A new SS-closedness in L-topological spaces

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***Abstract*—By means of strongly semi-open L-sets and their inequality, a new form of SS-closedness is introduced in L- topological spaces, where L is a complete De Morgan algebra. This new form does not depend on the structure of basis lattice L and L does not require any distributivity. When L is a completely distributive De Morgan algebra, its many characterizations are presented.**

Keywords-L-topology; strongly semi-open L -set; SS-closedness.

I. INTRODUCTION

As we know, compactness and its stronger and weaker forms play very important roles in general topology. In [1], we introduced the concepts of strongly semi-open sets and strongly semi-continuous mappings in [0,1]-topological spaces. Based on this, in [2], we introduced the concept of SR-compactness in L-topological spaces long Wang's [11] and Zhao's [13] compactness. In [3], we introduced the concept of SS-closedness in L-topological spaces along Kudri's compactness in [7].

However, Wang's compactness and Kudri's compactness, as well as the SR-compactness and the SS-closedness rely on the structure of L and L is required to be completely distributive. In [10], by means of open L-sets and their inequality, Shi introduced a new definition of fuzzy compactness in L-topological spaces, where L is a complete de Morgan algebra. This new definition doesn't depend on the structure of L . In [4], we introduced the concept of SR-compactness in L -topological spaces along Shi's compactness in [10].

In this paper, following the lines of [10], we'll introduce a new form of SS-closedness in L -topological spaces by means of strongly semi-open L -sets and their inequality. It is weaker form of SR-compactness. It can also be characterized by strongly semi-closed L -sets and their inequality. This new form of SS-closedness has many characterizations if L is completely distributive De Morgan algebra.

II. PRELIMINARIES

Throughout this paper, (L, $\vee$, $\wedge$, ') is a complete De Morgan algebra, X a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. The smallest element and the largest element in L^X are denoted by 0 and 1. An element a in L is called prime element if $b\wedge c\le a$ implies $b\le a$ or $c\le a$. a in L is called a co-prime element if a' is a prime element [6]. The set of nonunit prime elements in L is denoted by P(L). The set of nonzero co-prime elements in L is denoted by M(L).

The binary relation $\prec$ in L is defined as follows: or $a,b\in L$, $a\prec b$ iff for every subset $D\subseteq L$, the relation $b\le \sup D$ always implies the existence of $d\in D$ with $a\le d$ [5]. In a completely distributive De Morgan algebra L, each element b is a sup of $\{a\in L\mid a\prec b\}$. $\{a\in L\mid a\prec b\}$ is called the greatest minimal family of b in the sense of [8,12], in symbol $\beta(b)$. Moreover for $b\in L$, define $\beta^*(b)=\beta(b)\bigcap M(L)$, $\alpha(b)=\{a\in L\mid a'\prec b'\}$ and . $\alpha^*(b)=\alpha(b)\wedge P(L)$. For $a\in L$ and $A\in L^X$, we denote $A^{(a)}=\{x\in X\mid A(x)\not\le a\}$, $A_{(a)}=\{x\in X\mid a\in\beta(A(x))\}$, $A^{[a]}=\{x\in X\mid a\notin\alpha(A(x))\}$ and $A_{[a]}=\{x\in X\mid A(x)\ge a\}$. For a subfamily $\psi\subseteq L^X$, $2^{(\psi)}$ denotes the set of all finite subfamilies of ψ , and $2^{[\psi]}$ denotes the set of all countable subfamilies of ψ . An L-topological space denotes L -ts for short. Let (X,τ) be a weakly induce L -ts, $a\in L$, $A\in\tau$ .Then $A_{(a)}$ is an open set in $[\tau]$, where $[\tau]$ is the topology formed by all crisp sets in τ [8].

Let (X,δ) be an L -ts and $A\in L^X$. Then A is called a strongly semi-open set iff there is a $B\in\delta$ such that $B\le A\le B^{-o}$, and A is called a strongly semi-closed set iff there is a $B\in\delta'$ such that $B^{o-}\le A\le B$ [1,2], where B^o and B^- are the interior and closure of B , respectively. The sets $A^\Delta=\vee\{B:B$ is strongly semi-open, $B\le A\}$ and $A^\sim=\wedge\{B:B$ is strongly semi-closed, $B\ge A\}$ are called the strong semi-interior and the strong semi-closure of A [1,2], respectively.

Definition 2.1([4]) Let (X,δ) be an L-ts. $A\in L^X$ is called SR-compact if for every family μ of strongly semi-open L-sets, it follows that

$$\underset{x\in X}{\wedge}\left(A'(x)\vee\underset{B\in\mu}{\vee}B(x)\right)\le\underset{\nu\in 2^{(\mu)}}{\vee}\underset{x\in X}{\wedge}\left(A'(x)\vee\underset{B\in\nu}{\vee}B(x)\right)$$

Definition 2.2([9,10]) Let (X,δ) be an L-ts, $a\in L\setminus\{1\}$ and $A\in L^X$. A family $\mu\subseteq L^X$ is called

(1) an a-shading of A if for any $x\in X$,

$(A'(x)\vee\vee_{B\in\mu}B(x))\not\le a$.

(2) a strong a-shading (briefly S-a-shading) of A if

$\wedge_{x\in X}(A'(x)\vee\vee_{B\in\mu}B(x))\not\le a$.

(3) an a-R-neighborhood family (briefly a-R-NF) of A if for any $x\in X,(A(x)\wedge\wedge_{B\in\mu}B(x))\not\ge a$.

(4)a strong a-R-neighborhood family (briefly S-a-R-NF) of A if $\vee_{x\in X}(A(x)\wedge\wedge_{B\in\mu}B(x))\not\ge a$.

Definition 2.3([9]) Let (X,δ) be an L-ts, $a\in L\setminus\{0\}$ and $A\in L^X$. A family $\mu\subseteq L^X$ is called

(1) a β_a-cover of A if for any $x\in X$, it follows that

$a\in\beta(A'(x)\vee\vee_{B\in\mu}B(x))$.

(2) a strong β_a-cover (briefly S-β_a-cover) of A if

$a\in\beta(\wedge_{x\in X}(A'(x)\vee\vee_{B\in\mu}B(x)))$

(3) a Q_a-cover of A if for any $x\in X$, it follows that

$A'(x)\vee\vee_{B\in\mu}B(x)\ge a$.

It is obvious that an S-a-shading of A is an a-shading of A, that an S-a-R-NF of A is an a-R-NF of A, that μ is an S-a-R-NF of A iff μ' is an S-a-shading of A, that an S-β_a-cover of A must be a β_a-cover of A, and that a β_a-cover of A must be a Q_a-cover of A.

Lemma 2.4 ([9]) Let L be a complete Heyting algebra, $f:X\to Y$ be a map and $f_L^{\to}:L^X\to L^Y$ is the extension of f, then for any family $\psi\subseteq L^Y$

$$\underset{y\in Y}{\vee}\left(f_L^{\to}(A)(y)\wedge\underset{B\in\psi}{\wedge}B(y)\right)=\underset{x\in X}{\vee}\left(A(x)\wedge\underset{B\in\psi}{\wedge}f_L^{\to}B(x)\right)$$

Definition 2.5([2]) Let (X,δ) and (Y,τ) be two L-ts's. A map $f:(X,\delta)\to(Y,\tau)$ is called S-irresolute if $f_L^{\leftarrow}(B)$ is strongly semi-open in (X,δ) for every strongly semi-open L set B in (Y,τ).

Theorem 2.6 ([2]) A map $f:(X,\delta)\to(Y,\tau)$ is S-irresolute iff $f_L^{\leftarrow}(B^{\Delta})\le(f_L^{\to}(B))^{\Delta}$ for each $B\in L^Y$.

III. Definition and characterizations of SS-closedness

Definition 3.1 Let (X,δ) be an L-ts. $A\in L^X$ is called SS-closed if for every family μ of strongly semi-open L-sets, it follows that

$$\underset{x\in X}{\wedge}\left(A'(x)\vee\underset{B\in\mu}{\vee}B(x)\right)\le\underset{\nu\in 2^{(\mu)}}{\vee}\underset{x\in X}{\wedge}\left(A'(x)\vee\underset{B\in\mu}{\vee}B^{\sim}(x)\right).$$

Obviously we have the following result.

SR-compactness $\Rightarrow$ SS-closedness.

That the converse of above need not be true is shown by the Example 3.2.

Example 3.2 Let X be an infinite set(or X be a singleton); A and C be two [0, 1]-sets on X defined as $A(x)=0.6$, for all $x\in X$; $C(x)=0.8$, for all $x\in X$. Take $\delta=\{\phi,A,X\}$, then δ is a topology on X. We easily obtain that C is SS-closed. But C is not SR-compact.

From the Definition 3.1, we can obtain the following theorem by using quasi-complement.

Theorem 3.3 Let (X,δ) be an L-ts. $A\in L^X$ is SS-closed iff for every family μ of strongly semi-closed L-sets, it follows that

$$\underset{x\in X}{\vee}\left(A(x)\wedge\underset{B\in\mu}{\wedge}B(x)\right)\ge\underset{\nu\in 2^{(\mu)}}{\wedge}\underset{x\in X}{\vee}\left(A(x)\wedge\underset{B\in\nu}{\wedge}B^{\Delta}(x)\right).$$

From the Definition 3.1 and the Theorem 3.4, we immediately obtain the following the result.

Theorem 3.4 Let (X,δ) be an L-ts and $A\in L^X$. Then the following conditions are equivalent.

(1) A is SS-closed.

(2) For any $a \in L \setminus \{1\}$, each strongly semi-open S- a -shading μ of A has a finite subfamily ν such that $\{B^{\sim} \mid B \in \nu\}$ is an S- a -shading of A.

(3)For any $a \in L \setminus \{0\}$, each strongly semi-closed S- a -R-NF ψ of A has a finite subfamily φ such that $\{B^{\Delta} \mid B \in \varphi\}$ is an S- a -R-NF of A.

Theorem 3.5 If C is SS-closed and D is strongly semi-closed, then $C \wedge D$ is SS-closed.

Proof. For any family μ of strongly semi-closed L-sets, by the Theorem 3.4 we have that

$$\bigvee_{x\in X}\left((C\wedge D)(x)\wedge \bigwedge_{B\in\mu} B(x)\right) = \bigvee_{x\in X}\left(C(x)\wedge \bigwedge_{B\in\mu\cup\{D\}} B(x)\right)$$
$$\geq \bigwedge_{\nu\in 2^{(\mu\cup\{D\})}} \bigvee_{x\in X}\left(C(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)$$
$$=\left\{\bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left(C(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)\right\}$$
$$\wedge\left\{\bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left(C(x)\wedge D(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)\right\}$$

$$\left\{\bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left(C(x)\wedge D(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)\right\}$$
$$=\left\{\bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left((C\wedge D)(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)\right\}.$$

This shows that $C \wedge D$ is SS-closed.

Theorem 3.6 Let L be a complete Heyting algebra. If

both C and D are SS-closed, then $C \vee D$ is SS-closed.

Proof. For any family μ of strongly semi-closed L-sets,

by the Theorem 3.4 we have that

$$\bigvee_{x\in X}\left((C\vee D)(x)\wedge \bigwedge_{B\in\mu} B(x)\right)$$
$$=\left\{\bigvee_{x\in X}\left(C(x)\wedge \bigwedge_{B\in\mu} B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(D(x)\wedge \bigwedge_{B\in\mu} B(x)\right)\right\}$$
$$\geq\left\{\bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left(C(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)\right\}$$
$$\vee\left\{\bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left(D(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)\right\}$$
$$= \bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left((C\vee D)(x)\wedge \bigwedge_{B\in\nu} B^{\Delta}(x)\right)$$

This shows that $C \vee D$ is SS-closed.

Theorem 3.7 Let L be a complete Heyting algebra and $f:(X,\delta)\to(Y,\tau)$ be an S-irresolute map. If A is an SS-closed L-set in (X,δ), then so is $f_L^{\to}(A)$ in (Y,τ).

Proof. Suppose that μ is a family of strongly semi-closed L-sets in (Y,τ), by the Lemma 2.4, the Theorem 2.6 and SS-closed of A, we have that

$$\bigvee_{y\in Y}\left(f_L^{\to}(A)(y)\wedge \bigwedge_{B\in\mu} B(y)\right)$$
$$= \bigvee_{x\in X}\left(A(x)\wedge \bigwedge_{B\in\mu} f_L^{\leftarrow}B(x)\right)$$
$$\geq \bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left(A(x)\wedge \bigwedge_{B\in\nu} \left(f_L^{\leftarrow}(B)\right)^{\Delta}(x)\right)$$
$$\geq \bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{x\in X}\left(A(x)\wedge \bigwedge_{B\in\nu} f_L^{\leftarrow}\left(B^{\Delta}\right)(x)\right)$$
$$= \bigwedge_{\nu\in 2^{(\mu)}} \bigvee_{y\in Y}\left(f_L^{\to}(A)(y)\wedge \bigwedge_{B\in\nu} B^{\Delta}(y)\right).$$

Therefore $f_L^{\to}(A)$ is SS-closed.

IV. Further Properties of SS-Closedness

In this section, we assume that L is a completely distributive de Morgan algebra.

Theorem 4.1 Let (X,δ) be an L-ts and $A \in L^X$.

Then the following conditions are equivalent.

(1) A is SS-closed.

(2) For any $a \in L \setminus \{0\}$, each strongly semi-closed S- a -R-NF ψ of A has a finite subfamily φ such that $\{B^{\Delta} \mid B \in \varphi\}$ is an a -R-NF (S- a -R-NF) of A.

(3) For any $a \in L \setminus \{0\}$ and any strongly semi-closed S-a-R-NF ψ of \$A\$, there is a finite subfamily φ of ψ and $b \in \beta(a)$ such that $\{B^{\Delta} \mid B \in \varphi\}$ is a b-R-NF (an S-b-R-NF) of A.

(4) For any $a \in L \setminus \{1\}$, each strongly semi-open S-a-shading μ of A has a finite subfamily ν such that $\{B^{\sim} \mid B \in \nu\}$ is an (a-shading) S-a-shading of A.

(5) For any $a \in L \setminus \{1\}$ and any strongly semi-open S-a-shading μ of A, there is a finite subfamily ν of μ and $b \in \alpha(a)$ such that $\{B^{\sim} \mid B \in \nu\}$ is a b-shading (an S-b-shading) of A.

(6) For any $a \in L \setminus \{0\}$, each strongly semi-open S-β_a-cover μ of A has a finite subfamily ν such that $\{B^{\sim} \mid B \in \nu\}$ is a β_a-cover (an S-β_a-cover) of A.

(7) For any $a \in L \setminus \{0\}$ and any strongly semi-open S-β_a-cover μ of A, there is a finite subfamily ν of μ and $b \in L$ with $a \in \beta(b)$ such that $\{B^{\sim} \mid B \in \nu\}$ is a β_a-cover (an S-β_a-cover) of A.

(8)For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each strongly semi-open Q_a-cover μ of A has a finite subfamily ν of μ such that $\{B^{\sim} \mid B \in \nu\}$ is a Q_b-cover of A.

(9) For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each strongly semi-open Q_a-cover μ of A has a finite subfamily ν of μ such that $\{B^{\sim} \mid B \in \nu\}$ is a β_b-cover (an S-β_b-cover) of A.

Proof. This is immediate from the Definition 3.1 and the

Theorem 3.5.

Lemma 4.2([4]). Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). If A is a strongly semi-open L-set in (X, τ), then χ_A is a strongly semi-open set in $(X, \omega_L(\tau))$. If B is a strongly semi-open L-set in $(X, \omega_L(\tau))$, then $B_{(a)}$ is a strongly semi-open set in (X, τ) for every $a \in L$.

Since $B^{\sim}$ is the smallest strongly semi-closed L-set which contains B and B^{Δ} is the greatest strongly semi-open L-set which is contained in B, we can obtain the following lemma.

Lemma 4.3. Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(B_{[b]})^{\sim} \subseteq (B^{\sim})_{[b]}$ for any $b \in L$ for any $B \in L^X$, where $(B_{[b]})^{\sim}$ and $B^{\sim}$ denote the strongly semi-closures of $B_{[b]}$ and B in (X, τ) and $(X, \omega_L(\tau))$ respectively.

Theorem 4.4. Let (X, τ) be a topological space and $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ is SS-closed iff (X, τ) is SS-closed.

Proof. Necessity. Let μ be a strongly semi-open cover of

(X, τ). Then $\{\chi_A \mid A \in \mu\}$ is a family of strongly

semi-open L-sets in $(X, \omega_L(\tau))$ with $\wedge_{x \in X} \left(\vee_{A \in \mu} \chi_A(x)\right) = 1$.

From SS-closedness of $(X, \omega_L(\tau))$, we have that

$$\bigvee_{\nu \in 2^{(\mu)}} \bigwedge_{x \in X} \left(\bigvee_{A \in \nu} \chi_{A^{\sim}}(x) \right) \geq \bigvee_{\nu \in 2^{(\mu)}} \bigwedge_{x \in X} \left(\bigvee_{A \in \nu} (\chi_A)^{\sim}(x) \right) = 1.$$

This implies that there exists $\nu \in 2^{(\mu)}$ such that

$$\wedge_{x \in X} \left(\vee_{A \in \nu} \chi_{A^{\sim}}(x)\right) = 1.$$

Hence, $\{A^{\sim} \mid A \in \nu\}$ is a cover of (X, τ). Therefore (X, τ) is SS-closed.

Sufficiency. Let μ be a family of strongly semi-open L-sets in $(X, \omega_L(\tau))$ and $\wedge_{x \in X} \left(\vee_{B \in \mu} B(x)\right) = a.$

If $a = 0$, obviously we have that

$$\bigwedge_{x \in X} \left(\bigvee_{B \in \mu} B(x) \right) \leq \bigvee_{\nu \in 2^{(\mu)}} \bigwedge_{x \in X} \left(\bigvee_{B \in \nu} B^{\sim}(x) \right).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$, we have that

$$b \in \beta\left(\bigwedge_{x \in X} \left(\bigvee_{B \in \mu} B(x) \right) \right) \subseteq \bigcap_{x \in X} \beta\left(\bigvee_{B \in \mu} B(x) \right) = \bigcap_{x \in X} \bigcup_{B \in \mu} \beta(B(x))$$

From the Lemma 4.2, this implies that $\{B_{(b)} \mid B \in \mu\}$ is a strongly semi-open cover of (X, τ). From SS-closedness of (X,τ), we know that there exists $\nu \in 2^{(\mu)}$ such that $\{(B_{(b)})^{\sim} \mid B \in \nu\}$ is a cover of (X,τ). Obviously $\{(B^{\sim})_{[b]} \mid B \in \nu\}$ is a cover of (X,τ) since $(B_{(b)})^{\sim} \subseteq (B_{[b]})^{\sim} \subseteq (B^{\sim})_{[b]}$. Hence $b \le \wedge_{x\in X}\left(\vee_{B\in\nu} B^{\sim}(x)\right)$

Further, we have that

$$b \le \bigwedge_{x\in X}\left(\bigvee_{B\in\nu} B^{\sim}(x)\right) \le \bigvee_{\nu\in 2^{(\mu)}} \bigwedge_{x\in X}\left(\bigvee_{B\in\nu} B^{\sim}(x)\right).$$

This implies that

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mu} B(x)\right) = a = \vee\{b \mid b \in \beta(a)\} \le \bigvee_{\nu\in 2^{(\mu)}} \bigwedge_{x\in X}\left(\bigvee_{B\in\nu} B^{\sim}(x)\right).$$

Therefore, $(X,\omega_L(\tau))$ is SS-closed.

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REFERENCES

[1] S.Z. Bai, " Fuzzy strongly semiopen sets and fuzzy strong semicontinuity,"Fuzzy Sets and Systems, 52(1992) 345-351.

[2] S.Z. Bai, "The SR-compactness in L -fuzzy topological spaces,"Fuzzy Sets and Systems, 87(1997) 219-225.

[3] S.Z.Bai, "SS-closedness in L -topological spaces,"Indian J.Pure Appl.Math, 34(2003) 1697-1706 .

[4] S.Z. Bai, "A new form of SR-compactness in L -topological Spaces,"Fuzzy Systems and Mathematics, 21(2007) 57-62 .

[5] P. Dwinger, "Characterizations of the complete homomorphic images of a completely distributive complete lattice,"I, Nederl. Akad. Wetensch. Indag. Math, 44(1982) 403-414.

[6] G. Gierz, et al, ",A Compendium of Continuous Lattices,"Springer Verlag, Berlin, 1980.

[7] S.R.T.Kudri, Compactness in L -fuzzy topological spaces",''Fuzzy Sets and Systems, 67(1994) 329-336 .

[8] Y.M. Liu and M.K. Luo, "Fuzzy Topology,World Scienti.c, Singapore, 1998".

[9] F.G.Shi, "A new notion of fuzzy compactness in L -topological spaces,"Information Sciences, 173(2005) 35-48.

[10] F.G.Shi, "A new definition of fuzzy compactness,"Fuzzy Sets and Systems, 158(13)(2007) 1486-1495.

[11] G.J.Wang, A new fuzzy compactness defined by fuzzy nets",''J.Math.Anal.Appl, 94(1983) 1-23.

[12] G.J.Wang, "Theory of L-Fuzzy Topological Space," .Shaanxi Normal University Press, Xian, 1988.

[13] D.S.Zhao, "The N-compactness in L -fuzzy topological spaces,"J.Math.Anal.Appl, 128(1987) 64-79 .