations for the single and the product moments of progressive Type-II right censored order statistics from Marshall-Olkin exponential distribution, utilizing the characterizing differential equation (1.6).

3.1. *Single Moments*

Theorem 3.1. *For $2 \le m \le n$ and $k \ge 0$,*

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} = \frac{1}{(R_1+1)}\Big[(k+1)\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} - (n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k+1)}} \\ - \frac{1-\alpha}{\alpha}\{a_1\mu_{1:m-1:n+1}^{(R_1+R_2+2,R_3,\ldots,R_m)^{(k+1)}} + (R_1+2)a_2\mu_{1:m:n+1}^{(R_1+1,R_2,\ldots,R_m)^{(k+1)}}\}\Big],\end{aligned} \tag{3.3}$$

where $a_1 = \frac{c}{(n+1)(n-S_2-2)\prod_{i=3}^{m-1}[n-S_i-i]}$, $a_2 = \frac{c}{(n+1)(n-S_1-1)\prod_{i=2}^{m-1}[n-S_i-i]}$

and $c = n(n-S_1-1)(n-S_2-2)\ldots(n-S_{m-1}-(m-1))$.

And, for $m = 1, n = 1,2,\ldots$ and $k > -1$,

$$\mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{1}{n}\Big[(k+1)\mu_{1:1:n}^{(n-1)^{(k)}} - \frac{1-\alpha}{\alpha}\mu_{1:1:n+1}^{(n)^{(k+1)}}\Big]. \tag{3.4}$$

Proof. Relation (3.3) may be proved by following exactly the same steps as those in proving Theorem 3.2, which is presented next to Remark 3.1. □

Remark 3.1. Putting $k = 0$ in Eq. (3.4), we get

$$\mu_{1:1:n+1}^{(n)} = \frac{\alpha}{1-\alpha}\Big[1-n\mu_{1:1:n}^{(n-1)}\Big]. \tag{3.5}$$

Changing n to $(n-1)$ in Eq. (3.5) and then substituting the value of $\mu_{1:1:n}^{(n-1)}$ in (3.5) itself, we get

$$\mu_{1:1:n+1}^{(n)} = \frac{\alpha}{1-\alpha}\Big[1-n\frac{\alpha}{1-\alpha}+n(n-1)\Big(\frac{\alpha}{1-\alpha}\Big)\mu_{1:1:n-1}^{(n-2)}\Big], \tag{3.6}$$

following similar iterations $n-1$ times, we get

$$\mu_{1:1:n+1}^{(n)} = \frac{\alpha}{1-\alpha}\Big[1-n\Big(\frac{\alpha}{1-\alpha}\Big)+n(n-1)\Big(\frac{\alpha}{1-\alpha}\Big)^2-n(n-1)(n-2)\Big(\frac{\alpha}{1-\alpha}\Big)^3+\ldots$$
$$\ldots+(-1)^{n-1}n!\Big(\frac{\alpha}{1-\alpha}\Big)^{n-1}+(-1)^n n!\Big(\frac{\alpha}{1-\alpha}\Big)^{n-1}\mu_{1:1:1}^{(0)}\Big],$$

where $\mu_{1:1:1}^{(0)} = \mu_{1:1} = \frac{\alpha\log\alpha}{\alpha-1}$ (cf. Salah et al., 2009, p.82). Thus,

$$\mu_{1:1:n+1}^{(n)} = \sum_{i=1}^{n}(-1)^{i-1}\frac{n!}{(n-i+1)!}\Big(\frac{\alpha}{1-\alpha}\Big)^i+(-1)^{n-1}n!\Big(\frac{\alpha}{1-\alpha}\Big)^{n+1}\log\alpha.$$

Changing n to $n-1$, the above equation becomes

$$\mu_{1:1:n}^{(n-1)} = \sum_{i=1}^{n-1}(-1)^{i-1}\frac{(n-1)!}{(n-i)!}\Big(\frac{\alpha}{1-\alpha}\Big)^i+(-1)^{n-2}(n-1)!\Big(\frac{\alpha}{1-\alpha}\Big)^{n}\log\alpha. \tag{3.7}$$

Similarly, for $k = 1$ from (3.4), we get

$$\mu_{1:1:n+1}^{(n)^{(2)}} = \sum_{i=1}^{n}(-1)^{i-1}2\beta\frac{n!}{(n-i+1)!}\Big(\frac{\alpha}{1-\alpha}\Big)^i+(-1)^{n-1}n!\Big(\frac{\alpha}{1-\alpha}\Big)^{n+1}\log\alpha,$$

where $\beta = \mu_{1:1:n}^{(n-1)}$.
Changing n to $n-1$, the above equation becomes

$$\mu_{1:1:n}^{(n-1)^{(2)}} = \sum_{i=1}^{n-1}(-1)^{i-1}2\beta'\frac{(n-1)!}{(n-i)!}\Big(\frac{\alpha}{1-\alpha}\Big)^i+(-1)^{n-2}(n-1)!\Big(\frac{\alpha}{1-\alpha}\Big)^{n}\log\alpha, \tag{3.8}$$

where $\beta' = \mu_{1:1:n-1}^{(n-2)}$, which can be calculated from (3.7).

Theorem 3.2. *For $2 \le r \le m-1,\ m \le n$ and $k \ge 0$,*

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} = \frac{1}{(R_r+1)}\Big[(k+1)\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} - (n-S_r-r)\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(k+1)}}$$
$$+[n-S_{r-1}-(r-1)]\mu_{r-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(k+1)}}$$
$$-\frac{1-\alpha}{\alpha}\{b_1\mu_{r:m-1:n+1}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+2,R_{r+2},\ldots,R_m)^{(k+1)}}$$
$$-b_2\mu_{r-1:m-1:n+1}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+2,R_{r+1},\ldots,R_m)^{(k+1)}}$$
$$+(R_r+2)b_3\mu_{r:m:n+1}^{(R_1,\ldots,R_{r-1},R_r+1,R_{r+1},\ldots,R_m)^{(k+1)}}\}\Big], \qquad (3.9)$$

where
$b_1 = \frac{c}{\prod_{i=0}^{r-1}[(n+1)-S_i-i]\prod_{t=r+1}^{m-1}[n-S_t-t]}$, $b_2 = \frac{c}{\prod_{i=0}^{r-2}[(n+1)-S_i-i]\prod_{t=r}^{m-1}[n-S_t-t]}$
and $b_3 = \frac{c}{\prod_{i=0}^{r-1}[(n+1)-S_i-i]\prod_{t=r}^{m-1}[n-S_t-t]}$.

Proof. From equation (3.2), we have

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = A(n,m-1)\int\limits_{0<x_1<x_2<\ldots<x_m<\infty}\int\cdots\int x_r^k \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t$$

$$= A(n,m-1)\int\limits_{0<x_1<x_2<\cdots<x_{r-1}<x_{r+1}<\ldots<x_m<\infty}\int\cdots\int I(x_{r-1},x_{r+1})\prod_{t=1,t\neq r}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t, \qquad (3.10)$$

where

$$I(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^k f(x_r)[1-F(x_r)]^{R_r}dx_r.$$

Using equation (1.6), we get

$$I(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^k[1-F(x_r)]^{R_r}\Big\{1-F(x_r)+\Big(\frac{1-\alpha}{\alpha}\Big)[1-F(x_r)]^2\Big\}dx_r$$

$$= \int_{x_{r-1}}^{x_{r+1}} x_r^k[1-F(x_r)]^{R_r+1}dx_r + \Big(\frac{1-\alpha}{\alpha}\Big)\int_{x_{r-1}}^{x_{r+1}} x_r^k[1-F(x_r)]^{R_r+2}dx_r$$

$$= I_0(x_{r-1},x_{r+1}) + \Big(\frac{1-\alpha}{\alpha}\Big)I_1(x_{r-1},x_{r+1}), \qquad (3.11)$$

where

$$I_a(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^k[1-F(x_r)]^{R_r+a+1}dx_r, \quad a = 0,1.$$

Integrating by parts yields,

$$I_a(x_{r-1},x_{r+1}) = \frac{x_{r+1}^{k+1}}{k+1}[1-F(x_{r+1})]^{R_r+a+1} - \frac{x_{r-1}^{k+1}}{k+1}[1-F(x_{r-1})]^{R_r+a+1} + (R_r+a+1)\int_{x_{r-1}}^{x_{r+1}} \frac{x_r^{k+1}}{k+1}[1-F(x_r)]^{R_r+a}dx_r. \tag{3.12}$$

Upon substituting for $I_0(x_{r-1},x_{r+1})$ and $I_1(x_{r-1},x_{r+1})$ from (3.12) in (3.11) and then substituting the resultant expression for $I(x_{r-1},x_{r+1})$ in equation (3.10) and simplifying, it leads to (3.9).

Proceeding on similar lines, one can derive the following recurrence relation. □

Theorem 3.3. *For $2 \le r \le n$ and $k > -1$,*

$$\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} = \frac{1}{R_m+1}\Big[(k+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} + [n-S_{m-1}-m+1]\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} - \frac{1-\alpha}{\alpha}\Big\{-c_1\mu_{m-1:m-1:n+1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+2)^{(k+1)}} + c_2(R_m+2)\mu_{m:m:n+1}^{(R_1,\ldots,R_{m-1},R_m+1)^{(k+1)}}\Big\}\Big], \tag{3.13}$$

where

$c_1 = \frac{c}{\prod_{i=0}^{m-2}[(n+1)-S_i-i][n-S_m-m+1]}$, $c_2 = \frac{c}{\prod_{i=0}^{m-2}[(n+1)-S_i-i][n-S_m-m+2]}$.

3.2. *Product Moments*

To obtain the recurrence relations for the product moments of progressive Type-II right censored order statistics from Marshall-Olkin exponential distribution, we have from equation (3.1),

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} = E\left[\left\{X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^i\left\{X_{s:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^j\right]$$

$$= A(n,m-1)\int\int_{0<x_1<x_2<\ldots<x_m<\infty}\cdots\int x_r^i x_s^j \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t. \tag{3.14}$$

Theorem 3.4. *For $1 \le r < m$ and $m \le n-1$,*

$$\begin{aligned}\mu_{r,r+1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+1)}} = \frac{1}{R_{r+1}+1}\Big[&(j+1)\mu_{r,r+1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} \\ &-[n-S_{r+1}-(r+1)]\mu_{r,r+1:m-1:n}^{(R_1,\ldots,R_r,R_{r+1}+R_{r+2}+1,R_{r+3},\ldots,R_m)^{(i,j+1)}} \\ &+(n-S_r-r)\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(i,j+1)}} \\ &-\frac{1-\alpha}{\alpha}\Big\{d_1\mu_{r,r+1:m-1:n+1}^{(R_1,\ldots,R_r,R_{r+1}+R_{r+2}+2,R_{r+3},\ldots,R_m)^{(i,j+1)}} \\ &-d_2\mu_{r:m-1:n+1}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+2,R_{r+2},\ldots,R_m)^{(i,j+1)}} \\ &+d_3(R_{r+1}+2)\mu_{r,r+1:m:n+1}^{(R_1,\ldots,R_r,R_{r+1}+1,R_{r+2},\ldots,R_m)^{(i,j+1)}}\Big\}\Big],\end{aligned} \tag{3.15}$$

where

$d_1 = \frac{c}{\prod_{i=0}^{r}[(n+1)-S_i-i]\prod_{t=r+2}^{m-1}[n-S_t-t]}$, $d_2 = \frac{c}{\prod_{i=0}^{r-1}[(n+1)-S_i-i]\prod_{t=r+1}^{m-1}[n-S_t-t]}$

and $\quad d_3 = \frac{c}{\prod_{i=0}^{r}[(n+1)-S_i-i]\prod_{t=r+1}^{m-1}[n-S_t-t]}$.

Proof. Relation (3.15) may be proved by following exactly the same steps as those in proving Theorem 3.5, which is presented below.

□

Theorem 3.5. *For $1 \le r < s \le m$ and $m \le n$,*

$$\begin{aligned}\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+1)}} = \frac{1}{R_s+1}\Big[&(j+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} - (n-S_s-s)\mu_{r,s:m-1:n}^{(R_1,\ldots,R_{s-1},R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(i,j+1)}} \\ &+[n-S_{s-1}-(s-1)]\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(i,j+1)}} \\ &-\frac{1-\alpha}{\alpha}\Big\{e_1\mu_{r,s:m-1:n+1}^{(R_1,\ldots,R_{s-1},R_s+R_{s+1}+2,R_{s+2},\ldots,R_m)^{(i,j+1)}} \\ &-e_2\mu_{r,s-1:m-1:n+1}^{(R_1,\ldots,R_{s-2},R_{s-1}+R_s+2,R_{s+1},\ldots,R_m)^{(i,j+1)}} \\ &+(R_s+2)e_3\mu_{r,s:m:n+1}^{(R_1,\ldots,R_s+1,\ldots,R_m)^{(i,j+1)}}\Big\}\Big],\end{aligned} \tag{3.16}$$

where

$e_1 = \frac{c}{\prod_{i=0}^{s-1}[(n+1)-S_i-i]\prod_{t=s+1}^{m-1}[n-S_t-t]}$, $e_2 = \frac{c}{\prod_{i=0}^{s-2}[(n+1)-S_i-i]\prod_{t=s}^{m-1}[n-S_t-t]}$

and $\quad e_3 = \frac{c}{\prod_{i=0}^{s-1}[(n+1)-S_i-i]\prod_{t=s}^{m-1}[n-S_t-t]}$.

Proof. From equation (3.14), we have

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} = A(n,m-1)\underset{0<x_1<x_2<\ldots<x_m<\infty}{\int\int\cdots\int} x_r^i I(x_{s-1},x_{s+1})\prod_{t=1,t\neq s}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t, \tag{3.17}$$

where $I(x_{s-1},x_{s+1}) = \int_{x_{s-1}}^{x_{s+1}} x_s^j[1-F(x_s)]^{R_s}f(x_s)dx_s$.

Making use of characterizing differential equation (1.6), and splitting the integral accordingly into two, we get,

$$I(x_{s-1},x_{s+1}) = I_0(x_{s-1},x_{s+1}) + \Big(\frac{1-\alpha}{\alpha}\Big) I_1(x_{s-1},x_{s+1}), \tag{3.18}$$

where

$$I_a(x_{s-1},x_{s+1}) = \int_{x_{s-1}}^{x_{s+1}} x_s^j [1-F(x_s)]^{R_s+a+1} dx_s, \quad a = 0,1.$$

Integrating by parts, we get

$$\begin{aligned} I_a(x_{s-1},x_{s+1}) = \frac{1}{j+1}\Big[& x_{s+1}^{j+1}\ [1-F(x_{s+1})]^{R_s+a+1} - x_{s-1}^{j+1}\ [1-F(x_{s-1})]^{R_s+a+1} \\ & + (R_s+a+1)\int_{x_{s-1}}^{x_{s+1}} x_s^{j+1}[1-F(x_s)]^{R_s+a} dx_s \Big]. \end{aligned} \tag{3.19}$$

Upon substituting for $I_0(x_{s-1},x_{s+1})$ and $I_1(x_{s-1},x_{s+1})$ from (3.19) in (3.18) and then substituting the resultant expression for $I(x_{s-1},x_{s+1})$ in equation (3.17) and simplifying, it leads to (3.16).

Likewise, one can easily derive the recurrence relations given in the following theorems. □

Theorem 3.6. *For* $1 \le r \le m-1$, *and* $m \le n$,

$$\begin{aligned} \mu_{r,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+1)}} = \frac{1}{(R_m+1)} \Big[& (j+1)\mu_{r,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} \\ & + (n-S_{m-1}-(m-1))\mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i,j+1)}} \\ & - \frac{1-\alpha}{\alpha}\Big\{ -t_1\mu_{r,m-1:m-1:n+1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+2)^{(i,j+1)}} \\ & + (R_m+2)t_2\mu_{r,m:m:n+1}^{(R_1,\ldots,R_{m-1},R_m+1)^{(i,j+1)}} \Big\}\Big], \end{aligned} \tag{3.20}$$

where $t_1 = \frac{c}{\prod_{i=0}^{m-2}[(n+1)-S_i-i][n-S_m-m+1]}$, $t_2 = \frac{c}{\prod_{i=0}^{m-2}[(n+1)-S_i-i][n-S_m-m+2]}$.

Theorem 3.7. *For* $1 \le r < s \le m$ *and* $m \le n$,

$$\begin{aligned} \mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+1,j)}} = \frac{1}{(R_1+1)} \Big[& (i+1)\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} - (n-S_1-1)\mu_{1,s-1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(i+1,j)}} \\ & - \frac{1-\alpha}{\alpha}\Big\{ p_1\mu_{1,s-1:m-1:n+1}^{(R_1+R_2+2,R_3,\ldots,\ldots,R_m)^{(i+1,j)}} \\ & + (R_1+2)p_2\mu_{1,s:m:n+1}^{(R_1+1,R_2,\ldots,R_m)^{(i+1,j)}} \Big\}\Big], \end{aligned} \tag{3.21}$$

where

$p_1 = \frac{c}{(n+1)[n-S_2-2]\prod_{i=3}^{m-1}[n-Si-i]}$, $p_2 = \frac{c}{(n+1)[n-S_1-1]\prod_{i=2}^{m-1}[n-Si-i]}$.

Theorem 3.8. *For $1 \le r < s \le m$ and $m \le n$,*

$$
\begin{aligned}
\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+1,j)}} = \frac{1}{(R_r+1)}\Big[(i+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} \\
-(n-S_r-r)\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(i+1,j)}} \\
+[n-S_{r-1}-(r-1)]\mu_{r-1,s-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(i+1,j)}} \\
-\frac{1-\alpha}{\alpha}\Big\{q_1\mu_{r,s-1:m-1:n+1}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+2,R_{r+2},\ldots,R_m)^{(i+1,j)}} \\
-q_2\mu_{r-1,s-1:m-1:n+1}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+2,R_{r+1},\ldots,R_m)^{(i+1,j)}} \\
+(R_r+2)q_3\mu_{r,s:m:n+1}^{(R_1,\ldots,R_{r-1},R_r+1,R_{r+1}\ldots,R_m)^{(i+1,j)}}\Big\}\Big],
\end{aligned}
\tag{3.22}
$$

where

$$q_1 = \frac{c}{\prod_{i=0}^{r-1}[(n+1)-S_i-i]\prod_{t=r+1}^{m-1}[n-S_t-t]},\quad q_2 = \frac{c}{\prod_{i=0}^{r-2}[(n+1)-S_i-i]\prod_{t=r}^{m-1}[n-S_t-t]}$$

and $$q_3 = \frac{c}{\prod_{i=0}^{r-1}[(n+1)-S_i-i]\prod_{t=r}^{m-1}[n-S_t-t]}.$$

Remark 3.2. Substituting $\alpha = 1$ in Theorems 3.1-3.8, we obtain recurrence relations for single and product moments of progressive Type-II right censored order statistics from standard exponential distribution, which are in agreement with the results established by Aggarwala and Balakrishnan (1996), Balakrishnan and Aggarwala (2000, pp. 42-49).

Remark 3.3. For the special case $R_1 = \ldots = R_m = 0$ so that $m = n$ in which case the progressive censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, whose single moments are denoted by $\mu_{r:n}^{(k)}$ for $1 \le r \le n$ and product moments are denoted by $\mu_{r,s:n}^{(i,j)}$ for $1 \le r < s \le n$, the relations established in Subsections 3.1 and 3.2 reduce to the following:

From (3.3): For $k > -1$,

$$\mu_{1:n}^{(k+1)} = (k+1)\mu_{1:n}^{(k)} - (n-1)\mu_{1:n-1:n}^{(1,0,\ldots,0)^{(k+1)}} - \frac{1-\alpha}{\alpha(n+1)}\Big\{n(n-1)\mu_{1:n-1:n+1}^{(2,0,\ldots,0)^{(k+1)}} + 2\mu_{1:n:n+1}^{(1,0,\ldots,0)^{(k+1)}}\Big\}. \tag{3.23}$$

From (3.9): For $2 \le r \le n-1$ and $k > -1$,

$$
\begin{aligned}
\mu_{r:n}^{(k+1)} = (k+1)\mu_{r:n}^{(k)} - (n-r)\mu_{r:n-1:n}^{(0,\ldots,0,1,0,\ldots,0)^{(k+1)}} + (n-r+1)\mu_{r-1:n-1:n}^{(0,\ldots,0,1,0,\ldots,0)^{(k+1)}} \\
-\frac{1-\alpha}{\alpha}\left(\frac{n-r+1}{n+1}\right)\Big\{(n-r)\mu_{r:n-1:n+1}^{(0,\ldots,0,2,0,\ldots,0)^{(k+1)}} - (n-r+2)\mu_{r-1:n-1:n+1}^{(0,\ldots,0,2,0,\ldots,0)^{(k+1)}} \\
+2\mu_{r:n:n+1}^{(0,\ldots,0,1,0,\ldots,0)^{(k+1)}}\Big\},
\end{aligned}
\tag{3.24}
$$

where, in the superscript of the second term of the right hand side , the 1 is in the r^{th} position, in the superscript of third term, the 1 is in the $(r-1)^{th}$ position, in the superscript of fourth term, the 2 is in the r^{th} position, in the superscript of fifth term, the 2 is in the $(r-1)^{th}$ position and in the superscript of the sixth term of the right hand side, the 1 is in the r^{th} position.

From (3.13): For $2 \le r \le n$ and $k > -1$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1,n-1:n}^{(0,\dots,0,1)^{(k+1)}} - \frac{1-\alpha}{\alpha}\Big(\frac{2}{n+1}\Big)\Big\{-\mu_{n-1:n-1:n+1}^{(0,\dots,0,2)^{(k+1)}} + \mu_{n:n:n+1}^{(0,\dots,0,1)^{(k+1)}}\Big\}. \quad (3.25)$$

From (3.16): For $1 \le r < s \le n$,

$$\begin{aligned}\mu_{r,s:n}^{(i,j+1)} &= (j+1)\mu_{r,s:n}^{(i,j)} - (n-s)\mu_{r,s:n-1:n}^{(0,\dots,0,1,0,\dots,0)^{(i,j+1)}} + (n-s+1)\mu_{r,s-1:n-1:n}^{(0,\dots,0,1,0\dots,0)^{(i,j+1)}}\\ &\quad -\frac{1-\alpha}{\alpha}\Big(\frac{n-s+1}{n+1}\Big)\Big\{(n-s)\mu_{r,s:n-1:n+1}^{(0,\dots,0,2,0,\dots,0)^{(i,j+1)}}\\ &\quad -(n-s+2)\mu_{r,s-1:n-1:n+1}^{(0,\dots,0,2,0,\dots,0)^{(i,j+1)}} + 2\mu_{r,s:n:n+1}^{(0,\dots,0,1,0,\dots,0)^{(i,j+1)}}\Big\}, \end{aligned} \quad (3.26)$$

where, in the superscript of the second term of the right hand side , the 1 is in the s^{th} position, in the superscript of third term, the 1 is in the $(s-1)^{th}$ position, in the superscript of fourth term, the 2 is in the s^{th} position, in the superscript of fifth term, the 2 is in the $(s-1)^{th}$ position and in the superscript of the sixth term of the right hand side, the 1 is in the s^{th} position.

Now, if $R_1 = R_2 = \ldots = R_{s-1} = 0$ so that there is no censoring before the time of the s-th failure, then the first s progressive Type-II right censored order statistics are simply the first s usual order statistics. Thus, the relations above reduce to:

From (3.4): For $n \ge 1,\ k \ge 0,\ \alpha > 0,\ \alpha \ne 1$,

$$\mu_{1:n+1}^{(k+1)} = \frac{\alpha}{1-\alpha}\Big[\frac{(k+1)}{n}\mu_{1:n}^{(k)} - \mu_{1:n}^{(k+1)}\Big], \quad (3.27)$$

with $\mu_{1:n}^{(0)} = 1$ and $\mu_{1:1} = \frac{\alpha log\alpha}{\alpha-1}, \quad \alpha > 0,\ \alpha \ne 1.$

From (3.9): For $2 \le r \le n, n \ge 1,\ k \ge 0$ *and* $\alpha > 0$,

$$\begin{aligned}(k+1)\mu_{r:n}^{(k)} &= (n-r+1)\Big\{\mu_{r:n}^{(k+1)} - \mu_{r-1:n}^{(k+1)}\Big\}\\ &\quad + \frac{1-\alpha}{\alpha}\Big[\frac{(n-r+1)(n-r+2)}{n+1}\Big\{\mu_{r:n+1}^{(k+1)} - \mu_{r-1:n+1}^{(k+1)}\Big\}\Big]. \end{aligned} \quad (3.28)$$

From Relation 1 of David and Nagaraja (2003, p.44) for $r = 1,2,\dots,n-1$ and for $k \ge 1$, viz.

$$r\mu_{r+1:n}^{(k)} + (n-r)\mu_{r:n}^{(k)} = n\mu_{r:n-1}^{(k)}, \quad (3.29)$$

we have

$$(n+1)\mu_{r:n}^{(k+1)} = r\mu_{r+1:n+1}^{(k+1)} + (n-r+1)\mu_{r:n+1}^{(k+1)}, \quad (3.30)$$

and

$$(n+1)\mu_{r-1:n}^{(k+1)} = (r-1)\mu_{r:n+1}^{(k+1)} + (n-r+2)\mu_{r-1:n+1}^{(k+1)}, \quad (3.31)$$

which gives

$$\mu_{r-1:n+1}^{(k+1)} = \frac{1}{n-r+2}\Big[(n+1)\mu_{r-1:n}^{(k+1)} - (r-1)\mu_{r:n+1}^{(k+1)}\Big]. \quad (3.32)$$

Utilizing Eqs. (3.30) and (3.32) into Eq. (3.28), we obtain for $2 \leq r \leq n, n \geq 1, k \geq 0, \alpha > 0$,

$$\mu_{r+1:n+1}^{(k+1)} = \frac{1}{r}\Big[\frac{(n+1)(k+1)}{n-r+1}\mu_{r:n}^{(k)} - \frac{n-\alpha r+1}{\alpha}\mu_{r:n+1}^{(k+1)} + \frac{n+1}{\alpha}\mu_{r-1:n}^{(k+1)}\Big]. \tag{3.33}$$

It is worth mentioning here that equations (3.27) and (3.33) are in agreement with the corresponding results obtained by Salah et al. (2009) in their equations (17) and (20), respectively, for the single moments of order statistics from MOE distribution.

From (3.13): For $2 \leq r \leq n$ and $k > -1$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n}^{(k+1)} - \frac{2(1-\alpha)}{\alpha(n+1)}\Big\{\mu_{n:n+1}^{(k+1)} - \mu_{n-1:n+1}^{(k+1)}\Big\}. \tag{3.34}$$

From (3.16): For $1 \leq r < s \leq n$,

$$\begin{aligned}(j+1)\mu_{r,s:n}^{(i,j)} &= (n-s+1)\Big\{\mu_{r,s:n}^{(i,j+1)} - \mu_{r,s-1:n}^{(i,j+1)}\Big\} \\ &\quad + \frac{(1-\alpha)(n-s+1)(n-s+2)}{\alpha(n+1)}\Big[\mu_{r,s:n+1}^{(i,j+1)} \qquad -\mu_{r,s-1:n+1}^{(i,j+1)}\Big]. \end{aligned} \tag{3.35}$$

4. Recursive Algorithm

Using the recurrence relations established in Subsections 3.1 and 3.2, the means, variances and covariances of all progressive Type-II right censored order statistics from the Marshall-Olkin exponential distribution can be readily computed as follows.

Eq. (3.7) will give us the values $\mu_{1:1:n}^{(n-1)}$, $n = 1, 2, \ldots$ and Eq. (3.8) will give us the values $\mu_{1:1:n}^{(n-1)^{(2)}}$, $n = 1, 2, \ldots$. Thus, all first and second moments with $m = 1$ for all sample sizes Ł will be obtained. Next, using (3.3), applying the same method as we did to obtain $\mu_{1:1:n}^{(n-1)^{(2)}}$ in (3.8), we can determine all moments of the form $\mu_{1:2:n}^{(R_1,R_2)}$ for all $R_1, R_2,\ n = 2, 3, \ldots$, in a simple iterative manner, which can in turn be used, with (3.3), to determine all moments of the form $\mu_{1:2:n}^{(R_1,R_2)^{(2)}}$, $n = 2, 3, \ldots$. Similarly, Eq. (3.13) can then be used to obtain $\mu_{2:2:n}^{(R_1,R_2)}$ for all R_1, R_2, and $n \geq 2$, and these values can be used to obtain all moments of the form $\mu_{2:2:n}^{(R_1,R_2)^{(2)}}$ for $n \geq 2$ using (3.13) again. Eq. (3.3) can now be used again to obtain $\mu_{1:3:n}^{(R_1,R_2,R_3)}$ and $\mu_{1:3:n}^{(R_1,R_2,R_3)^{(2)}}$ for all R_1, R_2 and R_3, and $n \geq 3$, and (3.9) can be used next to obtain all moments of the form $\mu_{2:3:n}^{(R_1,R_2,R_3)}$ and $\mu_{2:3:n}^{(R_1,R_2,R_3)^{(2)}}$. Finally, (3.13) can be used to obtain all moments of the form $\mu_{3:3:n}^{(R_1,R_2,R_3)}$ and $\mu_{3:3:n}^{(R_1,R_2,R_3)^{(2)}}$. This process can be continued until all desired first and second order moments (and therefore all variances) are obtained for all sample sizes and all censoring schemes. In a similar manner, by utilizing the results of Subsection 3.2, one can obtain in a recursive manner, all the product moments (and therefore all the covariances) for all sample sizes and all censoring schemes.

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