Nonlinear Mathematical Physics 1997, V.4, N 1–2, 141–145.

The Algebra $A\tilde{P}(1,3)$ Invariants and Their Application to the Theory of Born-Infeld Field

Iryna A. MISHCHENKO

The Technical University, Pershotravnevyj prospect, 24, Poltava, Ukraïna

Abstract

The algebra $A\tilde{P}(1,3)$ invariants were found. These invariants allowed to reduce the Born-Infeld equation. After the reduction some solutions of the equation were found.

Let us consider the Born-Infeld equation

$$(1 - u_\nu u^\nu)\,\Box u + u^\mu u^\nu u_{\mu\nu} = 0; \tag{1}$$

where $u = u(x)$; $x = (x_0, x_1, x_2)$; $u_\mu = \dfrac{\partial u}{\partial x_\mu}$; $u_{\mu\nu} = \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$; $u^\mu = g^{\mu\nu} u_\nu$; $g^{00} = -g^{11} = -g^{22} = 1$, $g^{\mu\nu} = 0$ for $\mu \neq \nu$, $\mu, \nu = \overline{0,2}$. We suppose a summation over repeated indices in formula (1) and further.

The extended Poincaré algebra $A\tilde{P}(1,3)$ is a maximal invariance Lie algebra for equation (1) [1]. The basis operators of the algebra are following:

$$\partial_A = \frac{\partial}{\partial x_A},\ J_{AB} = x^A \partial_B - x^B \partial_A,\ D = x_A \partial_A, \tag{2}$$

where $x^A = g^{AB} x_B$; $A, B = \overline{0,3}$; $x_3 \equiv u$; g^{AB} is the metrical tensor of space R_{1+3} with the signature $(+,-,-,-)$.

The symmetry of one- and many-dimensional equation (1) was researched in the articles [1–4]. The symmetry properties of the current equation were particularly used for determination for its precise solution.

Here the full set of algebra (2) invariants in two-dimensional case is used for reduction of equation (1) to a partial differential equation with two variables and is given further as Table. We used the following notation in Table:

$$ax = a_A x^A = g^{AB} a_A x_B = a_0 x_0 - a_1 x_1 - a_2 x_2 - a_3 x_3;$$

$$x^2 = x_A x^A = g^{AB} x_A x_B = x_0^2 - x_1^2 - x_2^2 - x_3^2;$$

$$a^2 = -b^2 = -c^2 = -d^2 = 1, \quad \alpha = a - d, \quad \beta = a + d;$$

$$ab = ac = ad = bc = bd = cd = 0; \quad k, l, m, n \text{ are constants; } A, B = \overline{0,3}.$$

The algebra representation (2) gives us that the invariant solutions of equation (1) should have the following form

$$z = \varphi(\omega, w), \tag{3}$$

where $\omega = \omega(x_0, x_1, x_2, u)$, $w = w(x_0, x_1, x_2, u)$, $z = z(x_0, x_1, x_2, u)$ are current algebra invariants, φ is some new unknown function.

Table. **The Algebra $A\tilde{P}(1,3)$ Invariants**

N	ω	w	z
1	$m(bx)-k(ax)$	$m(cx)-l(ax)$	$m(dx)-n(ax)$
2	βx	$m(bx)+(\beta x)(cx)$	$\frac{1}{2}(bx)^2-(ax)(\beta x)$
3	$m(cx)+k(bx)$	$(\alpha x)(\beta x)$	$\frac{m}{2}\ln\frac{\alpha x}{\beta x}+bx$
4	$k(ax)-m(bx)$	$(cx)^2+(dx)^2$	$m\arctan\frac{cx}{dx}-ax$
5	$k(cx)-m(\beta x)$	$(\beta x)^2+2k(bx)$	$\frac{(\beta x)^3}{3}+k(bx)\times$ $\times(\beta x)+k^2(ax)$
6	$\frac{bx}{cx}$	$\frac{(\alpha x)(\beta x)}{(cx)^2}$	$(\alpha x)^{1-\kappa}(\beta x)^{1+\kappa}$
7	$\frac{cx}{\beta x}$	$\frac{2(ax)}{\beta x}-\frac{(bx)^2}{(\beta x)^2}$	$\frac{1}{\kappa}\ln(\beta x)+\frac{bx}{\beta x}$
8	$\frac{bx}{ax}$	$\frac{x^2}{(ax)^2}$	$\arctan\frac{cx}{dx}+\frac{1}{\kappa}\ln(ax)$
9	$\frac{cx}{bx}$	$\frac{\alpha x}{(bx)^2}$	$\beta x-m\ln(bx)$
10	$\frac{bx}{ax}$	$\frac{cx}{ax}$	$\frac{bx}{ax}$
11	$(\alpha x)\mathrm{e}^{-\beta x}$	$\left[(bx)^2+(cx)^2\right]\mathrm{e}^{-\beta x}$	$m\arctan\frac{bx}{cx}-\beta x$
12	$(\alpha x)^{1-\kappa}(\beta x)^{1+\kappa}$	$\frac{x^2}{(\alpha x)(\beta x)}$	$\arctan\frac{cx}{bx}+\frac{1}{2}\ln\frac{\alpha x}{\beta x}$

If we substitute the formula (3) in equation (1), we get the new equation:

$$\begin{aligned}&\left[(M_A\omega^A)^2-M_AM^A\omega_B\omega^B\right]\varphi_{\omega\omega}+2\left[M_A\omega^AM_Bw^B-M_AM^A\omega_Bw^B\right]\varphi_{\omega w}+\\&\left[(M_Aw^A)^2-M_AM^Aw_Bw^B\right]\varphi_{ww}+\left[M^AM^B\omega_{AB}-M_AM^A\Box\omega\right]\varphi_\omega+\\&\left[M^AM^Bw_{AB}-M_AM^A\Box w\right]\varphi_w+M_AM^A\Box z-M^AM^Bz_{AB}=0;\end{aligned}\tag{4}$$

where $M_A=\omega_A\varphi_\omega+w_A\varphi_w-z_A$; $\omega_A=\dfrac{\partial\omega}{\partial x_A}$; $w_A=\dfrac{\partial w}{\partial x_A}$; $z_A=\dfrac{\partial z}{\partial x_A}$; $\omega_{AB}=\dfrac{\partial^2\omega}{\partial x_A\partial x_B}$; $w_{AB}=\dfrac{\partial^2 w}{\partial x_A\partial x_B}$; $z_{AB}=\dfrac{\partial^2 z}{\partial x_A\partial x_B}$; $\Box\omega=g^{AB}\omega_{AB}$; $\Box w=g^{AB}w_{AB}$; $\Box z=g^{AB}z_{AB}$; $A,B=\overline{0,3}$.

If ω, w, and z have values from Table then the function φ is a solution of the partial differential equation in one of the following cases:

1) $$\begin{aligned}&\left[c_1\varphi_w^2-2c_4\varphi_w+c_2\right]\varphi_{\omega\omega}+2\left[-c_1\varphi_\omega\varphi_w+c_4\varphi_\omega+c_5\varphi_w+c_6\right]\varphi_{\omega w}+\\&\left[c_1\varphi_\omega^2-2c_5\varphi_\omega+c_3\right]\varphi_{ww}=0;\end{aligned}$$

where $c_1=k^2+l^2-m^2$, $c_2=k^2+n^2-m^2$, $c_3=n^2+l^2-m^2$, $c_4=nl$, $c_5=nk$, $c_6=kl$.

2) $$\begin{aligned}&\omega^2\varphi_{\omega\omega}+2\omega\left[w-(m^2+\omega^2)\varphi_w\right]\varphi_{\omega w}+\left[w^2+(m^2+\omega^2)\left(\omega^2+2(\omega\varphi_\omega-\varphi)\right)\right]\varphi_{ww}+\\&(2m^2+\omega^2)\varphi_w^2-4\omega\varphi_\omega-4w\varphi_w-\omega^2+4\varphi=0;\end{aligned}$$

3) $$\left[4w(m^2+k^2)\varphi_w^2-\frac{m^2}{w}(m^2+k^2)-m^2\right]\varphi_{\omega\omega}+8w\left[k-(m^2+k^2)\varphi_\omega\right]\varphi_w\varphi_{\omega w}+$$
$$\left[4w(m^2+k^2)\varphi_\omega^2-8kw\varphi_\omega+4(w+m^2)\right]\varphi_{ww}+$$
$$\left[4(m^2+k^2)\varphi_\omega^2-8k\varphi_\omega-\frac{2m^2}{w}+4\right]\varphi_w=0;$$

4) $$\left[4w(k^2-m^2)\varphi_w^2+(k^2-m^2)\left(\frac{m^2}{w}+1\right)+k^2\right]\varphi_{\omega\omega}+8w\left[(m^2-k^2)\varphi_\omega-k\right]\varphi_w\varphi_{\omega w}+$$
$$\left[4w(k^2-m^2)\varphi_\omega^2+8wk\varphi_\omega-4(m^2+w)\right]\varphi_{ww}+$$
$$4(k^2-m^2)\varphi_\omega^2\varphi_w-8w\varphi_w^3+8k\varphi_\omega\varphi_w-\left(\frac{6m^2}{w}+4\right)\varphi_w=0;$$

5) $$\left[m^2+w-4\varphi_w^2\right]\varphi_{\omega\omega}+8\varphi_w(\varphi_\omega-m)\varphi_{\omega w}++4\left[w+2m\varphi_\omega-\varphi_\omega^2\right]\varphi_{ww}+6\varphi_w=0;$$

6) $$\left[w(\omega^2-w+1)\varphi_w^2-2\varphi(1+\omega^2)\varphi_w+(1-\kappa^2)(1+\omega^2)\frac{\varphi^2}{w}\right]\varphi_{\omega\omega}+$$
$$2\left[w(w-\omega^2-1)\varphi_\omega\varphi_w+\varphi(1+\omega^2)\varphi_\omega-2\varphi\omega w\varphi_w+\frac{1}{2}\omega\varphi^2(1-\kappa^2)\right]\varphi_{\omega w}+$$
$$\left[w(\omega^2-w+1)\varphi_\omega^2+4\varphi\omega w\varphi_\omega+4\varphi^2\left(w(1-\kappa^2)+\kappa^2\right)\right]\varphi_{ww}+$$
$$(\omega^2+\frac{5}{2}\omega+1)\varphi_\omega^2\varphi_w+2\omega w\varphi_\omega\varphi_w^2-4w(1+5w)\varphi_w^3+(\kappa^2-1)(1+\omega^2)\frac{\varphi}{w}\varphi_\omega^2+$$
$$4\omega\varphi(\kappa^2-1)\varphi_\omega\varphi_w+2\varphi(2w\kappa^2+8w-2\kappa^2+3)\varphi_w^2+$$
$$2(1-\kappa^2)\frac{\varphi^2\omega}{w}\varphi_\omega+2(1-\kappa^2)(1-3w\varphi)\frac{\varphi}{w}\varphi_w+2(1-\kappa^2)\frac{\varphi^3}{w^3}=0;$$

7) $$\left[(\omega^2-w+1)\varphi_w^2-\frac{1}{\kappa}\varphi_w-\frac{1}{4}\right]\varphi_{\omega\omega}+\left[2(w-\omega^2-1)\varphi_\omega\varphi_w+\frac{1}{\kappa}\varphi_\omega-\frac{\omega}{\kappa}\varphi_w\right]\varphi_{\omega w}+$$
$$\left[(\omega^2-w+1)\varphi_\omega^2+\frac{2\omega}{\kappa}\varphi_\omega+\frac{1}{\kappa^2}+1-w\right]\varphi_{ww}+\frac{1}{2}\varphi_\omega^2\varphi_w+\omega\varphi_\omega\varphi_w^2+$$
$$2(w-1)\varphi_w^3+\frac{1}{\kappa}\varphi_w^2-2\varphi_w=0;$$

8) $$\left[4w(\omega^2+w-1)\varphi_w^2+\frac{4}{\kappa}(\omega^2+w-1)\varphi_w+\frac{1}{\kappa^2}+\frac{1-\omega^2}{\omega^2+w-1}\right]\varphi_{\omega\omega}+$$
$$4\left[2w(1-\omega^2-w)\varphi_\omega\varphi_w+\frac{1}{\kappa}(1-\omega^2-w)\varphi_\omega-\omega\left(\frac{1}{\kappa^2}+\frac{w}{\omega^2+w-1}\right)\right]\varphi_{\omega w}+$$
$$4\left[w(\omega^2+w-1)\varphi_\omega^2+(1-w)\left(\frac{1}{\kappa^2}+\frac{w}{\omega^2+w-1}\right)\right]\varphi_{ww}+2(1-\omega^2)\varphi_\omega^2\varphi_w-$$
$$8\omega w\varphi_\omega\varphi_w^2+8(1-w)(1-2w)\varphi_w^3-\frac{1+2\omega^2}{\kappa}\varphi_\omega^2+\frac{4\omega}{\kappa}(1-2w)\varphi_\omega\varphi_w+$$
$$\frac{12}{\kappa}(1-w)\varphi_w^2-\frac{2\omega}{\omega^2+w-1}\varphi_\omega-\left(\frac{6}{\kappa^2}+\frac{1+w}{\omega^2+w-1}\right)\varphi_w-\frac{1}{\kappa(\omega^2+w-1)}=0;$$

9) $\left[-w^2\varphi_w^2+(mw-\omega^2-1)\varphi_w-\dfrac{m^2}{4}\right]\varphi_{\omega\omega}+\left[2w^2\varphi_\omega\varphi_w+(\omega^2+1-mw)\varphi_\omega-\right.$

$\left.2\omega w\right]\varphi_{\omega w}+\left[-w^2\varphi_\omega^2+2\omega w\varphi_\omega+1-2mw\right]\varphi_{ww}+$

$\dfrac{w}{2}\varphi_\omega^2\varphi_w-\dfrac{m}{4}\varphi_\omega^2-2w\varphi_w^2-m\varphi_w=0;$

10) $\left[(\omega^2+w^2-1)\varphi_w^2-2w\varphi\varphi_w+\omega^2+\varphi^2-1\right]\varphi_{\omega\omega}+2\left[(1-\omega^2-w^2)\varphi_\omega\varphi_w+w\varphi\varphi_\omega+\right.$

$\left.\omega\varphi\varphi_w+\omega w\right]\varphi_{\omega w}+\left[(\omega^2+w^2-1)\varphi_\omega^2-2\omega\varphi\varphi_\omega+w^2+\varphi^2-1\right]\varphi_{ww}=0;$

11) $\left[4\omega w\varphi_w^2+\dfrac{\omega m^2}{w}-1\right]\varphi_{\omega\omega}+4w(1-2\omega\varphi_\omega)\varphi_w\varphi_{\omega w}+\left[4\omega w\varphi_\omega^2-4w\varphi_\omega+m^2\right]\varphi_{ww}+$

$3\omega\varphi_\omega^3+\omega\varphi_\omega^2\varphi_w+2w\varphi_w^3-3\varphi_\omega^2-4\varphi_\omega\varphi_w+\dfrac{m^2}{w}\varphi_\omega+\dfrac{m(m+2)}{2w}\varphi_w=0;$

12) $\left[2\omega^2(w-1)(\kappa^2w-1)\varphi_w^2+2\kappa\omega^2(w-1)\varphi_w+\dfrac{\omega^2(w+\kappa^2-2)}{2(w-1)}\right]\varphi_{\omega\omega}+$

$2\left[2\omega^2(1-w)(\kappa^2w-1)\varphi_\omega\varphi_w+\kappa\omega^2(1-w)\varphi_\omega+\kappa\omega w(1-w)\varphi_w+\dfrac{2-w}{8(w-1)}\right]\varphi_{\omega w}+$

$\left[2\omega^2(w-1)(\kappa^2\omega-1)\varphi_\omega^2+2\kappa\omega w(w-1)\varphi_\omega+\dfrac{w}{2}(w-2)\right]\varphi_{ww}+(\kappa^2-1)\omega^3\varphi_\omega^3+$

$\omega^2(w\kappa^2+3w+\kappa^2-5)\varphi_\omega^2\varphi_w+\omega(w-1)(2\kappa^2-3w+1)\varphi_\omega\varphi_w^2+$

$w(w-1)(w-2)\varphi_w^2+\kappa\omega^2\varphi_\omega^2-\kappa\omega(3w-1)\varphi_\omega\varphi_w+$

$\dfrac{\omega(2\kappa^2+3w-5)}{4(w-1)}\varphi_\omega-\dfrac{w}{4(w-1)}\varphi_w=0.$

Let us determine some solution of the new reduced partial differential equations.

The equation (2) has a following solution

$$\varphi=\frac{w^2-m^2\omega^2}{2m^2}, \tag{5}$$

where m is constant. We will find a solution of equation (5) in the following form

$$\varphi=B(w)\omega+C(w) \tag{6}$$

If we substitute (6) in to equation (5), we get a system of equations.

$$8B\dot{B}^2-8m\dot{B}^2+\ddot{B}(-4B^2+8mB+4w)+6\dot{B}=0, \tag{7}$$

$$8B\dot{C}\dot{B}-8m\dot{C}\dot{B}+\ddot{C}(-4B^2+8mB+4w)+6C=0. \tag{8}$$

It is obviously that any constant function is a solution of equation (7), i.e., $B=C_1$. So we have got from (8) a following equation for determining the function $C(w)$:

$$\ddot{C}(w+C_1)+C=0,$$

then

$$C(w)=C_2\exp\left(\int f(w)\mathrm{d}w\right),$$

where function $f(w)$ is a solution of the equation $f' + f^2 + \dfrac{1}{w + C_1} = 0$; C_2 is an arbitrary constant. In this way the solution of equation (5) is the following function:

$$\varphi = C_1\omega + C_2\mathrm{exp}\left(\int f(w)\mathrm{d}w\right). \tag{9}$$

The equation (9) has the following solution

$$\varphi = 2C_1\omega - C_1^2 w + C_2, \tag{10}$$

where C_1, C_2 are constants.

At last we are able to determine the solution of equation (1) by using the algebra invariants and the solutions of equations (5), (9) and (10):

1. $(\beta x)(cx)^2 + 2m(bx)(cx) + 2m^2(bx) - m^2(\beta x) = 0;$

2. $\dfrac{(\beta x)^3}{3} + k(bx)(\beta x) + k^2(ax) - C_1\left(k(cx) - m(\beta x)\right) - C_2\mathrm{exp}\left(\int f(w)\mathrm{d}w\right) = 0;$

where f is a solution of the equation $f' + f^2 + \frac{1}{w+C_1} = 0$;

3. $m(bx)^2 \ln(bx) - (\beta x)(bx)^2 + C_2(bx)^2 + 2C_1(cx)(bx) - C_1^2(\alpha x) = 0.$

Let us remind that the desired function $u = u(x_0, x_1, x_2)$ is implicit as the last coordinate of the vector x, $x \in R_{1+3}$.

References

[1] Fushchych W.I., Serov N.I., About some exact solutions of many-dimensional nonlinear Euler-Lagrange equation, *Doklady AN SSSR,* 1984, V.278, N 4, 847–851.

[2] Fushchych W.I., Serov N.I., About exact solutions of Born-Infeld equation, *Doklady AN SSSR,* 1981, V.263, N 3, 582–586.

[3] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993.

[4] Serov N.I., Reduction and exact solutions of one-dimensional Born-Infeld equation, *Doklady AN USSR,* 1991, N 10, 26–28.