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# On Relativistic Mass Spectra of a Two-Particle System

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**Abstract**

A relativistic two-particle system with time-asymmetric scalar and vector interactions in the two-dimensional space-time is considered within the frame of the front form of dynamics using the dynamical symmetry approach. The mass-shell equation may be represented in terms of the nonlinear canonical realization of the Lie algebra of the group $SO(2,1)$. This allows us to quantize the system and to obtain a closed form for the mass spectrum.

## Introduction

The classical relativistic direct-interactions theory (RDIT) [1–2] allows to construct a wide class of various models of interacting particle systems [3–5]. The models connected with the field theory via Fokker-type action integrals evoke a particular interest [5–7]. Usually they lead to the functional-differential equations of motion. But there exists an exception connected with choice of a time-asymmetric Green's function of the d'Alembert equation in the Fokker-type [5–7] action integrals. For a two-particle system such a choice gives an ordinary second-order differential equations of motions . In this case one particle responds only to advanced fields and the other responds only to retarded fields. On the classical level, such models have been considered in the four-dimensional space-time [8, 9] as well as in the two-dimensional one [5, 6, 10]. These models can also be considered as some approximation of the time-symmetric theories.

After quantization of the classical RDIT, the canonical generators of the Lie algebra of the Poincaré group $P(1,3)$ are replaced with the operators, determining the unitary representation of this group in a certain Hilbert space. The Poincaré-invariance conditions lead to complicate dependence of interaction potentials on coordinates and momenta. Therefore, relativistic Hamiltonians usually cannot be represented as a sum of terms depending only on commutative operators . As a result, the quantization problem does not have a unique solution. Different quantization methods may result in different expressions for observable quantities. The quantization of the classical Hamiltonian description of the N-particle system in the front form of dynamics in a two-dimensional space-time has been carried out in [11] according to the Weyl rule. Such a treatment which is based on the positivity condition of momentum variables coordinates with the quantum results [16, 7]. But this condition is violated by vector interaction on the classical level in the attraction case. According to the Weyl rule, the Heisenberg algebra is a basic algebraic structure, realized on a classical level by canonical variables in terms of the Poisson brackets $\{x_a, p_b\} = \delta_{ab}$. On a quantum level, this algebra is realized by Hermitean operators in terms of the commutator. As was demonstrated in [12], such a treatment is not suitable in the case, where some of canonical variables are determined only on the finite or half-infinite interval. The matter is that discussed variables cannot be generators of regular canonical transformations in the whole phase space. In the quantum case corresponding operators are not self-adjoint. Therefore some Lie algebra which is not isomorphic to the Heisenberg algebra can be chosen as a basic algebraic structure [12].

In this paper we consider quantization procedure of the two-particle system in the two-dimensional space-time in the front form of dynamics with interactions corresponding to scalar and vector massless fields. The classical mass-shell equation can be represented as a linear one on the Lie algebra $so(2{,}1)$ realized by the functions of canonical variables in terms of Poisson brackets. We consider this algebraic structure as a basic one. The quantization procedure consists in substitution of classical functions, forming the $so(2,1)$ algebra basis, by corresponding quantum-mechanical operators. Using only the commutation relation of the algebra $so(2,1)$, we obtain mass spectra by means of Barut's algebraic method [13,14] without specifying a realization of operators, and we will discuss construction of the state vectors in a abstract Hilbert space.

# 1 Scalar and vector interactions

The front form of dynamics in the two-dimensional space-time $\mathrm{M}_2$ with coordinates $(x^0, x)$ corresponds to the foliation $\mathrm{M}_2$ by isotropic hyperplanes

$$x^0 + x = \tau. \tag{1.1}$$

The quantity $\tau$ is the evolution parameter of the system [6,10]. The motion of particles is described by functions $x_a(\tau)$, and the parametric equations of world lines have the form $x = x_a(\tau)$, $x^0 = \tau - x_a(\tau)$. The functions $x_a(\tau)$ are defined as solutions of the Hamilton principle $\delta S = 0$ with an action integral

$$S = \int d\tau L. \tag{1.2}$$

The general structure of the Lagrange function $L$ is determined by the Poincaré-invariance conditions. This permits in the present case the solutions, which do not contain derivatives higher than first order. Thus, the Lagrangian for a N-particle system can be written in the form [10]

$$L = -\sum_{a=1}^{N} m_a k_a + \sum_{a<b} r_{ab} V_{ab}(r_{ab} k_a^{-1}, r_{ab} k_b^{-1}), \tag{1.3}$$

where $k_a = \sqrt{1 - 2v_a}, v_a = dx_a/d\tau, r_{ab} = x_a - x_b, a, b = \overline{1, N}$, and $V_{ab}$ are arbitrary functions of indicated arguments. As a consequence of general properties of the Lagrangian mechanics, invariance under the Poincaré group $P(1{,}1)$ leads to the three conservation laws: of the energy $E$, of the momentum $P$, and of the center-of-inertia integral of motion $K$. They have the form [10]

$$E = \sum_{a=1}^{N} v_a \frac{\partial L}{\partial v_a} - L, \quad P = \sum_{a=1}^{N} \frac{\partial L}{\partial v_a} + E, \quad K = -\tau P - \sum_{a=1}^{N} x_a \frac{\partial L}{\partial v_a}. \tag{1.4}$$

In the two-dimensional space-time in the front form for the system of N particles interacting through a local relativistic field of rank $\ell$, in such a way that one particle responds only to retarted field and the other particle responds to advanced field, Fokker-type action integrals [11–13] lead to Lagrangian [10]

$$L = -\sum_{a=1}^{N} m_a k_a - \sum_{a<b} \frac{g_a g_b}{|r_{ab}|} \Big(\frac{\delta_{ab}}{2}\Big)^{\ell}; \quad \delta_{ab} = \frac{k_a}{k_b} + \frac{k_b}{k_a}. \tag{1.5}$$

Let us consider two-particle systems. In this case momenta are defined by formulae

$$p_a = \frac{m_a}{k_a} + \frac{\alpha}{2|r|}\Big(1 + \ell + (1+\ell)\frac{k_{\bar a}^2}{k_a^2}\Big)\Big(\frac{\delta}{2}\Big)^{\ell-1}; \quad a = 1, 2; \quad \bar a = 3 - a \tag{1.6}$$

where $\alpha = g_1 g_2, r = r_{12}, \delta = \delta_{12}$. Solving the system (1.6) with respect to velocities $v_a$ and substituting the result in the expressions (1.4) yield canonical generators of the Poincaré group $P(1,1)$. Further we consider the cases $\ell = 0, 1$, for which such a procedure can be carried out explicitly. Using quantities $P_\pm = P_0 \pm P$ more convenient in the front form , we obtain

$$P_+ = p_1 + p_2, \quad P_- = \frac{m_1^2 p_2 + m_2^2 p_1 + A_\ell \alpha/|r|}{p_1 p_2 + (-1)^\ell \alpha^2/|r|^2 + B_\ell \alpha P_+/|r|}, \quad K = x_1 p_1 + x_2 p_2, \tag{1.7}$$

where

$$A_0 = 2m_1 m_2; \quad A_1 = -m_1^2 - m_2^2; \quad B_0 = 0; \quad B_1 = -1. \tag{1.8}$$

Quantities (1.7) satisfy the following Poisson brackets relations

$$\{P_+, P_-\} = 0, \qquad \{K, P_\pm\} = \pm P_\pm. \tag{1.9}$$

The classical expression for the total mass squared function $M^2 = P_+ P_-$ has vanishing Poisson brackets with all generators (1.7).

The separation of external and internal motion is carried out by the choice $P_+$ and $Q = K/P_+$ as new external canonical variables. As internal variables, we choose

$$\xi = \frac{m_2 p_1 - m_1 p_2}{P_+}, \quad q = r\frac{P_+}{m}; \quad \{q, \xi\} = 1, \tag{1.10}$$

where $m = m_1 + m_2$. Since $sign(r)$ is an integral of motion, one can only consider the case $r > 0$ $(q > 0)$. Then, in terms of variables (1.10) the function $M^2$ which determines the inner motion of the system has the form

$$M^2 = X/Y, \tag{1.11}$$

where

$$X = m(m m_1 m_2 q + m(m_2 - m_1) q\xi + \alpha A_\ell), \tag{1.12}$$

$$Y = m_1 m_2 q + (m_2 - m_1) q\xi - \Big(q\xi^2 + (-1)^\ell \frac{\alpha^2}{q}\Big) + \alpha m B_\ell. \tag{1.13}$$

## 2 The quantization procedure

The standard approach to quantization of the present classical problem consists in the establishing of correspondence of the $P(1.1)$ Poincaré group generators (1.2) with the Hermitean operators, determining unitary representation of this group in some Hilbert space. This determines a squared mass operator $\hat{M}^2$ of the system. The equation

$$\hat{M}^2\psi = M^2\psi \tag{2.1}$$

describes the stationary states of inner motion. This method has been used in the two-dimensional space-time in front form for a number of simple systems [11]. In these papers the Weyl quantization rule and the momentum representation in the Hilbert space [4,11] $H_N^F = \mathrm{L}^2(\mathrm{R}_+^N; d\mu_N^F), d\mu_N^F = \prod_{a=1}^{N} \Theta(p_a)dp_a/p_a$ have been used. But a number of difficulties arise when one applies this quantization rule for particle systems with the field-type interactions. The first of them is the violation of the positivity condition of momentum variables $p_a \geq 0$ for the vector interaction. The second difficulty is a too cumbersome form of the integral equations which are derived from Eq. (2.1).

We avoid this difficulties writing down the mass-shell equation (1.11) in the form

$$Y(q,\xi)M^2 - X(q,\xi) = 0. \tag{2.2}$$

Now we introduce the following functions of canonical variables

$$J_0 = \frac{1}{2}\Big(\beta q\xi^2 + \frac{q}{\beta} + (-1)^\ell\frac{\beta\alpha^2}{q}\Big), \quad J_1 = \frac{1}{2}\Big(\beta q\xi^2 - \frac{q}{\beta} + (-1)^\ell\frac{\beta\alpha^2}{q}\Big), \quad J_2 = q\xi, \tag{2.3}$$

where $\beta$ is an arbitrary constant that one needs for dimensional reasons and vanishing in the expressions for observable quantities. Using (2.3) turns (2.2) into

$$aJ_0 + bJ_1 + dJ_2 + C_\ell = 0, \tag{2.4}$$

where following notation are introduced

$$\begin{gathered} a = (M^2 + \beta^2 m_1 m_2(m^2 - M^2))/\beta, \quad b = (M^2 - \beta^2 m_1 m_2(m^2 - M^2))/\beta, \\ d = (m_2 - m_1)(m^2 - M^2), \quad C_\ell = 2\alpha(-1)^\ell m m_1 m_2\Big(\frac{m_1^2 + m_2^2 + M^2}{2m_1 m_2}\Big)^\ell. \end{gathered} \tag{2.5}$$

Functions (2.3) span, under Poisson bracketing, the Lie algebra of the group $SO(2,1)$

$$\{J_0, J_1\} = J_2, \qquad \{J_1, J_2\} = -J_0, \qquad \{J_2, J_0\} = J_1. \tag{2.6}$$

We demand the preservation of linear relation (2.4) among generators of the group $SO(2.1)$ after quantization. Hence, we replace functions (2.3) with Hermitean operators obeying the commutation relations of the $so(2,1)$ Lie algebra

$$[\hat{J}_0, \hat{J}_1] = i\hat{J}_2, \qquad [\hat{J}_1, \hat{J}_2] = -i\hat{J}_0, \qquad [\hat{J}_2, \hat{J}_0] = i\hat{J}_1, \tag{2.7}$$

and obtain the following quantum-mechanical equation:

$$(a\hat{J}_0 + b\hat{J}_1 + d\hat{J}_2 + C_\ell)\psi = 0. \tag{2.8}$$

A general structure of the mass spectrum can be found on the base of the relations (2.7) without specifying the realization of the operators $\hat{J}_0, \hat{J}_1, \hat{J}_2$. We shall use Barut's dynamical group method [13,14]. If we put

$$\psi = e^{-i\chi_1(\hat{J}_0-\hat{J}_1)}e^{-i\chi_2\hat{J}_2}\psi' \tag{2.9}$$

and choose, for the case $|m_1 - m_2| < M < m$,

$$\chi_1 = \frac{d}{a+b}, \qquad \tanh\chi_2 = R/K, \tag{2.10}$$

where

$$K = a - \frac{d^2}{2(a+b)}, \qquad R = b + \frac{d^2}{2(a+b)}, \tag{2.11}$$

then we obtain

$$(\sqrt{a^2-b^2-d^2}\hat{J}_0 + C_\ell)\psi' = 0. \tag{2.12}$$

Operator $\hat{J}_0$ as the generator of the compact subgroup $SO(2)$ has a discrete spectrum,

$$\hat{J}_0|n>= n|n> . \tag{2.13}$$

The Casimir operator of the group $SO(2,1)$

$$\hat{Q} = \hat{J}_0^2 - \hat{J}_1^2 - \hat{J}_2^2, \tag{2.14}$$

and its eigenvalues $Q = \varphi(\varphi+1)$ determine the eigenvalue $n$ of $\hat{J}_0$ [15]. In the classical case $Q = (-1)^\ell\alpha^2$. Quantity $Q$ is the only element of theory which remains undetermined in the framework of a purely algebraic approach. Thus, if $\psi'$ is an eigenstate of $\hat{J}_0$, we obtain

$$mn\sqrt{(m^2-M^2)(M^2-(m_1-m_2)^2)} = -C_\ell. \tag{2.15}$$

Solutions of Eq.(2.15) exist in the case $n > 0$ for $\ell = 0$ when $\alpha = g_1g_2 < 0$ and for $\ell = 1$ when $M^2 > m_1^2 + m_2^2, \alpha < 0$ or $M^2 < m_1^2 + m_2^2, \alpha > 0$. They have the form

$$(M_n^\pm)_\ell^2 = m_1^2 + m_2^2 \pm 2m_1m_2(1-(-1)^\ell\alpha^2/n^2)^{(-1)^\ell/2}. \tag{2.16}$$

The scattering states are also contained in Eq. (2.8). In this case we diagonalize the operator in the l.h.s. of Eq. (2.8) to $\hat{J}_1$, which has a continuous spectrum. We put $\tanh\chi_2 = K/R$, that gives

$$(M_\lambda^\pm)_l^2 = m_1^2 + m_2^2 \pm 2m_1m_2(1-(-1)^\ell\alpha^2/\lambda^2)^{(-1)^\ell/2}. \tag{2.17}$$

The vector-type spectra have been obtained in [13, 14, 16] on the base of an infinite-component wave equation with the dynamical group $O(4,2)$ in the four-dimensional space-time. It is interesting to denote that such an equation in the rest frame has the form (2.8) and can be obtained in our approach by immediate quantization of the time-asymmetric electromagnetic interaction in the frame of Hamiltonian description of a directly interacting particle system in the front form of dynamics. The correspondence condition with the one-particle problem in an external field fixes a classical value of $Q$ on the quantum level. The branch $(M^+_{n,\lambda})^2_\ell$ has a correct nonrelativistic limit contrary to the branch $(M^-_{n,\lambda})^2_\ell$ .

Using the operators $\hat{J}^{\pm} = \hat{J}_1 \pm i\hat{J}_2$ and commutation relation (2.7) only, one can construct normalized eigenvectors of $\hat{J}_0$, corresponding to the discrete spectrum states in some abstract Hilbert space [15]. This gives us the normalized solutions of Eq.(2.8) in the form

$$|\psi_s>= \frac{\exp\left(-i\chi_1(\hat{J}_0-\hat{J}_1)\right)\exp\left(-i\chi_2\hat{J}_2\right)(\hat{J}^+)^s|n_1>}{\sqrt{\prod\limits_{j=0}^{s-1}(j+1)(j+1+\sqrt{1+4Q})}}, \tag{2.18}$$

where $|n_1>$ is the normalized solution of the equation $\hat{J}^-|n_1>=0$. Using commutation relation (2.7) easy proves that vectors $|\psi_s>$ form an orthogonalized system: $<\psi_{s'}|\psi_s>=\delta_{s's}$. In such a purely algebraic way, mean values for some quantities can be calculated :

$$<\psi_s|\hat{q}|\psi_s>=-2n_s^2M_s^2/C_l, \qquad <\psi_s|\hat{q}^2|\psi_s>=<\psi_s|\hat{q}|\psi_s>^2(3-Q/n_s^2)/2. \tag{2.19}$$

The nondiagonal elements can be obtained too.

## Conclusion

The quantization problem of the relativistic Hamiltonian theory does not have a unique solution. This is related to the complicated dependence of interaction potentials on coordinates and momenta. The Hamiltonian formalism of the front form in the two-dimensional space-time is not an exception. One of possible quantization approaches in this case is the Weyl rule. But for the two-particle system with time-asymmetric scalar and vector interactions such a way leads to difficulties. They are avoided by writing down the mass-shell equation as a linear one on the Lie algebra $so(2.1)$ . We consider this algebraic structure as a basic one. The quantization procedure replaces classical $SO(2,1)$ generators (2.3) with the Hermitean operators, preserving commutation relations of the Lie algebra $so(2,1)$. Using these relations and Barut's algebraic method [13,14] without specifying realization of the operators, we obtain eigenvalue and eigenstates corresponding to a discrete spectrum. Such a way permits even to obtain manifest expressions for mean values of internal coordinates. It is necessary to point out that the linear equations on Lie algebras of dynamical groups have been postulated for description of composite electromagnetic systems by analogy with the Dirac equation. In our approach such equations arise in a natural way from classical Lagrangian or Hamiltonian description of a two-particle system with the time-asymmetric scalar and vector interactions in the two-dimensional model of the front form of dynamics.

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