

# Contact Transformations in Classical Mechanics

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## Abstract

Transformations of coordinates of points in an infinite-dimensional graded vector space, the so-called contact transformations, are examined. An infinite jet prolongation of the extended configuration space of  $N$  spinless particles is the subspace of this vector space. The dynamical equivalence among Lagrangian  $N$ -body systems connected by an invertible jet transformation is established. As an example of the invertible jet transformations, a class of gauge transformations of Lagrangian variables is investigated. The method of contact transformations is applied to the Wheeler-Feynman electrodynamics for two point charges.

## Introduction

The present paper is mainly concerned with the problem of how the change of Lagrangian variables including higher derivatives transforms the dynamics of an arbitrary  $N$ -body system. As a rule, the set of new motions is wider than that of the original. Indeed, both the order of the transformed Lagrangian and the order of the corresponding equation of motion are higher than those of the original lower-derivative theory. Nevertheless, we show that Lagrangian systems connected by an invertible contact transformation are dynamically equivalent. A constrained Hamiltonian formalism corresponds to Lagrangian theory obtained by an invertible contact transformation. We present the application of the method of contact transformations to two-body nonlocal systems admitting local perturbative expansions (e.g., Wheeler-Feynman electrodynamics [1]). In the specific case of time-asymmetric theory [2,3] we propose an invertible contact transformation which allows to construct a first-order Lagrangian function. The time-symmetric theory turns into a lower-derivative Lagrangian system with the double number of variables related to each other by the time reflection.

## 1 Contact Transformations in Lagrangian Formalism of Classical Mechanics

First of all we would like to introduce several notations. The  $x_a(t)$  denote the three-dimensional position variables of the various particles (bodies) at time  $t$  with  $a = 1, \dots, 3N \equiv \overline{1, N}$ . According to the rule  $a = 3(\alpha - 1) + i$ , the Latin index  $a$  (or  $b, c$ ) replaces the spatial index  $i=1,2,3$  as well as the label  $\alpha$  numbering the  $N$  particles ( $\alpha = \overline{1, N}$ ). The coordinates  $x_a$  span the  $N$ -dimensional configuration space, say  $E^N$ , of our dynamical system.

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Four symbols  $(Z_0, p_0, R, E^N)$  mean the fiber bundle where  $E^N$  is a typical fiber in the extended configuration space  $R \times E^N$ , which is  $Z_0$ , and the base manifold  $R$  is the time axis. Projection  $p_0 : Z_0 \rightarrow R$  relates the point  $(t, x_a) \in Z_0$  with the point  $t$  on the axis  $R$ . A section of  $Z_0$ , say  $s(t)$ , is an inverse to  $p_0$  map  $s : I \rightarrow Z_0$  given by  $t \mapsto (t, x_a(t))$ . Here  $I \subset R$  is the time interval between  $t_0$  and  $t_1$ .

Higher-derivative Lagrangian dynamics [4] is based on the Lagrangian function defined on the bundle  $J^k(Z_0)$  of  $k$ -jets  $j^k(s)(t)$  of sections  $s(t) = (t, x_a(t))$  [5]:

$$L = L(t, x_a(t), \dot{x}_a(t), \dots, \overset{k}{x}_a(t)). \quad (1.1)$$

We shall call a "motion" [6] a section  $s(t) = (t, x_a(t))$ , where the coordinates  $x_a(t)$  are solutions of the Euler-Lagrange equations

$$\frac{\delta S}{\delta x_a} \equiv \sum_{r=0}^k \left(-\frac{d}{dt}\right)^r \frac{\partial L}{\partial \overset{r}{x}_a} = 0, \quad (1.2)$$

obtained by applying the variational principle to the action integral  $S = \int_I dt L$ . A set of all motions of this particle system will be denoted as  $\Gamma(s)$ .

We define the category [7], say  $\mathbf{J}$ , which consists of bundles of infinite-order jets, such as  $J^\infty(Z_0)$ ,  $J^\infty(Z'_0)$  (objects of  $\mathbf{J}$ ) and contact transformations (morphisms of  $\mathbf{J}$ ). The contact transformation [8,6] given by the expressions

$$\begin{aligned} x_a &= f_a(t, y_b, \dot{y}_b, \dots, \overset{n}{y}_b), \\ \overset{s}{x}_a &= \left(\frac{d}{dt}\right)^s f_a, \quad s \geq 1, \end{aligned} \quad (1.3)$$

we call  $n$ -order jet transformation and denote as  $\mathcal{F} : J^\infty(Z'_0) \rightarrow J^\infty(Z_0)$ . An identity map  $id_{Z_0} : J^\infty(Z_0) \rightarrow J^\infty(Z_0)$  is given by the relations  $x_a = x_a$  and  $\overset{k}{x}_a = d^k x_a / dt^k$ .

Having carried out the  $n$ -order jet transformation in  $k$ -order Lagrangian (1.1), we construct the Lagrangian function which is defined on the bundle  $J^{k+n}(Z'_0)$  of  $(k+n)$ -jets  $j^{k+n}(s')(t)$  of sections  $s'(t) = (t, y_b(t))$ . The Lagrangian (1.1) behaves as a scalar [6] because the time variable does not change. In this paper we indicate the initial Lagrangian functions, Euler-Lagrange equations, etc., by the adjective "original" and those transformed by jet transformation by the adjective "new".

In ref. [6] the transformation law of the Euler-Lagrange equations with respect to the jet transformation (1.3) was obtained:

$$\frac{\delta \tilde{S}}{\delta y_b} = \sum_{l=0}^n \left(-\frac{d}{dt}\right)^l \frac{\partial f_a}{\partial \overset{l}{y}_b} \left\{ \frac{\delta S}{\delta x_a} \Big|_{\mathcal{F}} \right\}. \quad (1.4)$$

It suggests the formulation of the Euler-Lagrange equations  $\delta \tilde{S} / \delta y_b$  in the following form:

$$T_{ab} \chi_a \equiv \sum_{l=0}^n \left(-\frac{d}{dt}\right)^l \frac{\partial f_a}{\partial \overset{l}{y}_b} \chi_a = 0 \quad (\text{a}), \quad \frac{\delta S}{\delta x_a} \Big|_{\mathcal{F}} \equiv \sum_{r=0}^k \left(-\frac{d}{dt}\right)^r \frac{\partial L}{\partial \overset{r}{x}_a} \Big|_{\mathcal{F}} = \chi_a \quad (\text{b}). \quad (1.5)$$

Here  $\chi_a(t, y_b, \dot{y}_b, \dots, \overset{n}{y}_b)$  is a function belonging to the kernel of the matrix operator  $\hat{T}$  with components  $T_{ab}$ . The problem of correspondence between the original motions and new ones can be solved by investigation of the kernel  $ker \hat{T}$  of the operator  $\hat{T}$ .

We define the category  $\mathbf{L}$  which consists of the set of Lagrangian functions (objects of  $\mathbf{L}$ ) and jet transformations (morphisms of  $\mathbf{L}$ ). The following diagram

$$\begin{array}{ccccc}
 L(J^k(Z_0)) & \xrightarrow{\mathcal{F}} & \tilde{L}(J^{k+n}(Z'_0)) & & \\
 \downarrow E.-L. & \searrow \mathcal{H} & \downarrow E.-L. & \swarrow \mathcal{G} & \\
 & \tilde{L}(J^{k+n+m}(Z''_0)) & & & \\
 & \downarrow E.-L. & & & \\
 \frac{\delta S}{\delta x_a} & \xrightarrow{\hat{T}_h} & \frac{\delta \tilde{S}}{\delta z_c} & \xrightarrow{\hat{T}_f} & \frac{\delta \tilde{S}}{\delta y_b} \\
 & \nearrow \hat{T}_g & & \nwarrow \hat{T}_g & \\
 & & & & 
 \end{array}$$

is commutative. It shows that the category  $\mathbf{E}$  which consists of the set of expressions of Euler-Lagrange equations (objects of  $\mathbf{E}$ ) and differential matrix operators  $\hat{T}$  (morphisms of  $\mathbf{E}$ ) can be introduced. Such categories are mentioned in Vinogradov's epilogue to Russian edition of [5] and in [9]. The "Euler-Lagrange derivative"  $E.-L.$  is the functor [7] from the category  $\mathbf{L}$  to the category  $\mathbf{E}$ .

**Theorem.** *If jet transformation  $\mathcal{F} : J^\infty(Z'_0) \rightarrow J^\infty(Z_0)$  is isomorphism, then N-particle Lagrangian systems related by this transformation are dynamically equivalent:  $\Gamma(s') \cong \Gamma(s)$ .*

**Proof:** Any invertible operator  $\hat{T}$  has a trivial kernel:  $\ker \hat{T} = \{0\}$ . In this case, the transformed Euler-Lagrange equations (1.5) get simplified:

$$\sum_{r=0}^k \left( -\frac{d}{dt} \right)^r \frac{\partial L}{\partial x_a^r} \Big|_{\mathcal{F}} = 0. \quad (1.6)$$

It is evident that their solutions  $y_b(t)$  have to satisfy the system of  $n$ -order differential equations:

$$f_a(t, y_b(t), \dot{y}_b(t), \dots, y_b^{(n)}(t)) = x_a(t), \quad (1.7)$$

where functions  $x_a(t)$  are solutions of the original Euler-Lagrange equations (1.2). Having used the functions  $g_b$  inverted to the functions  $f_a$ , we obtain the motions

$$y_b(t) = g_b(t, x_a(t), \dot{x}_a(t), \dots, x_a^{(n)}(t)), \quad (1.8)$$

which are specified by the same initial data as the original motions  $x_a(t)$ . Thus, if Lagrangians are related by an invertible jet transformation, the set  $\Gamma(s')$  of new motions is isomorphic to the set  $\Gamma(s)$  of original motions. **Q.E.D.**

We face the problem of how the balance between numbers of degrees of freedom in the original and the new dynamics is achieved. Having applied the Hamiltonian formalism for higher-derivative theories [4], we can demonstrate that hamiltonization of the Lagrangian theory, obtained from non-degenerate Lagrangian by an invertible jet transformation, leads to a Hamiltonian constraint formalism. Usage of the method of mathematical induction

gives the relations between the original generalized momenta [4]  $p_{a,r}$  ( $r = \overline{0, k-1}$ ) and new generalized momenta  $\pi_{b,j}$ , ( $j = \overline{0, k+n-1}$ ):

$$\pi_{b,n+i} = \sum_{r=i}^{k-1} p_{a,r} \sum_{l=i}^{\beta} C_r^l \frac{d^{r-l}}{dt^{r-l}} \left( \frac{\partial f_a}{\partial y_b^{n+i-l}} \right), \quad \beta = \begin{cases} r, r < n+i \\ n+i, r \geq n+i \end{cases}; \quad (1.9a)$$

$$\begin{aligned} \pi_{b,j} = & \sum_{r=0}^{k-1} p_{a,r} \sum_{l=0}^{\gamma} C_r^l \frac{d^{r-l}}{dt^{r-l}} \left( \frac{\partial f_a}{\partial y_b^{j-l}} \right) + \\ & \sum_{l=j}^{n-1} \left( -\frac{d}{dt} \right)^{l-j} \frac{\partial f_a}{\partial y_b^{l+1}} \left\{ \frac{\delta S}{\delta x_a} \Big|_{\mathcal{F}} \right\}, \quad \gamma = \begin{cases} r, r < j \\ j, r \geq j \end{cases}. \end{aligned} \quad (1.9b)$$

Here index  $i$  runs from 0 to  $k-1$  and index  $j$  runs from 0 to  $n-1$ . According to eqs.(1.5) if  $\ker \widehat{T} = \{0\}$ , the transformed Euler-Lagrange expressions  $\delta S / \delta x_a |_{\mathcal{F}}$  are equal to zero. Thus on the Hamiltonian level we have the set of primary constraints obtained by excluding the original momenta  $p_{a,r}$  from the modified relations (1.9).

As an example of the invertible jet transformations, we investigate a class of gauge transformations of Lagrangian variables. Primary first-class constraints are their counterparts in the Hamiltonian formalism [4]. We examine a many-body system based on a Lagrangian function  $L$  which is defined on the bundle  $J^k(Z_0)$  (e.g.(1.1)). Let us assume that  $\mathcal{R}$  Lagrangian variables  $\mu_p$  are not contained in  $L$  explicitly. Corresponding variational derivatives  $\delta S / \delta \mu_p$  are identically equal to zero. Hence time-dependent functions  $x_a(t)$  which are solutions of eqs.(1.2) together with arbitrary time-dependent functions  $\mu_p(t)$  are coordinates of motion  $s(t)$ . Having carried out the invertible jet transformation, say  $\mathcal{A}$

$$\begin{aligned} x_a &= f_a(t, y_b; \nu_p, \dot{\nu}_p, \dots, \overset{m}{\nu}_p), \quad \mu_q = \nu_q; \\ \overset{s}{x}_a &= \frac{d^s f_a}{dt^s} \equiv \overset{s}{f}_a, \quad \overset{s}{\mu}_q = \overset{s}{\nu}_q, \end{aligned} \quad (1.10)$$

where indices  $p, q$  run from 1 to  $\mathcal{R}$ , we obtain the Lagrangian  $\tilde{L}(t, y_b, \dot{y}_b, \dots, \overset{k}{y}_b; \nu_p, \dot{\nu}_p, \dots, \overset{k+m}{\nu}_p)$  which is determined on the bundle  $J^{k+m}(Z'_0)$  of  $(k+m)$ -jets  $j^{k+m}(s')(t)$  of sections  $s' = (t, y_b(t), \nu_p(t))$ . Using eqs.(1.4), we write the relations between the original equations of motion (1.2) and the new ones in the form

$$\frac{\delta \tilde{S}}{\delta y_b} = \frac{\partial f_a}{\partial y_b} \frac{\delta S}{\delta x_a} \Big|_{\mathcal{A}} \quad (\text{a}), \quad \frac{\delta \tilde{S}}{\delta \nu_p} = \sum_{l=0}^m \left( -\frac{d}{dt} \right)^l \frac{\partial f_a}{\partial \nu_p^l} \left\{ \frac{\delta S}{\delta x_a} \Big|_{\mathcal{A}} \right\} \quad (\text{b}). \quad (1.11)$$

It is obvious that  $\Gamma(s') \cong \Gamma(s)$ .

If the new Lagrangian  $\tilde{L}$  does not depend on variables  $\nu_p$  and their time derivatives, then jet transformation (1.10) becomes the gauge transformation [4]. In this case the left-hand side in (1.11b) vanishes identically. Whence we obtain  $\mathcal{R}$  relations including the original Euler-Lagrange equations which are transformed by the substitution (1.10). Taking the  $\nu_p \rightarrow 0$  limits, we find correlations [4] among "pure" original equations of motion (1.2).

If the jet transformation is irreversible, and the kernel of the corresponding operator  $\widehat{T}$  is nontrivial, any set of functions  $(\chi_a|a = \overline{1, N}) \in \ker \widehat{T}$  correlates with the subset  $\Gamma_\chi(s')$  of the set  $\Gamma(s')$  of new motions. It consists of motions which are solutions of the differential equations (1.5b). In the particular case  $\chi_a = 0$ , the  $Nn$ -parametric family of curves  $y_b(t)$  constitutes solutions of the  $n$ -order differential eqs.(1.7) and corresponds to some concrete original motion  $x_a(t)$ .

## 2 Two-Body Problem in Three-Dimensional Formalism of the Fokker-Type Relativistic Dynamics

In this Section we deal with the Fokker action integral for two-body systems:

$$S = - \sum_{a=1}^2 m_a \int_R d\tau_a \sqrt{\dot{x}_a^2} - e_1 e_2 \iint_R d\tau_1 d\tau_2 \dot{x}_{1\mu} \dot{x}_2^\mu \delta(\sigma^2). \quad (2.1)$$

Here  $m_a$  ( $a = 1, 2$ ) are the rest masses of the particles,  $\tau_a$  is an invariant parameter of their world-lines  $x_a(\tau_a)$ ,  $\dot{x}_a^\mu = dx_a^\mu/d\tau_a$  are four-velocities of the particles, and  $\sigma^2 = (x_{2\mu} - x_{1\mu})(x_2^\mu - x_1^\mu)$  is the square of the interval between points  $x_1$  and  $x_2$  lying on the world-lines of the particles. We choose a metric tensor of the Minkowski space  $\eta_{\mu\nu} = (1, -1, -1, -1)$ ; velocity of light  $c = 1$ . The action integral (2.1) describes the interaction between two point charges  $e_1$  and  $e_2$  [1]. The nonlocal Wheeler-Feynman theory does not explain the particle creation and annihilation. Consequently, it is not valid for large particle velocities. An average velocity  $v$  will be implied as a small parameter in all the expansions of this Section.

Our consideration will be based on the three-dimensional form of the action (2.1) which is given in ref.[10]. It can be obtained from the expression (2.1) by replacement of the "instant" parameters  $t_a = x_a^0$  for own parameters  $\tau_a$  and substitution of the pair  $(t, \theta)$  for integration variables  $(t_1, t_2)$ , where

$$t_1 = t - (1 - \lambda)\theta, \quad t_2 = t + \lambda\theta. \quad (2.2)$$

Here  $\lambda$  is an arbitrary real number. The double sum (2.1) which describes interaction between particles becomes

$$S_{int} = -\frac{1}{2} \iint_R dt d\theta \frac{e_1 e_2}{r} \left(1 - (\dot{\vec{x}}_1 \dot{\vec{x}}_2)\right) [\delta(\theta - r(t, \theta)) + \delta(\theta + r(t, \theta))], \quad (2.3)$$

where coordinates  $\vec{x}_1(t_1) \equiv \vec{x}_1(t - (1 - \lambda)\theta)$ ,  $\vec{x}_2(t_2) \equiv \vec{x}_2(t + \lambda\theta)$ , velocities  $\dot{\vec{x}}_a(t_a) \equiv d\vec{x}_a(t_a)/dt_a$  and  $r(t, \theta) = |\vec{x}_2(t_2) - \vec{x}_1(t_1)|$  is the nonlocal distance between charges. Now we introduce the functions  $\vec{y}_a^{(\kappa)}(t) = \vec{x}_a(t_a)|_{C^\kappa}$ , where symbols  $|_{C^+}$  and  $|_{C^-}$  mean that the parameter  $\theta$  is a root of either algebraic equation

$$\theta - r(t, \theta) = 0 \quad (C^+) \quad \text{or} \quad \theta + r(t, \theta) = 0 \quad (C^-), \quad (2.4)$$

respectively. Let the integer  $\kappa$  be equal to +1 for the advanced cone  $C^+$  and to -1 for the retarded cone  $C^-$ . Having integrated (2.3) over the parameter  $\theta$  and having transformed

the free particle term of the action (2.1), we obtain the single-time Lagrangian of the investigated two-body system:

$$L = \frac{1}{2} \left( L^{(+)} + L^{(-)} \right), \quad (2.5)$$

where

$$L^{(\kappa)} = -m_1 k_1^{(\kappa)} - m_2 k_2^{(\kappa)} - \frac{e_1 e_2 (1 - \kappa(1 - \lambda)\dot{y}^{(\kappa)})(1 + \kappa\lambda\dot{y}^{(\kappa)}) - (\dot{\vec{y}}_1^{(\kappa)} \dot{\vec{y}}_2^{(\kappa)})}{y^{(\kappa)} J^{(\kappa)}}. \quad (2.6)$$

Here

$$k_1^{(\kappa)} = \sqrt{\left(1 - \kappa(1 - \lambda)\dot{y}^{(\kappa)}\right)^2 - \dot{\vec{y}}_1^{(\kappa)2}}, \quad k_2^{(\kappa)} = \sqrt{\left(1 + \kappa\lambda\dot{y}^{(\kappa)}\right)^2 - \dot{\vec{y}}_2^{(\kappa)2}}, \quad (2.7)$$

$$J^{(\kappa)} = 1 - \kappa\lambda \frac{(\vec{y}^{(\kappa)} \dot{\vec{y}}_1^{(\kappa)})}{y^{(\kappa)}} - \kappa(1 - \lambda) \frac{(\vec{y}^{(\kappa)} \dot{\vec{y}}_2^{(\kappa)})}{y^{(\kappa)}}. \quad (2.8)$$

The functions  $y_a^{i(\kappa)}(t)$  can be written as the power series defined on the bundle  $J^\infty(R \times E^6)$  of the infinite-order jets  $j^\infty(s)(t)$  of the sections  $s = (t, x_1^i(t), x_2^i(t))$ :

$$y_a^{i(\kappa)}(t) = \sum_{s=0}^{\infty} \frac{(\lambda D - D_1)^s}{s!} (\kappa r)^s \left( 1 - \kappa(1 - \lambda) \frac{(\vec{r}\vec{x}_1)}{r} - \kappa\lambda \frac{(\vec{r}\vec{x}_2)}{r} \right) x_a^i(t). \quad (2.9)$$

Here  $D = \partial/\partial t + D_1 + D_2$  is the total time derivative, the symbol  $D_a$  signifies differentiation with respect to  $t$  of the variable  $\vec{x}_a$  only, and  $r = |\vec{x}_2(t) - \vec{x}_1(t)|$ . Thus, the Lagrangian function (2.5) and corresponding equations of motion are defined on the  $J^\infty(R \times E^6)$ . This Lagrangian can be expanded into a Taylor series. It differs from Kerner's Lagrangian [11] in terms which are the total time derivatives.

Having used the results of Section 1 (see (1.4)), infinite-order Euler-Lagrange equations

$$\sum_{s=0}^{\infty} \left( -\frac{d}{dt} \right)^s \frac{\partial L}{\partial \dot{x}_a^i} = 0 \quad (2.10)$$

can be written in the following form

$$T_{abi}^{(+j)} \left( \frac{\partial L^{(+)}}{\partial y_b^{j(+)}} - \frac{d}{dt} \frac{\partial L^{(+)}}{\partial \dot{y}_b^{j(+)}} \right) + T_{abi}^{(-j)} \left( \frac{\partial L^{(-)}}{\partial y_b^{j(-)}} - \frac{d}{dt} \frac{\partial L^{(-)}}{\partial \dot{y}_b^{j(-)}} \right) = 0. \quad (2.11)$$

Here

$$T_{abi}^{(\kappa)j} = \sum_{s=0}^{\infty} \left( -\frac{d}{dt} \right)^s \frac{\partial f_{bj}^{(\kappa)}}{\partial \dot{x}_a^i} \quad (2.12)$$

are the components of the differential operators  $\widehat{T}^{(+)}$  and  $\widehat{T}^{(-)}$ .

We can construct the expansions which are inverted to the expressions (2.9). Therefore, both operator  $\widehat{T}^{(+)}$  and operator  $\widehat{T}^{(-)}$  have a trivial kernel.

In ref.[2] the time-asymmetric Wheeler-Feynman electrodynamics for two point charges was investigated. In this model the first particle moves in the retarded (advanced) Lienard-Wiechert potential of the second particle, while the second particle moves in the advanced (retarded) Lienard-Wiechert potential of the first particle. Four-dimensional "coordinates

on the cone” were used as the Lagrangian variables. In order to get three-dimensional formalism of the instant form of dynamics [10], it is necessary to restrict the gauge reparametrization group by imposing the constraint  $x^0 = t$ . As a result, a single-time Lagrangian function coincides with the expression (2.6), where the value  $\kappa = +1$  corresponds to Fokker-type action with advanced Green’s function of the d’Alembert equation in [2], and  $\kappa = -1$  — for retarded Green’s function in [2]. Hence, in case of the time-asymmetric model, the set  $\Gamma(s)$  of motions  $s(t) = (t, x_a^i(t))$  which are solutions of the infinite-order Euler-Lagrange equations, is isomorphic to the set  $\Gamma(s')$  which consists of the motions  $s'(t) = (t, y_a^{i(+)}(t))$  or  $s'(t) = (t, y_a^{i(-)}(t))$  satisfying the equations of motion corresponding to the first-order Lagrangian  $L^{(+)}$  or  $L^{(-)}$ , respectively. If we change the time direction, the trajectory  $y_a^{i(+)}(t)$  becomes  $y_a^{i(-)}(t)$  and vice versa. An analogous result was obtained in ref.[3] where Wheeler-Feynman N-body electrodynamics in two-dimensional Minkowski space was examined. On the Hamiltonian level, the constrained Hamiltonian formalism is expected for the time-asymmetric Lagrangian function defined on the bundle  $J^\infty(R \times E^6)$ . This Hamiltonian theory has to contain an infinite set of constraints (see (1.9) where  $\delta S/\delta x_a = 0, k = 1, n \rightarrow \infty$ ).

As far as time-symmetric Lagrangian (2.5) is concerned, it would be interesting to consider the functions  $y_a^{i(+)}(t)$  and  $y_a^{i(-)}(t)$  as Lagrangian variables. They are not independent quantities because they have to satisfy the conditions  $y_a^{i(+)}(-t) = y_a^{i(-)}(t)$ ,  $y_a^{i(-)}(-t) = y_a^{i(+)}(t)$ . Admissible relativistic motions are considered to be specified by initial coordinates and velocities only. Therefore it is natural to find the (local) constraints which allow to eliminate six redundant degrees of freedom. This constrained first-order Lagrangian theory must yield the time-symmetric solutions which are analytical with respect to the expansion parameter.

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