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On the Symmetry of Some Nonlinear Generalization of a Vector Subsystem of the Maxwell Equations

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Abstract

The problem of studying the maximal Lie symmetry of some nonlinear generalization of the vector subsystem of the Maxwell equations is completely solved.

Let us consider the following nonlinear system of vector equations:

$$D_t\vec{E} = \lambda_1 \operatorname{rot} \vec{H}, \qquad D_t\vec{H} = \lambda_2 \operatorname{rot} \vec{E}$$
$$D_t \equiv \frac{\partial}{\partial t} + \sigma E_i \frac{\partial}{\partial x_i} + \rho H_i \frac{\partial}{\partial x_i} + \omega V_i \frac{\partial}{\partial x_i}, \quad i = \overline{1,3} \tag{1}$$

where $\sigma, \rho, \omega, \lambda_1, \lambda_2 \in \mathbf{R}/\{0\}$ are arbitrary constants; $t, x_i \equiv (x_0, x_1, x_2, x_3)$ are independent variables; $\vec{E}(x_0, \vec{x})$; $\vec{H}(x_0, \vec{x})$; $\vec{V}(x_0, \vec{x})$ are arbitrary vector functions, which satisfy system (1).

System (1) can be interpreted as a nonlinear generalization of the vector subsystem of the Maxwell equations. This follows from (1) at $\lambda_1 = -\lambda_2 = 1, \quad \sigma = \rho = \omega = 0$.

The problem of investigation of the maximal symmetry of system (1), which has been suggested in [1], is completely solved in the present paper.

The following statements are proved by means of the Lie method [2].

Theorem 1 *A maximal invariance group of system (1) for $\omega \neq 0$ is generated by the operator*

$$X = \xi^\mu(x,u)\frac{\partial}{\partial x_\mu} + \eta^n(x,u)\frac{\partial}{\partial u_n}, \tag{2}$$

where $x = (x_0, x_1, x_2, x_3), u = (\vec{E}, \vec{H}, \vec{V}), \mu = \overline{0,3}, n = \overline{1,9}$,

$$\xi^0 = \xi^0(x_0),$$

$$\xi^1 = \xi^0_0(x_0)x_1 + b_3x_2 - b_2x_3 + Q^1(x_0),$$

$$\xi^2 = \xi^0_0(x_0)x_2 - b_3x_1 + b_1x_3 + Q^2(x_0),$$

$$\xi^3 = \xi^0_0(x_0)x_3 + b_2x_1 - b_1x_2 + Q^3(x_0),$$

$$\eta^1 = cu_1 + b_3u_2 - b_2u_3 + c_1u_4 + d_1,$$

$$\eta^2 = cu_2 - b_3u_1 + b_1u_3 + c_1u_5 + d_2,$$

$$\eta^3 = cu_3 + b_2u_1 - b_1u_2 + c_1u_6 + d_3,$$

$$\eta^4 = cu_4 + b_3u_5 - b_2u_6 + \frac{\lambda_2}{\lambda_1}c_1u_1 + d_4,$$

$$\eta^5 = cu_5 - b_3u_4 + b_1u_6 + \frac{\lambda_2}{\lambda_1}c_1u_2 + d_5,$$

$$\eta^6 = cu_6 + b_2u_4 - b_1u_5 + \frac{\lambda_2}{\lambda_1}c_1u_3 + d_6,$$

$$\eta^7 = b_3u_8 - b_2u_9 - \frac{1}{\omega}\left[\left(\sigma c + \rho\frac{\lambda_2}{\lambda_1}c_1\right)u_1 + (\sigma c_1 + \rho c)u_4\right] +$$

$$\frac{\xi^0_{00}(x_0)x_1 + Q^1_0(x_0) - (\sigma d_1 + \rho d_4)}{\omega},$$

$$\eta^8 = -b_3u_7 + b_1u_9 - \frac{1}{\omega}\left[\left(\sigma c + \rho\frac{\lambda_2}{\lambda_1}c_1\right)u_2 + (\sigma c_1 + \rho c)u_5\right] +$$

$$\frac{\xi^0_{00}(x_0)x_2 + Q^2_0(x_0) - (\sigma d_2 + \rho d_5)}{\omega},$$

$$\eta^9 = b_2u_7 - b_1u_8 - \frac{1}{\omega}\left[\left(\sigma c + \rho\frac{\lambda_2}{\lambda_1}c_1\right)u_3 + (\sigma c_1 + \rho c)u_6\right] +$$

$$\frac{\xi^0_{00}(x_0)x_3 + Q^3_0(x_0) - (\sigma d_3 + \rho d_6)}{\omega},$$

here $c, c_1, b_1, b_2, b_3, d_1, d_2, d_3, d_4, d_5, d_6$ are real parameters, $\xi^0(x_0)$, $Q^1(x_0)$, $Q^2(x_0)$, $Q^3(x_0)$ are arbitrary twice differentiable functions of x_0.

Theorem 2 *For $\omega = 0$, under the condition $\frac{\lambda_2}{\lambda_1} = \frac{\sigma^2}{\rho^2}$, system (1) admits a group of transformations with the infinitesimal operator (2) which has the following coordinates:*

$$\begin{aligned}
\xi^0 &= cx_0 + c_0,\\
\xi^1 &= cx_1 + b_3x_2 - b_2x_3 + c_1x_0 + a_1,\\
\xi^2 &= cx_2 - b_3x_1 + b_1x_3 + c_2x_0 + a_2,\\
\xi^3 &= cx_3 + b_2x_1 - b_1x_2 + c_3x_0 + a_3,
\end{aligned} \tag{3}$$

$$
\begin{aligned}
\eta^1 &= -\sqrt{\frac{\lambda_2}{\lambda_1}}\gamma u_1 + b_3u_2 - b_2u_3 + \gamma u_4 + d_1,\\
\eta^2 &= -\sqrt{\frac{\lambda_2}{\lambda_1}}\gamma u_2 - b_3u_1 + b_1u_3 + \gamma u_5 + d_2,\\
\eta^3 &= -\sqrt{\frac{\lambda_2}{\lambda_1}}\gamma u_3 + b_2u_1 - b_1u_2 + \gamma u_6 + d_3,\\
\eta^4 &= -\sqrt{\frac{\lambda_2}{\lambda_1}}\gamma u_4 + b_3u_5 - b_2u_6 + \frac{\lambda_2}{\lambda_1}\gamma u_1 + \frac{1}{\rho}(c_1 - \sigma d_1),\\
\eta^5 &= -\sqrt{\frac{\lambda_2}{\lambda_1}}\gamma u_5 - b_3u_4 + b_1u_6 + \frac{\lambda_2}{\lambda_1}\gamma u_2 + \frac{1}{\rho}(c_2 - \sigma d_2),\\
\eta^6 &= -\sqrt{\frac{\lambda_2}{\lambda_1}}\gamma u_6 + b_2u_4 - b_1u_5 + \frac{\lambda_2}{\lambda_1}\gamma u_3 + \frac{1}{\rho}(c_3 - \sigma d_3)
\end{aligned}
\tag{4}
$$

here $c, c_0, c_1, c_2, c_3, a_1, a_2, a_3, b_1, b_2, b_3, d_1, d_2, d_3, \gamma$ *are real parameters.*

Under the condition $\frac{\lambda_2}{\lambda_1} \neq \frac{\sigma^2}{\rho^2}$ *(for* $\gamma = 0$*), the symmetry operator of system (1) is given by equations (2)–(4).*

Note 1. The maximal invariance group of system (1) for $\omega = 0$ is finite-parametric.

Note 2. For $\rho = 1$ algebras (2)–(4) contain the Lie algebra of the Galilei group $AG(1,3)$ as a subalgebra with the following basis elements

$$
\begin{aligned}
&P_0 = \frac{\partial}{\partial x_0}, \quad P_a = \frac{\partial}{\partial x_a},\\
&J_{ab} = x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a} + E_a\frac{\partial}{\partial E_b} - E_b\frac{\partial}{\partial E_a} + H_a\frac{\partial}{\partial H_b} - H_b\frac{\partial}{\partial H_a},\\
&G_a = x_0\frac{\partial}{\partial x_a} + \frac{\partial}{\partial H_a}.
\end{aligned}
$$

Hence, the Galilei relativity principle is valid for system (1).

References

[1] Fushchych W.I., Symmetry and solutions of nonlinear equations of mathematical physics, Institute of Mathematics, Kyiv, 1987, 4–16.

[2] Ovsyannikov L.V., Group Analysis of Differential Equations, Academic Press, New York, 1982, 400p.