Nonlinear Mathematical Physics 1997, V.4, N 1–2, 218–220.

Radiative Friction in the Lorentz-Dirac Equation and its Decomposition in the Interaction Constant

A.A. BORGHARDT and D.Ya. KARPENKO

Donetsk Physico-Technical Institute of the National Ukrainian Academy of Sciences

Abstract

The nonlinear Lorentz-Dirac equation of motion for charged particle, if one takes into account radiative friction can be written in dimensionless variables. Then, there is a possibility of introducing the constant of fine structure and following approximate solving it. This result may be used for more precise calculation of energy losses.

Existing methods of calculation of energy losses as yet are not available. We start from the equation of motion for a charged particle in the constant external field $F_{\mu\nu}$ with the radiative damping force [1]

$$\frac{dU_\mu}{ds} = \frac{e}{mc^2}F_{\mu\lambda}U_\lambda + \frac{2e^2}{3mc^2}\left[\frac{d^2U_\mu}{ds^2} - U_\mu\left(\frac{dU_\lambda}{ds}\right)^2\right]. \tag{1}$$

It is easy to find the energy which is emitted by the moving electric charge in the constant magnetic field. We take the fourth component of Eq. (1) without the Lorentz force [2]

$$\frac{dU_0}{ds} = \frac{2e^2}{3mc^2}\left[\frac{d^2U_0}{ds^2} - U_0\left(\frac{dU_\lambda}{ds}\right)^2\right], \tag{2}$$

where $U_0 = (1-\beta^2)^{-1/2} = \dfrac{E}{mc^2}$, $\beta = \dfrac{v}{c}$. In zero approximation $U_0 = \text{const}$ because there is no radiative friction and one should be suppose that for the value $\dfrac{d^2U_\mu}{ds^2}$. As follows from (2) we find

$$\frac{dE}{ds} \simeq -\frac{2e^2}{3mc^2}\varepsilon\left(\frac{dU_\lambda}{ds}\right)^2 \tag{3}$$

and one need to find the square of four-dimensional acceleration $\left(\dfrac{dU_\mu}{ds}\right)^2 = \left(\dfrac{eH}{mc^2}\right)^2 U_\perp^2$, but taking into account that $\vec{U}_\perp^2 = \dfrac{\beta^2}{(1-\beta^2)}$, the Eq.(3) has a form

$$\frac{dE}{dt} \simeq -\frac{2e^4H^2}{3m^2c^3}\frac{\beta}{1-\beta^2}. \tag{4}$$

Such a method of calculation is accepted but it does not contain knowledge about the degree of accuracy. For this reason it is desirable to develop any general method of the decomposition using the small constant $\alpha = \frac{e^2}{\hbar c}$. This is the constant of the fine structure. We transform Eq.(1) to new units introducing the dimensionless interval τ and dimensionless field $f_{\mu\nu}$ so that $s = s_0\tau$, $F_{\mu\nu} = F_0 f_{\mu\nu}$, by choosing the constants S_0 and F_0 in the form $s_0 = \frac{2\hbar}{3mc}, F_0 = \frac{3m^2c^3}{2e\hbar}$, where H_0 is a Schwinger field $H_0 = \frac{m^2c^3}{e\hbar}$. Then the Eq. (1) in the dimensionless form is

$$\frac{dU_\mu}{d\tau} = f_{\mu\nu}U_\lambda + \alpha\left[\frac{d^2U_\mu}{d\tau^2} - U_\mu\left(\frac{dU_\lambda}{d\tau}\right)^2\right]. \tag{5}$$

It should be noted that the parameter of decomposition can be made less than α. For example, it may be proportional to α^2 but it is not convenient because the value F_0 in this case is decreasing and one can not ignore more high degrees of $f_{\mu\nu}$.

So, we have the decomposition of $W_\mu^2 = \left(\frac{dU_\mu}{ds}\right)^2$ which is a nonlinear part of Eq.(1)

$$W_\lambda^2 = -Z_2 - \alpha^2(Z^4 + Z_2^2) + 0(\alpha^3), \tag{6}$$

where the notations are used

$$Z_2 = UffU, \quad Z_4 = UffffU, \dots \tag{7}$$

In Eq. (7) the sunmmation indices are omitted for briefness. From the point of view of Eq. (5) the values (7) are obeyed to the equations

$$\frac{dZ_2}{d\tau} \simeq 2\alpha(Z_4 + Z_2^2), \tag{8}$$

$$\frac{dZ_4}{d\tau} \simeq 2\alpha(Z_6 + Z_2Z_4), \tag{9}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

We restrict ourselves by the first approximation in the parameter α. We need to find the square of the acceleration in the same approximation and to solve the system of equations (8), (9) neglecting by Z_6 and Z_2Z_4. It is valid in a weak electromagnetic field. From Eq. (9) it follows that $Z_4(\tau) = Z_4(0) = A = \text{const}$ and for $Z = Z_2(\tau)$ we have

$$\frac{dZ}{d\tau} \simeq 2\alpha A + 2\alpha Z^2. \tag{10}$$

This equation may be a transformed to a linear one by the Calley transformation on the complex plane

$$V(\tau) = \frac{Z(\tau) + iA}{Z(\tau) - iA}, \qquad Z(\tau) = iA\frac{1 + V(\tau)}{1 - V(\tau)}. \tag{11}$$

We obtain the linear equation for the function $V(\tau)$

$$\frac{dV}{d\tau} - 4i\alpha AV = 0 \tag{12}$$

which has the solution $V(\tau) = V(0)\exp(4i\alpha A\tau)$ so that

$$Z(\tau) = iA\frac{Z(0) + iA + (Z(0) - iA)\exp(4iA\tau)}{Z(0) + iA - (Z(0) - iA)\exp(4iA\tau)}. \tag{13}$$

All the quadratic forms of the type (7) are real, and the real solution for the function $Z(\tau)$ obtained from (13) has the form in the limit $A \to 0$

$$Z(\tau) = \frac{U(0)ffU(0)}{f - 2\alpha U(0)ffU(0)\tau} \simeq U(0)ffU(0), \tag{14}$$

where in the general case of the constant field we have

$$U(0)ffU(0) = \frac{\beta_0^2\vec{H}^2 - (\vec{\beta}_0, \vec{H})^2 + \vec{E}^2 - (\vec{\beta}_0, \vec{E})^2 - 2\vec{\beta}_0[\vec{E}, \vec{H}]}{(1 - \beta_0^2)F_0^2}, \tag{15}$$

where $\beta = \dfrac{V(0)}{c}$. The approximate form of Eq.(5) is

$$\frac{dU_\mu}{d\tau} - \alpha\frac{d^2U_\mu}{d\tau^2} \simeq f_{\mu\lambda}U_\lambda - \alpha U(0)ffU(0)U_\mu. \tag{16}$$

Taking into account (16), one can obtain the more generally expression for energy losses

$$\frac{dE}{ds} \simeq \frac{2e^2}{3mc^2}\left(\frac{e}{mc^2}\right)^2 U(0)FFU(0) + \frac{e}{mc^2}\vec{E}, \vec{U} \tag{17}$$

The Eq. (17) can be integrated but we are restricted by the constant magnetic field (that is $\vec{E} = 0$) to compare with the known results. Then taking into account $q = \dfrac{\left(\frac{c\beta}{\Omega}\right)}{\sqrt{1-\beta^2}}$, $\Omega = \dfrac{eH}{mc}$, we obtain

$$E(s) = E(0)\exp\left\{-\frac{4\pi e^2}{3mc^2}\frac{1}{q}\left[\left(\frac{E(0)}{mc^2}\right)^2 - 1\right]\right\} \tag{18}$$

and supposing that $\Omega \ll \dfrac{3mc^3}{2e^2}$ the formula obtained in [3] by Picard method may be found as

$$\Delta E = \frac{4\pi e^2}{3q}\frac{E(0)}{mc^2}\left[\left(\frac{E(0)}{mc^2}\right)^2 - 1\right]^{\frac{3}{2}}. \tag{19}$$

The formula (19) in the ultrarelativistic limit gives the Schwinger result [4]

$$\Delta E \to \frac{4\pi e^2}{3q}\left(\frac{\varepsilon(0)}{mc^2}\right)^4. \tag{20}$$

It is not difficult to obtain the exact solution of the linearized equation (16). The characteristic equation has two roots for stable and self-accelerated solutions which is physically meaningless and must be ignored.

References

[1] Dirac P., Classical theory of radiating electrons, *Proc. Roy. Soc. of London,* 1938, V.167, 148–168.

[2] Ivanenko D., Sokolov A., Classical Theory of Field, Moscow, 1951 (in Russian).

[3] Plass G., Classical electrodynamics equations of motion with radiative reaction, *Rev. of Modern. Phys.,* 1961, V.33, N 1, 37–62.

[4] Schwinger J., On the classical theory of accelerating electrons, *Phys. Rev.,* 1949, V.75, N 12, 1912–1935.