

# Partially Mixed Duopoly and Oligopoly: Consistent Conjectural Variations Equilibrium (CCVE). Part 1.

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## Abstract

In this paper, we consider a model of partially mixed duopoly with conjectured variations equilibrium (CVE). The agents' conjectures concern the price variations depending upon their production output's increase or decrease. We establish existence and uniqueness results for the conjectured variations equilibrium (called an exterior equilibrium) for any set of feasible conjectures. To introduce the notion of an interior equilibrium, we develop a consistency criterion for the conjectures (referred to as influence coefficients) and prove the existence theorem for the interior equilibrium (understood as a CVE with consistent conjectures). To prepare the base for the extension of our results to the case of non-differentiable demand functions, we also investigate the behavior of the consistent conjectures in dependence upon a parameter representing the demand function's derivative with respect to the market price.

**Keywords:** Consistent Conjectural Variations Equilibrium, Partially Mixed Duopoly and Oligopoly.

## 1. Introduction

Nowadays, the topic of mixed oligopolies is referred to quite frequently in the literature. An oligopoly is usually called mixed if, apart from standard agents maximizing her/his net profit, at least one company participates with an objective function different from the net profit. Numerous mixed oligopoly models involve an agent maximizing domestic social surplus: see, [1]-[5], to mention only few. In others, an income-per-worker function replaces the standard net profit objective function: *cf.*, [6]-[9]. Papers [10] and [11] study the third type of the mixed duopoly, namely, where the special agent tries to maximize a convex combination of its net profit and domestic social surplus.

In the majority of the above-mentioned papers, the

mixed oligopoly is studied in the framework of classical Cournot, Hotelling or Stackelberg models. However, quite a long time ago, a concept of *conjectural variations equilibrium* (CVE) was proposed by Bowley [12] and Frisch [13] as another possible solution form in static games. According to this concept, players behave as follows: each agent chooses her/his most favorable action assuming that every rival's strategy is a *conjectural variation function* of her/his own strategy. For instance, as Laitner [14] puts it, "Although the firms make their output decisions simultaneously, plan changes are always possible before production begins. So each firm is aware that its choice of a production level will affect its rival's behavior." The thus arising expectation (or conjectural variation) function is what forms the base for the conjectural variations decision making, or conjectural variations equilibrium.

As is mentioned in [15] and [16], the concept of conjectural variations has been the subject of numerous theoretical controversies; e.g., *see* [17]. Nevertheless, economists have made extensive use of one form or the other of the CVE to predict the outcome of non-cooperative behavior in several areas of economics. The literature on conjectural variations has focused mainly on two-player games: *cf.*, [15], because a serious conceptual difficulty arises if the number of agents is greater than two: *see*, [15], [18].

To cope with this conceptual difficulty appearing in many-players models, Bulavsky in [19] proposed a completely new approach. Instead of assuming the identity of the agents in the conjectural variation model of a homogeneous good market, it is supposed that every player makes conjectures concerning not the (optimal) response functions of each of the other players but only about the variations of the market clearing price depending upon (infinitesimal) variations in her/his output volume. Knowing the rivals' conjectures (called influence coefficients), each agent can apply certain verification procedures and find out

if her/his influence coefficient is consistent with those of the other players.

In our recent papers [20] and [18], we extended the results of [19] to the mixed duopoly and oligopoly cases, respectively. In both papers, we defined the concept of an exterior equilibrium, i.e., a conjectural variations equilibrium (CVE) state with the influence coefficients fixed in an exogenous mode. Existence and uniqueness theorems for this kind of CVE were established, serving as a base for the concept of an interior equilibrium, which is defined as an exterior equilibrium state with consistent conjectures (influence coefficients). The consistency criteria, a consistency verification procedure, and the existence theorems for the interior equilibrium were also formulated and proved in [20] and [18].

In this paper, we extend the above-mentioned results to the case of a partially mixed duopoly and oligopoly, that is, such a model where the public company, similar to [10] and [11], maximizes a convex combination of the net profit and domestic social surplus. Results of numerical experiments with a test model of an electricity market from [21], with and without a public company among the agents, show the importance of CVE for the consumer.

The rest of the paper is organized as follows. Sections 2 to 6 deal with a duopoly framework. Namely, Section 2 formulates the model and different types of equilibrium we consider (exterior and interior ones). In Section 3, we establish the main theorem of existence and uniqueness of the exterior equilibrium for any set of feasible conjectures (influence coefficients), as well as the formulas of the derivative of the equilibrium price  $p$  with respect of the active demand variable  $D$ . Section 4 deals with the consistency criterion and the definition of an interior equilibrium (which can be treated as a consistent CVE, or CCVE); the CCVE existence theorem is also derived in that section. To provide the tools for the future research concerning the interrelationships between the demand structure (with not necessarily smooth demand function) and the CVEs with consistent conjectures (influence coefficients), the behavior of the latter as functions of certain parameter governed by the derivative by  $p$  of the demand function is studied in Theorem 4.2 completing Section 4. Section 5 deals with an important particular case of a linear demand function. In Section 6, we conduct a qualitative analysis of results of the numerical experiments. Concluding remarks (Section 7), acknowledgments and the list of references complete the paper.

## 2. Specification of the Duopoly Model

Consider two producers of a homogeneous good with the cost functions  $f_i(q_i)$ ,  $i = 0, 1$ , where  $q_i$  is the output by producer  $i$ . Consumers' demand is described by a demand function  $G(p)$ , whose argument  $p$  is the market price proposed by the producers. The value of an active demand  $D$  is nonnegative and does not depend upon the price. We will fix the equilibrium between the demand and supply for a given price  $p$  by the following balance equality

$$q_0 + q_1 = G(p) + D. \quad (1)$$

We assume the following properties of the model's data.

**A1.** The demand function  $G = G(p)$  defined for the price values  $p \in (0, +\infty)$  is non-increasing and continuously differentiable.

**A2.** For each agent  $i = 0, 1$ , the cost function  $f_i = f_i(q_i)$  is quadratic:

$$f_i(q_i) = \frac{1}{2}a_i q_i^2 + b_i q_i, \quad (2)$$

where  $a_i > 0, b_i > 0$ ,  $i = 0, 1$ . In addition, we assume that

$$b_0 \leq b_1. \quad (3)$$

**Remark 2.1** Although the assumption of  $a_i > 0$ ,  $i = 0, 1$ , may look as unacceptable in view of the scale effect often observed in real-life production economy, it is not uncommon in theory of both classical and mixed oligopoly, *see*, e.g., [3] – [5], [21], to mention only few. In the majority of cases, this assumption is the easiest way to provide for the concavity of each player's payoff function. However, this condition can be somewhat relaxed, like, for example, in [22], where the second derivative of the cost function is not assumed to be positive. Then the desired payoff function's concavity is achieved by another assumption combining the first derivative of the demand function and the second derivative of the cost function. Finally, the scale effect can be also modeled by permitting the first order coefficients  $b_i$ ,  $i = 0, 1$ , to be negative. We have already obtained the corresponding results for this more general case, and they will be published elsewhere very soon.

The private producer  $i = 1$  chooses its output volume  $q_1 \geq 0$  so as to maximize its profit function  $\pi_1(p, q_1) = p \cdot q_1 - f_1(q_1)$ . On the other hand, the public company with index  $i = 0$  produces  $q_0 \geq 0$  so as to maximize a convex combination of domestic social surplus (defined as the difference between the consumer surplus, the private company's total revenue, and the

public firm's production costs):

$$S(p; q_0, q_1) = \beta \left[ \int_0^{q_0+q_1} p(x) dx - p q_1 - b_0 q_0 - \frac{1}{2} a_0 q_0^2 \right] + (1 - \beta) (p q_0 - b_0 q_0 - \frac{1}{2} a_0 q_0^2), \quad (4)$$

where  $0 < \beta \leq 1$ . (Here, we follow the idea from [10] and [11].) Now we postulate that the agents (both public and private) assume that their choice of production volumes may affect the price value  $p$ . The latter assumption could be defined by a conjectured dependence of the price  $p$  upon the output values  $q_i$ . If so, the first order maximum condition to describe the equilibrium would have the form: for the public company ( $i = 0$ )

$$\frac{\partial S}{\partial q_0} = p - [(1 - \beta) q_0 + \beta q_1] \frac{\partial p}{\partial q_0} - f'_0(q_0) \begin{cases} = 0, & \text{if } q_0 > 0; \\ \leq 0, & \text{if } q_0 = 0; \end{cases} \quad (5)$$

and for the private firm ( $i = 1$ )

$$\frac{\partial \pi_1}{\partial q_1} = p + q_1 \frac{\partial p}{\partial q_1} - f'_1(q_1) \begin{cases} = 0, & \text{if } q_1 > 0; \\ \leq 0, & \text{if } q_1 = 0. \end{cases} \quad (6)$$

Thus, we see that to describe the agent's behavior, we need evaluate the behavior of the derivative  $\partial p / \partial q_i = -v_i$  rather than the dependence of  $p$  upon  $q_i$ ,  $i = 0, 1$ . Here, we introduce the minus to deal with nonnegative values of  $v_i$ . Of course, the conjectured dependence of  $p$  on  $q_i$  must provide (at least the local) concavity of the  $i$ -th agent's conjectured profit as a function of its output.

For instance, it is enough to assume the coefficient  $v_1$  (from now on referred to as the 1-st agent's *influence coefficient*) to be nonnegative and constant. Then the conjectured local dependence of the profit upon the production output's variation  $\eta_1$  has the form  $[p - v_1(\eta_1 - q_1)] \eta_1 - f_1(\eta_1)$ , while the maximum condition at  $\eta_1 = q_1$  is provided by the relationships

$$\begin{cases} p = v_1 q_1 + b_1 + a_1 q_1, & \text{if } q_1 > 0; \\ p \leq b_1, & \text{if } q_1 = 0. \end{cases} \quad (7)$$

Similarly, the public company conjectures the local dependence of domestic social surplus on its production output's variation  $\eta_0$  in the form

$$\beta \left\{ \int_0^{\eta_0+q_1} p(x) dx - [p - v_0(\eta_0 - q_0)] q_1 \right\} +$$

$$+ (1 - \beta) [p - v_0(\eta_0 - q_0)] \eta_0 - f'_0(\eta_0), \quad (8)$$

which allows one to write down the maximum condition at  $\eta_0 = q_0$  as follows

$$\begin{cases} p = -\beta v_0 q_1 + (1 - \beta) v_0 q_0 + b_0 + a_0 q_0, & \text{if } q_0 > 0; \\ p \leq -\beta v_0 q_1 + b_0, & \text{if } q_0 = 0. \end{cases} \quad (9)$$

Were the agents' conjectures about the model given exogenously like it was assumed in [23] and [24], we would allow the values  $v_i$  to be functions of  $q_i$  and  $p$ . However, here we use the approach from papers [18] and [19], where the conjecture parameters for equilibrium are determined simultaneously with the price  $p$  and the output values  $q_i$  by a special verification procedure. In the latter case, the influence coefficients are the scalar parameters determined only for the equilibrium. In what follows, such equilibrium is referred to as *interior* one and is described by the set of variables and parameters  $(p, q_0, q_1, v_0, v_1)$ .

### 3. Exterior Equilibrium in Duopoly

In order to present the verification procedure we need another notion of equilibrium called *exterior* (cf., [18] and [20]) with parameters  $v_i$  given exogenously. The set  $(p, q_0, q_1)$  is called an *exterior equilibrium state* for given influence coefficients  $(v_0, v_1)$ , if the market is balanced, i.e., condition (1) is satisfied, and the maximum conditions (7) and (9) are valid.

In what follows, we are going to consider only the case when the set of really producing participants is fixed (i.e., it does not depend upon the values  $v_i$  of the influence coefficients). To provide for this, we make the following assumption.

**A3.** For the price  $p_0 = b_1$ , the following estimate holds

$$\frac{p_0 - b_0}{a_0} < G(p_0). \quad (10)$$

The latter assumption, together with assumptions **A1** and **A2**, guarantees that for all nonnegative values of  $v_i$ ,  $i = 0, 1$ , there always exists a unique solution of the optimality conditions (7) and (9) satisfying the balance equality (1), i.e., an exterior equilibrium state. Moreover, conditions (1), (7) and (9) can hold simultaneously if, and only if  $p > p_0$ , that is, if and only if both outputs  $q_i$ ,  $i = 0, 1$ , are strictly positive.

**Lemma 3.1.** *Let assumptions **A1**–**A3** be valid. If  $(p, q_0, q_1)$  is an exterior equilibrium state, then  $p > p_0$ , which implies  $q_0 > 0$  and  $q_1 > 0$ .*

*Proof.* Assume, on the contrary, that although the vector  $(p, q_0, q_1)$  is an exterior equilibrium state, yet  $p \leq p_0 = b_1$ . The latter, together with (7), implies  $q_1 = 0$ . Then if  $q_0 = 0$ , as well, it immediately brings us to a contradiction. Indeed, assumption **A1** implies

$G(p) \geq G(p_0)$ . However, by (10),  $G(p_0) > 0$ , which makes the balance equality (1) impossible to hold, because  $q_0 + q_1 = 0 < G(p) + D$ . Otherwise, i.e., if  $q_0 > 0$ , relationships (9) get  $p = (1 - \beta)v_0q_0 + b_0 + a_0q_0 \geq b_0 + a_0q_0$ , hence  $q_0 \leq (p - b_0)/a_0$ . The latter, together with **A1** and (10), produces the same contradiction:

$$q_0 + q_1 = q_0 \leq \frac{p_0 - b_0}{a_0} < \\ < G(p_0) \leq G(p) + D,$$

which helps to conclude that the assumption  $p \leq p_0 = b_1$  was wrong. Now if, on the contrary,  $p > p_0 = b_1$ , then (7) implies  $q_1 > 0$ , and therefore (9), together with  $b_0 \leq b_1$  from **A2**, also guarantees  $q_0 > 0$ .

Now we are in a position to formulate the main result of this section. We have proven the following theorem 3.2 and the details of the very long proofs can be provided by the authors upon request.

**Theorem 3.2.** *Under assumptions **A1**, **A2** and **A3**, for any  $D \geq 0$ ,  $v_i \geq 0$ ,  $i = 0, 1$ , there exists uniquely an exterior equilibrium state  $(p, q_0, q_1)$  depending continuously upon the parameters  $(D, v_0, v_1)$ . The equilibrium price  $p = p(D, v_0, v_1)$  as a function of these parameters is differentiable with respect to both  $D$  and  $v_0, v_1$ . Moreover,  $p(D, v_0, v_1) > p_0$ , and*

$$\frac{\partial p}{\partial D} = \frac{1}{F(\beta, a, v, p)} \quad (11)$$

where

$$F(\beta, a, v, p) = \frac{1}{a_0 + (1 - \beta)v_0} + \\ + \frac{v_0 + a_0}{a_0 + (1 - \beta)v_0} \cdot \frac{1}{v_1 + a_1} - G'(p). \quad (12)$$

#### 4. Interior Equilibrium in Duopoly

Now we are ready to define an interior equilibrium state. To do that, we first describe the procedure of verification of the influence coefficients  $v_i$  as it was given in [18] and [20]. Assume that we have an exterior equilibrium state  $(p, q_0, q_1)$  that has occurred for some  $v_0, v_1$ , and  $D$ . One of the producers, say  $k$ , temporarily changes its behavior by abstaining from maximization of the conjectured profit (or domestic social surplus, as it is in case  $k = 0$ ) and making small fluctuations around its output volume  $q_k$ . In mathematical terms, it is tantamount to restricting the model agents' list to the subset  $\{i \neq k\}$  with the output  $q_k$  subtracted from the active demand.

A fluctuation of the production output by agent  $k$  is then equivalent to accepting the active demand varied as  $D_k := D - q_k$ . If we consider these variations as infinitesimal, we suppose that by observing the corresponding variations of the equilibrium price, agent

$k$  can estimate the derivative of the equilibrium price with respect to the active demand, which *coincides* with its own influence coefficient.

Applying formula (11) from Theorem 3.2 to calculate the derivatives, one has to remember that agent  $k$  is (temporarily) excluded from the equilibrium model, hence one has to eliminate the terms with number  $i = k$  from all the sums. Having that in mind, we come to the following criterion.

##### 4.1. Consistency Criterion

At an exterior equilibrium state  $(p, q_0, q_1)$ , the influence coefficients  $v_k$ ,  $k = 0, 1$ , are referred to as *consistent* if the following equalities hold:

$$v_0 = \frac{1}{\frac{1}{a_1 + v_1} - G'(p)}, \quad (13)$$

and

$$v_1 = \frac{1}{\frac{1}{a_0 + (1 - \beta)v_0} - G'(p)}. \quad (14)$$

Now we are ready to define the concept of an interior equilibrium state.

**Definition 4.1.** *The collection  $(p, q_0, q_1, v_0, v_1)$ , where  $v_k \geq 0$ ,  $k = 0, 1$ , is referred to as an interior equilibrium state if, for the considered influence coefficients, the collection  $(p, q_0, q_1)$  is an exterior equilibrium state, and the consistency criterion is satisfied for  $v_k$ ,  $k = 0, 1$ .*

**Remark 4.1.** If both agents  $i = 0$  and  $i = 1$  were net profit-maximizing companies, equations (13) and (14) would be reduced to the uniform ones obtained independently in [19] and [21]:

$$v_i = \frac{1}{\frac{1}{a_j + v_j} - G'(p)}, \quad i, j = 0, 1; i \neq j. \quad (15)$$

The following theorem is an extension of Theorem 4.2 from [20] to the case of a partially mixed duopoly.

**Theorem 4.1.** *Under assumptions **A1**, **A2** and **A3**, for any  $D \geq 0$ , there exists an interior equilibrium state.*

*Proof.* The proof is an evident extension of that of Theorem 4.2 in [20].

In our future research, we are going to extend the obtained results to the case of non-differentiable demand functions. However, some of the necessary techniques can be developed now, in the differentiable case. To do that, we denote the value of the demand function's derivative by  $\tau = G'(p)$  and rewrite the consistency equations (13)–(14) in the following form:

$$v_0 = \frac{1}{\frac{1}{a_1 + v_1} - \tau}, \quad (16)$$

and

$$v_1 = \frac{1}{\frac{1}{a_0 + (1-\beta)v_0} - \tau}, \quad (17)$$

where  $\tau = [-\infty, 0]$ . If  $\tau = -\infty$  then system (16)–(17) has the unique solution  $v_i(\tau) = 0$ ,  $i = 0, 1$ .

**Theorem 4.2.** *For any  $\tau \in (-\infty, 0]$  there exists a unique solution of equations (16)–(17), and this one-to-one correspondence is a continuous function of the variable  $\tau$ . Moreover,  $v_i(\tau) \rightarrow 0$  when  $\tau \rightarrow -\infty$ , and  $v_i(\tau)$  strictly increases up to  $v_i(0)$  as  $\tau$  grows and tends to zero,  $i = 0, 1$ .*

*Proof.* Differentiating (17) brings about the formula

$$v_1'(\tau) = \frac{(1-\beta)v_0'(\tau)}{\left[\frac{1}{a_0 + (1-\beta)v_0} - \tau\right]^2} > 0, \quad (18)$$

whenever  $v_0'(\tau) > 0$  for all  $\tau \leq 0$ . Manipulations with formulas (16)–(17) allow one to establish the latter property and thus to finish the proof of the theorem.

## 5. A Linear Demand Function

Let us consider a particular case of the linear demand function by introducing a new assumption instead of **A1**.

**A4.** The demand function is linear:  $G(p) = -Kp + T$  with  $K > 0, T > 0$ , and the ratio  $T/K > 0$  being large enough to provide that  $G(p) > 0$  for all equilibrium states that can happen in the model.

Now several interesting results concerning the behavior of the interior and exterior equilibria in dependence on the parameter  $\beta \in (0, 1]$  can be derived.

**Theorem 5.1.** *For each  $\beta \in (0, 1]$ , under assumptions **A2–A4**, there exists uniquely an interior equilibrium state  $(p^*, q_0^*, q_1^*, v_0^*, v_1^*) = (p^*(\beta), q_0^*(\beta), q_1^*(\beta), v_0^*(\beta), v_1^*(\beta))$ . Moreover, the consistent coefficients of influence  $v_i^* = v_i^*(\beta)$ ,  $i = 0, 1$ , the equilibrium production volumes  $q_i^* = q_i^*(\beta)$ ,  $i = 0, 1$ , as well as the equilibrium price  $p^* = p^*(\beta)$  treated as the (well-defined) functions of the variable  $\beta$ , are continuously differentiable over the feasible domain  $\beta \in (0, 1]$ .*

It is straightforward that the parameter  $\beta$  can be interpreted as a measure of “socialization” of company  $i = 0$  (cf., [11]). Indeed, the smaller the value of  $\beta$ , the higher the relative weight of the net profit in the company’s objective function (4). On the contrary, when  $\beta \rightarrow 1$ , the public company  $i = 0$  tends to behave more and more like the player maximizing domestic social surplus. Therefore, it is intuitively clear that when the parameter  $\beta$  grows, the output produced by firm  $i = 0$  must go up, whereas the private company

$i = 1$ , being downcast by the lowering price, should decrease its supply. Furthermore, it is also comprehensible that when  $\beta$  grows, the total (passive) demand  $G^* = G(p^*(\beta))$  must increase, thus dropping the clearing (equilibrium) price  $p^*(\beta)$ . The latter evidently leads both agents (private and public) of the market to the loss in their influence rates, i.e., the decrease in their influence coefficients  $v_i^*(\beta)$ ,  $i = 0, 1$ . For the particular case of the linear demand function, all these result can be demonstrated by mathematically rigorous arguments, as is illustrated with the following result.

**Theorem 5.2.** *Under assumptions **A2–A4**, the consistent coefficients of influence  $v_i^*(\beta)$ ,  $i = 0, 1$ , as well as the equilibrium price  $p^*(\beta)$  treated as the (well-defined) functions of the variable  $\beta \in (0, 1]$ , strictly decrease over the whole interval  $(0, 1]$ . In addition, there exists a value  $\bar{\beta} \in [0, 1)$  such that the interior equilibrium private volume function  $q_1^* = q_1^*(\beta)$  strictly decreases, while the interior equilibrium public volume function  $q_0^* = q_0^*(\beta)$  strictly increases over the (semi-closed) interval  $\beta \in (\bar{\beta}, 1]$ .*

**Remark 5.1.** The obtained results allow one to conclude that starting from a certain “degree of socialization”  $\bar{\beta} \in [0, 1)$  achieved, the private company is “crestfallen” and drops in both its production volume  $q_1$  and its self-evaluation parameter  $v_1$ . However, for the consumers, the growing of  $\beta$  is the good news, since the total production volume increases, whereas the clearing price  $p$ , vice versa, goes down.

## 6. Numerical Experiments: Duopoly

To illustrate the difference between the partially mixed, mixed, and classical duopoly cases related to the conjectural variations equilibrium with consistent conjectures (influence coefficients), we apply formulas (16)–(17) to a simple example of oligopoly in the electricity market from [20] and [21]. The only difference in our modified example from the instance of [21] is in the following: in their case, all six agents (suppliers) are private companies producing electricity and maximizing their net profits, and in our case, similar to [20], we assume that agent 0 (agent 2 in some instances) is a public enterprise seeking to maximize the convex combination of domestic social surplus and its profit described in (4), and the other generator is a private firm maximizing its net profit. On the other hand, similar numerical experiments were conducted and reported in [20] but only for  $\beta = 1$ . All the other parameters involved in the inverse demand function  $p = p(G, D)$  and the producers’ cost functions, are exactly the same as in [20].

Therefore, following the above-mentioned references, we select the IEEE 2-generator 30-bus system (cf., [21]) to illustrate our analysis. The inverse de-

mand function in the electricity market is accepted to have the form:

$$p(G, D) = 50 - 0.02(G + D) = 50 - 0.02(q_0 + q_1). \quad (19)$$

The cost functions parameters of suppliers (generators) are listed in Table 1. Here, agents 0, 1, and 2 will be combined pairwise in different examples listed below. In particular, Duopoly 1 will involve agents 0 (public) and 1 (private), whereas Duopoly 2 comprises agents 0 (public) and 2 (private).

Table 1: Cost functions' parameters

| Agent $i$ | $b_i$ | $a_i$   |
|-----------|-------|---------|
| 0         | 2.00  | 0.02000 |
| 1         | 1.75  | 0.01750 |
| 2         | 3.25  | 0.00834 |

To find the consistent influence coefficients in the classical duopoly market ( $\beta = 0$ ), [21] uses formulas (15) for both agents, while for the partially mixed duopoly models ( $\beta > 0$ ), we exploit formulas (13) for agent 0 and (14) for agent 1, with  $0 < \beta < 1$ . Of course, when  $\beta = 1$ , our model coincides with the mixed duopoly studied in [20]. With thus obtained influence coefficients, the (unique) equilibrium is found for Duopoly 1 and 2. The equilibrium results (influence coefficients, production outputs in MWh, equilibrium price, and the objective functions' optimal values in \$ per hour) are presented in Tables 2 through 9. To make our conjectures  $v_i$  comparable to those used in [20], [18], and [21], we divide them by  $[-p'(G)] = K^{-1} = 0.02$  and thus come to  $w_i := -v_i/p'(G) = Kv_i = 50v_i$ ,  $i = 0, 1$ , shown in Tables 2 and 6, where columns Cournot and Perfect comprise the influence coefficients for the Cournot and the perfect competition models, respectively.

Table 2: Coefficients of influence  $w_i$  for Duopoly 1

| Agent $i$ | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ | Cournot | Perfect |
|-----------|-------------|----------------|----------------|----------------|-------------|---------|---------|
| 0         | 0.5984      | 0.5945         | 0.5900         | 0.5849         | 0.5789      | 1.0     | 0.0     |
| 1         | 0.6152      | 0.5911         | 0.5643         | 0.5341         | 0.5000      | 1.0     | 0.0     |

Next, Tables 3 through 5 demonstrate the numerical results for Duopoly 1.

As Table 3 clearly reveals, the market clearing price (equilibrium price) in case of the classic duopoly ( $\beta = 0$ ) is quite high equaling  $p_1 = \$22.83$ , in comparison to the mixed duopoly equilibrium price  $p_2 = \$17.16$ , which is only 75% of the former. The assertions of Theorem 5.2 are also well-confirmed: the total production volume grows together with the public firm's

Table 3: Consistent equilibrium (production volumes  $q_i$ , the total volume  $G$ , price  $p$ , and the objective functions' values) for Duopoly 1

|         | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ |
|---------|-------------|----------------|----------------|----------------|-------------|
| $q_0$   | 651.47      | 746.03         | 848.83         | 960.62         | 1,082.07    |
| $q_1$   | 707.20      | 675.74         | 641.04         | 602.67         | 560.18      |
| $G$     | 1,358.67    | 1,421.77       | 1,489.87       | 1,563.29       | 1,642.25    |
| $p$     | 22.83       | 21.57          | 20.20          | 18.73          | 17.16       |
| $S$     | 9,323.80    | 14,083.76      | 19,344.35      | 25,176.38      | 31,659.88   |
| $\pi_1$ | 10,529.17   | 9,394.10       | 8,233.21       | 7,057.76       | 5,883.83    |

output and domestic social surplus, while the clearing price (as well as the private company's output and net profit) decreases when  $\beta$  increases from 0 to 1. A conclusion can be made: the higher the proportion of domestic social surplus in the public firm's objective, the greater the total production volume, hence, the lower the clearing price of electricity.

It is also interesting to compare the results in CVE with consistent conjectures against the production volumes and profits obtained for the same cases at the classic Cournot equilibrium (i.e., with both  $w_i = 1$ ,  $i = 0, 1$ ). Table 4 provides the numerical results, with  $p_3 = \$25.54$  in the classical duopoly ( $\beta = 0$ ) much higher than the market equilibrium price  $p_4 = \$17.57$  in the mixed duopoly ( $\beta = 1$ ).

Table 4: Cournot equilibrium (production volumes  $q_i$ , the total volume  $G$ , price  $p$ , and the objective functions' values) for Duopoly 1

|         | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ |
|---------|-------------|----------------|----------------|----------------|-------------|
| $q_0$   | 588.52      | 711.35         | 851.40         | 1,012.56       | 1,200.00    |
| $q_1$   | 634.43      | 591.70         | 542.99         | 486.93         | 421.74      |
| $G$     | 1,222.95    | 1,303.05       | 1,394.39       | 1,499.49       | 1,621.74    |
| $p$     | 25.54       | 23.94          | 22.11          | 20.01          | 17.57       |
| $S$     | 10,390.20   | 14,791.35      | 19,595.99      | 24,847.18      | 30,576.62   |
| $\pi_1$ | 11,571.20   | 10,066.37      | 8,475.44       | 6,816.71       | 5,115.61    |

Again, the total electricity production level is monotone growing as the parameter  $\beta$  increases starting from  $G_3 = 1358.67$  MWh when  $\beta = 0$  and ending with  $G_4 = 1642.25$  MWh for  $\beta = 1$ . Another interesting observation can be made by comparing Tables 3 and 4: when  $\beta$  is small ( $\beta \leq 0.5$ ), both companies have higher objective functions' values by making use of the Cournot conjectures  $w_i = 1$ ,  $i = 0, 1$ . However, for  $\beta$  greater than 0.5, the orderings are converse: by relying on the consistent conjectures calculated by formulas (13)–(14) instead of the Cournot conjectures, both companies improve their results significantly.

We also consider the perfect competition model (see Table 5) with  $w_i = 0$ ,  $i = 0, 1$ , which naturally gives the same results for all values of  $\beta$  and uses to be the best for consumers. However, in our example, this model is a runner-up to the mixed duopoly with consistent conjectures, both in the market price  $p_5 = \$17.18$  and in the total production volume  $G_5 = 1640.90$

MWh. Nevertheless, domestic social surplus (with  $\beta = 1$ ) is a bit higher in this case (of perfect competition), \$32,689.62 per hour, than that in the mixed duopoly with consistent conjectures (also  $\beta = 1$ ), which is \$31,689.62 per hour.

Table 5: Perfect competition equilibrium (production volumes  $q_i$ , the total volume  $G$ , price  $p$ , and the objective functions' values) for Duopoly 1

|         | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ |
|---------|-------------|----------------|----------------|----------------|-------------|
| $q_0$   | 759.10      | 759.10         | 759.10         | 759.10         | 759.10      |
| $q_1$   | 881.80      | 881.80         | 881.80         | 881.80         | 881.80      |
| $G$     | 1,640.90    | 1,640.90       | 1,640.90       | 1,640.90       | 1,640.90    |
| $p$     | 17.18       | 17.18          | 17.18          | 17.18          | 17.18       |
| $S$     | 5,760.81    | 12,493.01      | 19,225.21      | 25,957.42      | 32,689.62   |
| $\pi_1$ | 6,802.43    | 6,802.43       | 6,802.43       | 6,802.43       | 6,802.43    |

Next, we estimate numerically the other model, namely, Duopoly 2, where the same public company 0 competes with a much stronger private company 2 (*cf.*, Table 1 for its parameters). First, consistent coefficients of influence computed by (13)–(14) are shown in Table 6 below.

Table 6: Coefficients of influence  $w_i$  for Duopoly 2

| Agent $i$ | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ | Cournot | Perfect |
|-----------|-------------|----------------|----------------|----------------|-------------|---------|---------|
| 0         | 0.5044      | 0.4989         | 0.4929         | 0.4860         | 0.4783      | 1.0     | 0.0     |
| 2         | 0.6007      | 0.5778         | 0.5548         | 0.5286         | 0.5000      | 1.0     | 0.0     |

Now, Tables 7 through 9 demonstrate the numerical results for Duopoly 2.

Table 7: Consistent equilibrium (production volumes  $q_i$ , the total volume  $G$ , price  $p$ , and the objective functions' values) for Duopoly 2

|         | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ |
|---------|-------------|----------------|----------------|----------------|-------------|
| $q_0$   | 618.04      | 710.29         | 808.20         | 911.77         | 1,020.86    |
| $q_1$   | 852.19      | 815.31         | 775.57         | 732.78         | 686.82      |
| $G$     | 1,470.67    | 1,525.60       | 1,583.77       | 1,644.55       | 1,707.68    |
| $p$     | 20.60       | 19.49          | 18.33          | 17.11          | 15.85       |
| $S$     | 7,672.99    | 13,195.04      | 19,203.31      | 25,746.79      | 32,875.44   |
| $\pi_1$ | 11,753.21   | 10,467.00      | 9,183.19       | 7,916.39       | 6,684.36    |

About Duopoly 2 numerical results, similar comments may be formulated as for the previous Duopoly 1. For instance, as Table 7 evidently demonstrates, the market clearing price (equilibrium price) in case of the classic duopoly ( $\beta = 0$ ) is quite elevated reaching  $p_6 = \$20.60$ , in comparison to the mixed duopoly equilibrium price  $p_7 = \$15.85$ , which is 25% lower than the former one. The modes of behavior predicted by Theorem 5.2 and Remark 5.1, are confirmed, too: the total production volume grows together with the public firm's output and domestic social surplus, while the clearing price (as well as the private company's output and net profit) decreases when  $\beta$  grows from 0 to 1. Like in Duopoly 1, here, the same conclusion

can be made: the higher the proportion of domestic social surplus in the public firm's objective function, the greater the total production volume, hence, the lower the clearing price of electricity.

Again, it is worthy to compare the results in CVE with consistent conjectures versus the production volumes and profits obtained for the same cases at the classic Cournot equilibrium (i.e., with both  $w_i = 1$ ,  $i = 0, 1$ ). Table 8 presents the numerical results, with  $p_8 = \$24.16$  in the classical duopoly ( $\beta = 0$ ) essentially greater than the market equilibrium price  $p_9 = \$16.59$  in the mixed duopoly ( $\beta = 1$ ).

Table 8: Cournot equilibrium (production volumes  $q_i$ , the total volume  $G$ , price  $p$ , and the objective functions' values) for Duopoly 2

|         | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ |
|---------|-------------|----------------|----------------|----------------|-------------|
| $q_0$   | 554.04      | 686.42         | 835.73         | 1,005.56       | 1,200.00    |
| $q_1$   | 737.88      | 683.11         | 621.34         | 551.13         | 470.62      |
| $G$     | 1,291.92    | 1,369.53       | 1,457.07       | 1,556.56       | 1,670.62    |
| $p$     | 24.16       | 22.61          | 20.86          | 18.87          | 16.59       |
| $S$     | 9,208.79    | 14,124.02      | 19,391.53      | 25,023.08      | 31,014.87   |
| $\pi_1$ | 13,159.83   | 11,278.65      | 9,331.00       | 7,341.37       | 5,353.36    |

Similar to Duopoly 1, the total electricity production level is monotone growing as the parameter  $\beta$  increases starting from  $G_6 = 1470.23$  MWh when  $\beta = 0$  and ending with  $G_7 = 1707.68$  MWh for  $\beta = 1$ . A similar feature can found by comparing Tables 7 and 8: when  $\beta$  is small ( $\beta \leq 0.5$ ), both companies have higher objective function's values by making use of the Cournot conjectures  $w_i = 1$ ,  $i = 0, 1$ . However, for  $\beta$  greater than 0.5, the orderings are converse: by relying on the consistent conjectures calculated by formulas (13)–(14) instead of the Cournot conjectures, both companies improve their results significantly.

We also consider the perfect competition model (*see* Table 9) with  $w_i = 0$ ,  $i = 0, 1$ , which naturally gives the same results for all values of  $\beta$  and is known to be the best for consumers. Indeed, in contrast to Duopoly 1, in Duopoly 2, the perfect competition results are superior (from the consumers' point of view) to those of the mixed duopoly with consistent conjectures, both in the market clearing price  $p_{10} = \$13.60$  and in the total production volume  $G_8 = 1820.23$  MWh. In line with this, domestic social surplus (with  $\beta = 1$ ) is considerably higher in this case (of perfect competition), \$36,493.68 per hour, than that in the mixed duopoly with consistent conjectures (also  $\beta = 1$ ), which is \$32,875.44 per hour.

Finally, by comparing pairwise Tables 3 and 7, and Tables 4 and 8, we may see that the latter tables contain higher total production volumes and lower clearing prices than the former ones. These results may serve as a good example of how a strong private company may implicitly regulate the market price within a (mixed) duopoly: the stronger the private company,

Table 9: Perfect competition equilibrium (production volumes  $q_i$ , the total volume  $G$ , price  $p$ , and the objective functions' values) for Duopoly 2

|         | $\beta = 0$ | $\beta = 0.25$ | $\beta = 0.50$ | $\beta = 0.75$ | $\beta = 1$ |
|---------|-------------|----------------|----------------|----------------|-------------|
| $q_0$   | 579.77      | 579.77         | 579.77         | 579.77         | 579.77      |
| $q_1$   | 1240.46     | 1240.46        | 1240.46        | 1240.46        | 1240.46     |
| $G$     | 1,820.23    | 1,820.23       | 1,820.23       | 1,820.23       | 1,820.23    |
| $p$     | 13.60       | 13.60          | 13.60          | 13.60          | 13.60       |
| $S$     | 3,361.34    | 11,644.43      | 19,927.51      | 28,210.60      | 36,493.68   |
| $\pi_1$ | 6,416.53    | 6,416.53       | 6,416.53       | 6,416.53       | 6,416.53    |

the better results for consumers.

**Remark 6.1.** The latter interesting feature may look counter-intuitive, since in the real life, it often happens that the stronger the private company, the higher the chance for monopoly and negative results for consumers. However, it may be so in the classical duopoly but not in the mixed one. Indeed, it is easy to verify that the presence of a public company striving to maximize not its net profit but domestic social surplus completely excludes the possibility of the monopoly of the private company, no matter how strong it can be (cf., e.g., [20], where assumption **A3** is always valid if  $b_1 \leq b_0$ , i.e., if the private company is stronger than the public one; the latter implies that the public company never leaves the market.) It is quite apprehensible, because the public company strives to maximize not its net profit but domestic social surplus. On the other hand, a strong private company can produce more than a weak one, thus decreasing the market clearing price, which is beneficial for the consumer.

## 7. Concluding Remarks

In this Part 1 of the paper, we consider a model of partially mixed duopoly with Conjectural Variations Equilibrium (CVE). The agents' conjectures concern the price variations depending upon the increase or decrease of their production output. We establish the existence and uniqueness results for the conjectural variations equilibrium (called an exterior equilibrium) for any set of feasible conjectures. To introduce the notion of an interior equilibrium, we develop a consistency criterion for the conjectures (referred to as influence coefficients) and prove the existence theorem for the interior equilibrium (understood as a CVE with consistent conjectures).

To prepare the base for the extension of our results to the case of non-differentiable demand functions, we also investigate the behavior of the consistent conjectures in dependence upon a parameter representing the demand function's derivative with respect to the market price.

In our forthcoming papers, we are going to examine the qualitative behavior of prices and production outputs when the demand function is not necessarily

differentiable, and the cost functions are not necessarily quadratic.

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