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On Classes of Lie Solutions of MHD Equations, Expressed via the General Solution of the Heat Equation

Victor POPOVYCH

Institute of Mathematics of the National Ukrainian Academy of Sciences,
3 Tereshchenkivs'ka Street, Kyïv 4, Ukraïna

Abstract

Large classes of Lie solutions of the MHD equations describing the flows of a viscous homogeneous incompressible fluid of finite electrical conductivity are constructed. These classes contain a number of arbitrary functions of time and the general solutions of the heat equation.

The MHD equations describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity (the MHDEs) have the following form:

$$\begin{aligned}
&\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p + \tfrac{1}{4\pi}\vec{H}\times\operatorname{rot}\vec{H} = \vec{0},\\
&\vec{H}_t - \operatorname{rot}(\vec{u}\times\vec{H}) - \nu_m\triangle\vec{H} = \vec{0}, \qquad \operatorname{div}\vec{u} = 0, \qquad \operatorname{div}\vec{H} = 0.
\end{aligned} \tag{1}$$

In (1) and below $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{H} = \{H^a(t,\vec{x})\}$ denotes the magnetic intensity, ν_m is the coefficient of magnetic viscosity, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian. The kinematic coefficient of viscosity and fluid density are set equal to one.

The maximal Lie invariance algebra of the MHDEs (1) is an infinite-dimensional algebra $A(\mathrm{MHD})$ with the basis elements (see [1])

$$\begin{aligned}
&\partial_t, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - H^a\partial_{H^a} - 2p\partial_p,\\
&J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} + H^a\partial_{H^b} - H^b\partial_{H^a},\ a<b,\\
&R(\vec{m}) = R(\vec{m}(t)) = m^a\partial_a + m^a_t\partial_{u^a} - m^a_{tt}x_a\partial_p, \quad Z(\eta) = Z(\eta(t)) = \eta\partial_p,
\end{aligned} \tag{2}$$

where $m^a = m^a(t)$ and $\eta = \eta(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbf{R})$). Hereafter repeated indices denote summation; we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

Following [2, 3], in this paper we reduce the MHDEs by means of the algebra

$$< R(\vec{m}^1(t)),\ R(\vec{m}^2(t)) >, \quad \text{where} \quad \vec{m}^1_{tt}\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_{tt} = 0 \tag{3}$$

(that is, $\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t = C = \mathrm{const}$ and we may assume that $C\in\{0;1\}$) and $\nexists C_i\in\mathbf{R}:\ C_i\vec{m}^i\equiv\vec{0}$. An ansatz corresponding to this algebra can be obtained only for such t that $\operatorname{rank}(\vec{m}^1(t),\vec{m}^2(t)) = 2$, and has the form

$$\vec{u} = \vec{w} + \lambda^{-1}(\vec{n}^i \cdot \vec{x})\vec{m}^i_t - \lambda^{-1}(\vec{k}\cdot\vec{x})\vec{k}_t, \qquad \vec{H} = (4\pi)^{1/2}\vec{\xi},$$
$$p = s - \tfrac{1}{2}\lambda^{-1}(\vec{m}^i_{tt}\cdot\vec{x})(\vec{n}^i\cdot\vec{x}) - \tfrac{1}{2}\lambda^{-2}(m^i_{tt}\cdot\vec{k})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}), \tag{4}$$

where $\vec{w} = (w^1, w^2, w^3)$, $\vec{\xi} = (\xi^1, \xi^2, \xi^3)$; $w^a = w^a(z_1, z_2)$, $\xi^a = \xi^a(z_1, z_2)$, and $q = q(z_1, z_2)$ are new unknown functions; $z_1 = t$, $z_2 = \vec{k}\cdot\vec{x}$, $\vec{k} = \vec{m}^1 \times \vec{m}^2$, $\vec{n}^1 = \vec{m}^2 \times \vec{k}$, $\vec{n}^2 = \vec{k} \times \vec{m}^1$, and $\lambda = \lambda(t) = \vec{k}\cdot\vec{k} \neq 0$.

Substituting ansatz (4) into the MHDEs, we obtain the system of differential equations for the functions w^a, ξ^a, and q:

$$\vec{w}_1 + (\vec{k}\cdot\vec{w})(\vec{w}_2 - \lambda^{-1}\vec{k}_t) + \lambda^{-1}(\vec{n}^j\cdot\vec{w})\vec{m}^j_t - \lambda\vec{w}_{22} + s_2\vec{k} +$$
$$(\vec{\xi}\cdot\vec{\xi}_2)\vec{k} - (\vec{k}\cdot\vec{\xi})\vec{\xi}_2 + z_2\vec{e} = \vec{0}, \tag{5}$$

$$\vec{\xi}_1 + (\vec{k}\cdot\vec{w})\vec{\xi}_2 - (\vec{k}\cdot\vec{\xi})(\vec{w}_2 - \lambda^{-1}\vec{k}_t) - \lambda^{-1}(\vec{n}^j\cdot\vec{\xi})\vec{m}^j_t - \nu_m\lambda\vec{\xi}_{22} = \vec{0}, \tag{6}$$

$$\vec{k}\cdot\vec{w}_2 = 0, \qquad \vec{k}\cdot\vec{\xi}_2 = 0, \tag{7}$$

where $z_1 = t$ and $\vec{e} = \vec{e}(t) = 2\lambda^{-2}C\vec{k}_t \times \vec{k} + \lambda^{-2}(2\vec{k}_t\cdot\vec{k}_t - \vec{k}_{tt}\cdot\vec{k})\vec{k}$.

Equations (7) are integrated with respect to z_2 to the following expressions: $\vec{k}\cdot\vec{w} = \psi(t)$ and $\vec{k}\cdot\vec{\xi} = \chi(t)$. Here $\psi = \psi(t)$ is an arbitrary smooth function of $z_1 = t$, which can be transformed to zero by means of the transformation generated by the operator $R(\vec{l})$, where the vector-function $\vec{l}$ is a solution of the system

$$\vec{l}_{tt}\cdot\vec{m}^i - \vec{l}\cdot\vec{m}^i_{tt} = 0, \quad \vec{k}\cdot(\vec{l}_t - \lambda^{-1}(\vec{n}^i\cdot\vec{l})m^i_t + \lambda^{-1}(\vec{k}\cdot\vec{l})\vec{k}_t) + \psi = 0.$$

Therefore, without loss of generality we may assume that $\vec{k}\cdot\vec{w} = 0$. The scalar product of equation (6) by $\vec{k}$ gives $\chi_t = 0$, that is, $\chi = \text{const}$.

Let $f^i = f^i(z_1, z_2) := \vec{m}^i\cdot\vec{w}$, $g^i = g^i(z_1, z_2) := \lambda^{-1}\vec{n}^i\cdot\vec{\xi}$. Then

$$\vec{w} = \lambda^{-1}f^i\vec{n}^i, \quad \vec{\xi} = g^i\vec{m}^i + \lambda^{-1}\chi\vec{k},$$

and equation (5) multiplied by the scalar by $\vec{k}$ is integrated with respect to z_2 to the following expression for the function s:

$$s = 2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)\int f^i dz_2 - \tfrac{1}{2}(\vec{\xi}\cdot\vec{\xi}) + \tfrac{1}{2}\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k} - 2\vec{k}_t\cdot\vec{k}_t)z_2^2.$$

Let us multiply the scalar equation (5) by $\vec{m}^i$ and equation (6) by $\vec{n}^i$. As a result, we obtain the linear system of PDEs with variable coefficients in the functions f^i and g^i:

$$f^i_1 - \lambda f^i_{22} + C\lambda^{-1}((\vec{m}^i\cdot\vec{m}^2)f^1 - (\vec{m}^i\cdot\vec{m}^1)f^2) - \chi(\vec{m}^i\cdot\vec{m}^j)g^j_2 -$$
$$2C\lambda^{-2}((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i)z_2 = 0, \tag{8}$$
$$g^i_1 - \nu_m\lambda g^i_{22} - \chi\lambda^{-2}(\vec{n}^i\cdot\vec{n}^j)f^j_2 + 2\chi\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t) = 0$$

Let us consider particular cases.

1. $\chi = 0$, $C = 0$. Then $f^i_\tau = f^i_{\zeta\zeta}$, $g^i_\tau = \nu_m g^i_{\zeta\zeta}$. Hereafter $\tau = \int\lambda(t)dt$, $\zeta = z_2 = \vec{k}\cdot\vec{x}$.

2. $\chi = 0$, $C = 1$. Then $g^i_\tau = \nu_m g^i_{\zeta\zeta}$ and $f^i = \theta^{ij}(t)\tilde{f}^j(\tau, \zeta) + \theta^{i0}(t)\zeta$, where $\tilde{f}^i_\tau = \tilde{f}^i_{\zeta\zeta}$, $(\theta^{1i}(t), \theta^{2i}(t))$ are linearly independent solutions of the system

$$\theta^i_t + \lambda^{-1}(\vec{m}^i\cdot\vec{m}^2)\theta^1 - \lambda^{-1}(\vec{m}^i\cdot\vec{m}^1)\theta^2 = 0,$$

and $(\theta^{10}(t), \theta^{20}(t))$ is a particular solution of the inhomogeneous system

$$\theta^i_t + \lambda^{-1}(\vec{m}^i \cdot \vec{m}^2)\theta^1 - \lambda^{-1}(\vec{m}^i \cdot \vec{m}^1)\theta^2 = 2\lambda^{-2}((\vec{k} \times \vec{k}_t) \cdot \vec{m}^i).$$

3. $\chi \neq 0$, $C = 0$. Let $\nu_m = 1$ and $\vec{m}^i_t \cdot \vec{m}^j = 0$ additionally. Then we can assume that $|m^i| = 1$ and $\vec{m}^1 \cdot \vec{m}^2 = 0$. As a result, we construct the following class of solutions of the MHDEs with $\nu_m = 1$:

$$\vec{u} = (\varphi^{1i}(t, \zeta_+) - \varphi^{2i}(t, \zeta_-))\vec{m}^i - ((\vec{k} \cdot \vec{x})\vec{k})_t,$$
$$\vec{\xi} = (\varphi^{1i}(t, \zeta_+) + \varphi^{2i}(t, \zeta_-) + 2\chi \textstyle\int(\vec{m}^i_t \cdot \vec{k})dt)\vec{m}^i + \chi\vec{k},$$
$$p = -2(\vec{m}^i_t \cdot \vec{k})(\textstyle\int \varphi^{1i}(t, \zeta_+)d\zeta_+ - \int \varphi^{2i}(t, \zeta_-)d\zeta_-) - \frac{1}{2}(\vec{\xi} \cdot \vec{\xi}) +$$
$$\frac{1}{2}(\vec{k}_{tt} \cdot \vec{k} - 2\vec{k}_t \cdot \vec{k}_t)\zeta^2 - \frac{1}{2}(\vec{m}^i_{tt} \cdot \vec{x})(\vec{m}^i \cdot \vec{x}) - \frac{1}{2}(\vec{m}^i_{tt} \cdot \vec{k})(\vec{m}^i \cdot \vec{x})(\vec{k} \cdot \vec{x}),$$

where $\vec{m}^i = \vec{m}^i(t) : |m^i| = 1$, $\vec{m}^1 \cdot \vec{m}^2 = 0$, and $\vec{m}^1_t \cdot \vec{m}^2 = 0$; $\chi = \mathrm{const}$, $\vec{k} = \vec{m}^1 \times \vec{m}^2$, $\zeta = \vec{k} \cdot \vec{x}$, $\zeta_+ = \zeta + \chi t$, $\zeta_- = \zeta - \chi t$, $\varphi^{1i}_t = \varphi^{1i}_{\zeta_+\zeta_+}$, and $\varphi^{2i}_t = \varphi^{2i}_{\zeta_-\zeta_-}$.

References

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