

# Symmetries of Separating Nonlinear Schrödinger Equations

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## Abstract

We review here the main properties of symmetries of separating hierarchies of nonlinear Schrödinger equations and discuss the obstruction to symmetry liftings from  $(n)$ -particles to a higher number. We argue that for particles with internal degrees of freedom, new multiparticle effects must appear at each particle-number level.

## 1 Introduction

We say we have a hierarchy of nonlinear Schrödinger equations, if there is one equation for each number of particles. Such a hierarchy is said to be *separating* if tensor products evolve by separate evolution of factors.

Separating hierarchies are considered to describe systems of noninteracting particles. It is the nonlinearity that provides them with a rich structure. These types of equations were studied by Goldin and Svetlichny [1] and by Svetlichny [2]. The motivation for such equations comes from fundamental speculation about the nature of quantum mechanics, and, surprisingly enough, from representations of current algebras [3].

The main properties of such hierarchies are:

- There are two new universal physical constants with dimension of energy.
- True new multiparticle effects can appear for the first time at any particle number threshold.
- There are obstructions to lifting symmetries from  $n$ -particle equations to a higher number of particles.

Until July 1994 there were strong arguments against nonlinear Schrödinger equations [4, 5, 6], the main point being that these equations permit faster-than-light signals and even more strongly, they conflict with relativity.

There are strong plausability arguments [7] that the linearity of quantum mechanics can be deduced from three main premises:

1. The Lorentz causality structure of space-time.
2. The existence of self-subsisting physical states evolving independently of their creation process and subsequent observations.

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3. The ability to use EPR-type correlations to create any state at a space-like separated location.

It recently became clear that in principle there may be a way out of the mentioned difficulties via the consistent histories and decoherence approach to quantum mechanics [7]. Such an approach would essentially negate the third item above (and apparently the second one also has to be abandoned). Decoherence functionals that are not bilinear could maintain Lorentz covariance and exhibit nonlinear Schrödinger evolution in the nonrelativistic limit. In the relativistic theory one would not maintain the notion of a physical state, nor of evolution, there would only be consistent histories and decoherence. An explicit example of such a theory is in the process being developed.

This situation opens up again the possibility that non-linear Schrödinger equations may be relevant for fundamental physics. It thus becomes important to study them in greater detail. We shall here call attention to the last property mentioned of such hierarchies, that is, the obstruction to lifting symmetries. Details of proofs can be found in [1, 2].

## 2 Separating Hierarchies

The equations we study are of the form

$$i\hbar\partial_t\psi^{(n)} = F_n(t)\psi^{(n)}, \quad (1)$$

where

$$\psi^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n)$$

are  $n$ -particle wave functions and where

$$\mathbf{x} = (x, s) \in \mathbb{R}^d \times S,$$

where  $\mathbb{R}^d$  is a physical space and  $s \in S$  labels particle species and internal degrees of freedom. We consider the particles as distinguishable.

By a separating hierarchy of  $n$ -particle operators  $H_n$  we mean one that satisfies

$$H_{n_1}(\phi_1) \cdot H_{n_2}(\phi_2) \cdots H_{n_r}(\phi_r) = H_n(\phi_1 \cdot \phi_2 \cdots \phi_r). \quad (2)$$

A separating hierarchy satisfies *mixed-power homogeneity*

$$H(k\phi) = e^{a \ln |k| + ib \arg k} H(\phi). \quad (3)$$

If the time evolution of hierarchy (1) is separating, then:

$$F_{n_1}(\phi_1) \cdot \phi_2 \cdots \phi_r + \cdots + \phi_1 \cdots \phi_{r-1} \cdot F_{n_r}(\phi_r) = F_n(\phi_1 \cdots \phi_r),$$

or equivalently,

$$\frac{F_{n_1}(\phi_1)}{\phi_1} + \cdots + \frac{F_{n_r}(\phi_r)}{\phi_r} = \frac{F_n(\phi_1 \cdots \phi_r)}{\phi_1 \cdots \phi_r}.$$

This is the Leibnitz rule: one has a nonlinear derivation with respect to the tensor product. We call such hierarchies *tensor derivations*. The mixed-power homogeneity property of the time evolution expresses itself in the *mixed-logarithmic-homogeneity* of the generators:

$$F_n(k\phi) = kF_n(\phi) + k(p \ln |k| + iq \arg k)\phi.$$

Here,  $p$  and  $q$  are new universal physical constants with dimension of energy.

Some examples of one-particle equations are:

$$i\hbar\partial_t\psi = -(\hbar^2/2m)\nabla^2\psi + \kappa G(\psi)$$

where particular expressions for  $G$  are:

$$\begin{array}{ll} \text{DG} & \nabla^2\psi + (|\nabla\psi|^2/|\psi|^2)\psi \\ \text{BM} & \ln|\psi|\psi \\ \text{K} & \ln(\psi/\bar{\psi}) \end{array}$$

Here, DG is the Doebner-Goldin equation [3] for which  $p = 0$ ,  $q = 0$ , BM is the Bialynicki-Birula and Mycielski equation [8] for which  $p \neq 0$ ,  $q = 0$ , and K is the Kostin equation [9] for which  $p = 0$ ,  $q \neq 0$

### 3 Algebraic Structure

It's convenient to define the mixed  $(a, b)$ -power of the complex number  $z = re^{i\theta}$  by  $z^{(a,b)} = r^a e^{ib\theta} = e^{a \ln|z| + ib \arg z}$ . One has

$$z^{(a,b)} z^{(c,d)} = z^{(a+b, c+d)} \quad (4)$$

$$\left(z^{(c,d)}\right)^{(a,b)} = z^{(a,b)(c,d)}, \quad (5)$$

where

$$(a, b)(c, d) = (a \operatorname{Re} c + ib \operatorname{Im} c, b \operatorname{Re} d + ia \operatorname{Im} d).$$

From this one sees that mixed powers form an algebra isomorphic to the algebra of all real-linear endomorphisms of  $\mathbb{C}$ .

We shall need some definitions concerning non-linear operators. The *Frechet derivative* of an operator  $F$  is defined by

$$\mathbb{D}F(\phi) \cdot \psi = \left. \frac{d}{dt} F(\phi + t\psi) \right|_{t=0}.$$

The *Lie bracket* of two operators  $F$  and  $G$  is defined as

$$[F, G] = \mathbb{D}F \cdot G - \mathbb{D}G \cdot F.$$

One can lift an  $n$ -particle operator  $F$  to an  $m$ -particle operator for  $m > n$ . Let  $J = (j_1, \dots, j_n) \subset \{1, \dots, m\}$ . An  $m$ -particle function can be considered as a family of  $n$ -particle functions

$$\phi(\mathbf{x}_1, \dots, \mathbf{x}_m) = \phi_{\mathbf{y}}(\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n})$$

where  $\mathbf{y}$  stands for the variables  $\mathbf{x}_k$  for  $k \notin J$ . We then define the lift

$$F^J(\phi)(\mathbf{x}_1, \dots, \mathbf{x}_m) = F(\phi_{\mathbf{y}})(\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n})$$

One can construct separating hierarchies of Schrödinger equations (tensor derivations) from appropriate operators defined on a single particle level. We have:

1. If  $F$  is a one-particle mixed-logarithmic homogeneous operator, then

$$(F^\#)_n \phi = \sum_{j=1}^n F^{(j)} \phi - (n-1)(p, q) \cdot \ln \phi \phi,$$

constitutes a tensor derivation.

2. If  $F$  is a strictly homogeneous  $\ell$ -particle operator which vanishes on any tensor product, then for  $n \geq \ell$

$$(F^\#)_n = \sum_J F^J$$

constitute a tensor derivation.

A tensor derivation can be constructed from its canonical generators.

For each  $j$ , we are given a  $j$ -particle operator  $F_{(j)}$  satisfying:

1.  $F_{(1)}$  is mixed-logarithmic homogeneous.
2. For  $j > 1$ ,  $F_{(j)}$  is strictly homogeneous and vanishes on tensor product functions.

One shows that

$$F = \sum_{j=1}^{\infty} F_{(j)}^\#$$

is a tensor derivation and any one can be uniquely written in this form. The operators  $F_{(j)}$  are called *canonical generators* of the tensor derivation.

Tensor derivations have a Lie algebra structure. We have:

1. If  $F, G$  are mixed-logarithmic homogeneous operators, then so is  $[F, G]$  with

$$(p_{[F,G]}, q_{[F,G]}) = [(p_F, q_F), (p_G, q_G)].$$

2. If  $F, G$  are tensor derivations, then so is  $[F, G]$ .

The Lie algebra structure of tensor derivations does not have a simple relation to the canonical lifting operations. One has in general that

$$[F, G]^\# \neq [F^\#, G^\#].$$

This is a *true nonlinear effect* and is responsible for obstruction to symmetry lifting. For real-linear operators, the two sides of the above displayed inequality are equal.

## 4 Symmetries

We say  $V$  is a symmetry of an  $n$ -particle equation if

$$(i\hbar\partial_t - F(t))\psi = 0 \Rightarrow (i\hbar\partial_t - F(t))V\psi = 0.$$

We now consider symmetries of the form:

$$(V\psi)(t, \mathbf{x}) = (V(t)\psi(T(t)))(\mathbf{x}).$$

$V(t)$  acts on the variable  $\mathbf{x}$  and  $T: \mathbb{R} \rightarrow \mathbb{R}$  is a diffeomorphism such as  $T(t) = at + b$ .

The operator equation satisfied by a symmetry is

$$\hbar \frac{\partial V(t)}{\partial t} = \bar{i}F(t) \circ V(t) - T'(t) \mathbb{D}V(t) \cdot \bar{i}F(T(t)).$$

For  $T(t) = t$  and real-linear operators, the right-hand side is a usual commutator, whereas for  $T'(t) \neq 1$  and real-linear operators with  $F(t)$  time-independent, the right-hand side is a “quantum” commutator.

One shows that the infinitesimal generator of a one-parameter group of symmetries has the form:

$$(K\psi)(t, \mathbf{x}) = (K(t)\psi(t))(\mathbf{x}) + \tau(t)(\partial_t\psi)(t, \mathbf{x}). \quad (6)$$

If  $K, L$  are infinitesimal symmetries, then so is  $[K, L]$  with

$$\begin{aligned} [K, L](t) &= [K(t), L(t)] + \tau_K(t) \frac{\partial L(t)}{\partial t} - \tau_L(t) \frac{\partial K(t)}{\partial t}, \\ \tau_{[K, L]}(t) &= \tau_K(t)\tau'_L(t) - \tau_L(t)\tau'_K(t). \end{aligned}$$

One says a symmetry is *separating* if tensor products transform by separate transformation of the factors. It is quite natural to assume this for space-time symmetries.

## 5 Main Result

First we define for  $F$ , a mixed-logarithmic homogeneous operator with indices  $(a, b)$ , the strictly homogeneous operator  $F^{\natural}$  by

$$F\phi = F^{\natural}\phi + (a, b) \cdot \ln \phi \phi.$$

### Theorem 1

1. Let  $F$  be an  $\ell$ -particle generator and  $G$  an  $m$ -particle generator with  $1 \leq \ell \leq m$ . Let  $F^{\#}$  and  $G^{\#}$  be their respective canonical liftings. For any particle number  $n$  with  $n > m$ :

$$[F_n^{\#}, G_n^{\#}] - [F_m^{\#}, G_n^{\#}] = \sum_K \sum_{J \not\subseteq K} [F^{\natural J}, G^{\natural K}],$$

where  $J$  is an  $\ell$ -tuple  $(j_1, \dots, j_\ell)$  of elements of  $\{1, \dots, n\}$  in increasing order,  $K$  is an  $m$ -tuple of the same type, and where we write  $J \not\subseteq K$  to mean  $\{j_1, \dots, j_\ell\} \not\subseteq \{k_1, \dots, k_m\}$ .

2. *The obstruction to the equality*

$$[F^\#, G^\#] = [F_m^\#, G]^\#$$

is the set of operators on the right-hand side of (1) for particle numbers from  $m + 1$  to  $m + \ell$ . These operators are zero if and only if (2) holds.

For  $K$  defined by (6), we define a canonical lifting  $K^\#$  as the hierarchy of operators defined again through (6) by  $K(t)^\#$  and the same function  $\tau(t)$ .

**Corollary 1** *Let  $F(t)$  and  $K$  be one-particle generators, and suppose  $K$  be an infinitesimal symmetry of  $F(t)$ . The canonical lifting  $K^\#$ , defined in the previous paragraph, is a symmetry of  $F(t)^\#$  if and only if the two-particle operator  $K_2^\#$  is a symmetry of  $F(t)_2^\#$  and this happens if and only if the two-particle operator*

$$[F(t)^{\natural(1)}, K(t)^{\natural(2)}] + [F(t)^{\natural(2)}, K(t)^{\natural(1)}]$$

vanishes.

**Corollary 2** *Let  $F(t)$  be a tensor derivation and  $K$  a symmetry with only a one-particle generator. Let  $G(t)$  be an  $\ell$ -particle generator with  $\ell > 1$ , then  $K$  is a symmetry of  $F(t) + G(t)^\#$  if and only if  $K_\ell$  and  $K_{\ell+1}$  are symmetries of  $F_\ell(t) + G(t)$  and  $F_{\ell+1}(t) + G(t)_{\ell+1}^\#$ , respectively, and this happens if and only if  $[G(t), K_\ell(t)] = 0$  and the following  $(\ell + 1)$ -particle operator vanishes:*

$$\sum_{j=1}^{\ell+1} [G(t)^{j^\#}, K(t)^{\natural(j)}],$$

where  $j^\#$  is the  $\ell$ -tuple  $(1, \dots, j - 1, j + 1, \dots, \ell + 1)$ .

## 6 Point space-time symmetries

Assume first that particles have no internal degrees of freedom. We consider space-time transforms of the type

$$\Phi(t, x) = (T(t), X(t, x)),$$

and that the symmetry operator has the form

$$(V(t)\phi)(x) = H(\phi(X(t, x)), t, x)$$

for some complex function  $H$ , that is,  $V$  acts pointwise on the graph of  $\phi$ .

**Theorem 2** *A point space-time symmetry has the form:*

$$V(t)\phi = e^{i\left(\sum_{j=1}^n v^{(j)}(t)\right)} (\phi(X_1, \dots, X_n))^{(1+i\alpha(t), \beta(t))} |JX_1|^{\frac{1}{2}} \dots |JX_n|^{\frac{1}{2}}$$

for some real functions  $\alpha(t)$  and  $\beta(t)$  and  $v(t, x)$ . An infinitesimal point space-time symmetry has the form:

$$K(t)\phi = \sum_{j=1}^n \left( i\eta^{(j)} + (\xi \cdot \nabla)^{(j)} + \frac{1}{2}(\nabla \cdot \xi)^{(j)} \right) \phi + i(\gamma(t), \delta(t)) \cdot \ln \phi$$

for some real functions  $\gamma(t)$  and  $\delta(t)$  and  $\eta(t, x)$ .

**Theorem 3** *An infinitesimal point space-time symmetry is always a canonical lift from the one-particle generator.*

**Theorem 4** *The lifting obstructions vanish for point space-time symmetries for particles with no internal degrees of freedom.*

This last remarkable result does not hold for particles with internal degrees of freedom. For particles with spin, one has to necessarily introduce new generators at each particle number.

Consider an example of a two-particle generator for a scalar particle. Let

$$M(\phi) = \frac{\phi \nabla^{(1)} \cdot \nabla^{(2)} \phi - \nabla^{(1)} \phi \cdot \nabla^{(2)} \phi}{\phi^2}. \quad (7)$$

Any operator of the form

$$(k_1 \operatorname{Re} M(\phi) + k_2 \operatorname{Im} M(\phi))\phi$$

for real  $k_1, k_2$  is a two-particle generator.

To appreciate the difficulty for particles with spin, consider the case of vector particles  $\phi_i$  where  $i$  is a vector index. Consider a one-particle operator of the form

$$N_1 \psi_i = \frac{\sum_j |\nabla \psi_j|^2}{\sum_j |\psi_j|^2} \psi_i.$$

The canonical lifting of this to a two-particle operator is

$$N_2^\# \psi_{ij} = \frac{\sum_k |\nabla^{(1)} \psi_{kj}|^2}{\sum_k |\psi_{kj}|^2} \psi_{ij} + \frac{\sum_k |\nabla^{(2)} \psi_{ik}|^2}{\sum_k |\psi_{kj}|^2} \psi_{ij}.$$

One readily sees that *this is not a tensor*. For this to be a tensor, one has to add to this a two-particle canonical generator. One such generator, involving expressions similar to (7), would change the sums in the formula to sums over both the indices and this would restore the tensor character of the operator, but there are many other possible generators that do this.

$$\frac{\sum_{km} |\nabla^{(1)} \psi_{km}|^2}{\sum_{km} |\psi_{km}|^2} \psi_{ij} + \frac{\sum_{km} |\nabla^{(2)} \psi_{km}|^2}{\sum_{km} |\psi_{km}|^2} \psi_{ij}$$

A similar problem would now again appear on the three-particle level and so on. A nonlinear rotation-invariant separating hierarchy of particles with spin therefore has new multi-particle effects at each particle number. This rather striking result points out once again the radical nature of nonlinear quantum mechanics.

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