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On the Braided FRT-Construction

Yurij BESPALOV†

Bogolyubov Institute for Theoretical Physics,
Metrologichna str., 14-b Kyïv 143, 252143 Ukraïna
E-mail: *mmtpitp@gluk.apc.org*

Abstract

A fully braided analog of the Faddeev-Reshetikhin-Takhtajan construction of a quasitriangular bialgebra $A(X, R)$ is proposed. For a given pairing C, the factor-algebra $A(X, R; C)$ is a dual quantum braided group. Corresponding inhomogeneous quantum group is obtained as a result of generalized bosonization. Construction of a first order bicovariant differential calculus is proposed.

1 Introduction and preliminaries

Hopf algebras in braided categories (braided groups) have been extensively studied over the last few years and play an important role in q-deformed physics and mathematics [17],[18]. Examples, applications and the basic theory of braided groups have been introduced and developed by Majid. Some similar concepts arise independently in works of Lyubashenko inspired by results on conformal field theory. Crossed modules over braided Hopf algebras were introduced and studied in [3] and provide a useful technique for investigation of braided Hopf algebras. In particular, crossed product of braided Hopf algebras and generalized bosonization for quantum braided groups are defined in [3]. The theory of Hopf bimodules in braided categories is developed in [4] on grounds of [3]. Application of this theory is an analog of the Woronowicz construction of (bicovariant) differential calculi [20] developed in [5] for the case of braided Hopf algebras and quantum braided groups. Quantum braided group defined by Majid [14, 15] is a natural generalization of Drinfel'd's concept of (ordinary) quantum group (quasitriangular Hopf algebra) [7]. Basic examples of coquasitriangular bialgebras $A(R)$ are obtained as a result of the Faddeev-Reshetikhin-Takhtajan construction [8] applied to an arbitrary R-matrix. Analog of the FRT-construction for anyonic quantum groups is described in [19]. Majid proposed another construction of braided bialgebra $B(R)$ which can be obtained as a transmutation [14] of $A(R)$. Algebra $A(R, Z)$ defined in [10] generalizes both $A(R)$ and $B(R)$. In this paper we describe a fully braided analog of a FRT-construction of a quasitriangular bialgebra $A(X, R)$, where X is an object of an Abelian braided monoidal category $\mathcal{C}$ and $R : X \otimes X \to X \otimes X$ solution of the braid equation. This construction covers all mentioned above and can be considered as a coordinate-free version of [10]. For a given pairing C

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we define a factor-algebra $A(X,R;C)$ which is a dual quantum braided group. This is an analog of construction of quantum simple Lie groups of types B,C,D in [8]. Majid's definition of braided vectors $\mathrm{V}(R)$ is simply reformulated to our more abstract setting. In particular, our algebra $\mathrm{V}(X,R)$ is also a quantum braided group in the category of comodules over $A(X,R)$. Quantized analogs of inhomogeneous linear groups are studied in many papers (see [1, 6] and references therein). Generalized bosonization construction [3] allows us to define the quantum braided group $A(X,R) \ltimes \mathrm{V}(X,R)$. We propose construction of a first order bicovariant differential calculus on dual quantum braided group A related with any comodule X over A. In our special case $A = A(X,R)$, this is a generalization of construction [12].

In the rest of this part we give necessary preliminary results. The main results of the paper are presented in the second part.

1.1 We will suppose that $\mathcal{C}$ is *an Abelian* and *braided (monoidal) category* with tensor product $\otimes$, unit object $\underline{1}$ and braiding Ψ (without loss of generality by Mac Lane's coherence theorem we will assume that underlying monoidal category is strict, i.e., the functors $\_\otimes(\_\otimes\_)$ and $(\_\otimes\_)\otimes\_$ coincide and $\underline{1}\otimes X = X = X\otimes\underline{1}$). Compatibility conditions between tensor product and Abelian structure are the following [5]: functors $(-)\otimes X$ and $X\otimes(-)$ are right exact for any object X (this assumption is true if the category is closed); for any epimorphisms $X_i \xrightarrow{f_i} Y_i,\ \ i=1,2$, the diagram

$$\begin{array}{ccc} X_1\otimes X_2 & \xrightarrow{X_1\otimes f_2} & X_1\otimes Y_2 \\ {\scriptstyle f_1\otimes X_2}\downarrow & & \downarrow{\scriptstyle f_1\otimes Y_2} \\ Y_1\otimes X_2 & \xrightarrow{Y_1\otimes f_2} & Y_1\otimes Y_2 \end{array} \tag{1}$$

is push-out (the right-down part is a colimit of the left-up part). In this case there exist well-behaviored constructions of factor-algebra (coalgebra, bialgebra, Hopf algebra) by ideal (coideal, biideal, Hopf ideal). One can define an algebra by generator and relations. We mean under 'the ideal generated by relations $f_1 = f_2 : X \to A$' the subobject $\mathrm{Im}\,(\mu\circ(\mu\otimes A)\circ(A\otimes(f_1-f_2)\otimes A))$ of algebra A.

1.2 We will work with *graded and filtered algebras in* $\mathcal{C}$. A ($I\!N$-)graded algebra A means a collection of objects $A_k,\ \ k\in\mathcal{C}$, multiplications $m_{i,j} : A_i\otimes A_j \to A_{i+j}$ satisfying associativity conditions and unit $\eta : \underline{1}\to A_0$. A ($I\!N$-)filtered algebra A means a collection of objects $A_{(k)},\ \ k\in\mathcal{C}$, such that $A_{(i)}$ is a subobject of $A_{(j)}$ if $i<j$, multiplications $m_{(i),(j)} : A_{(i)}\otimes A_{(j)} \to A_{(i+j)}$ satisfying conditions of associativity and compatibility with restrictions on subobjects, and unit $\eta : \underline{1}\to A_{(0)}$. For any graded algebra $\{A_i\}$ the collection $\{A_{(k)} := \oplus_{i=0}^k A_i\}$ with natural multiplications is a filtered algebra. As shown in [5], graded or filtered algebra can be considered as a usual algebra in a certain category of 'graded spaces', i.e., functors from a certain category to the category $\mathcal{C}$. This category of 'graded spaces' is again an Abelian braided monoidal category. Similarly graded coalgebras, bialgebras, Hopf algebras can be defined. We will say briefly that a graded (filtered) algebra lives in the category $\mathcal{C}$ if its components A_n ($A_{(n)}$) live in $\mathcal{C}$. See [5] about more details.

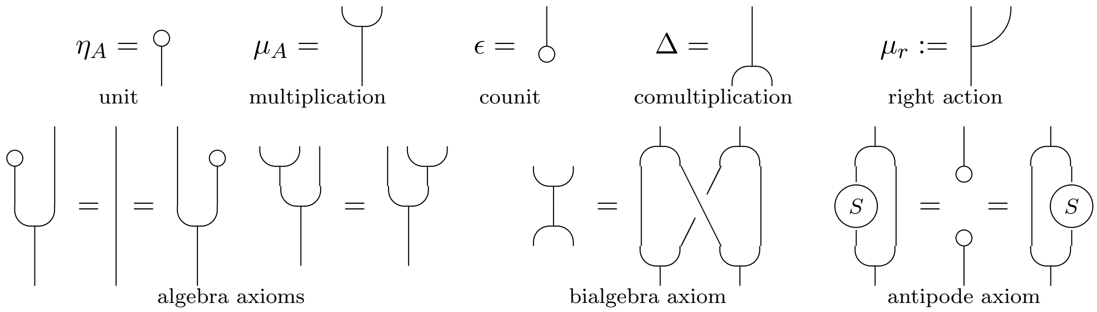
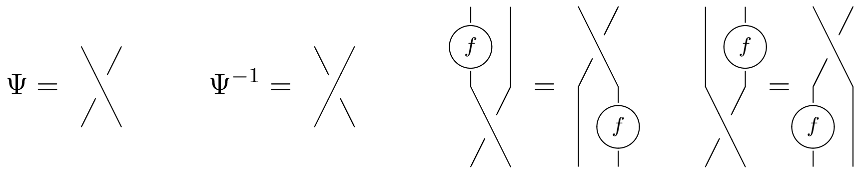


Figure 1: The basic algebraic structures in a braided category

1.3 We actively use diagrammatic calculus in braided categories [15, 17] (see [3] about our slight modifications). Morphisms Ψ and Ψ^{-1} are represented by under and over crossing and algebraic information 'flows' along braids and tangles according to functoriality and the coherence theorem for braided categories [11]:



(2)

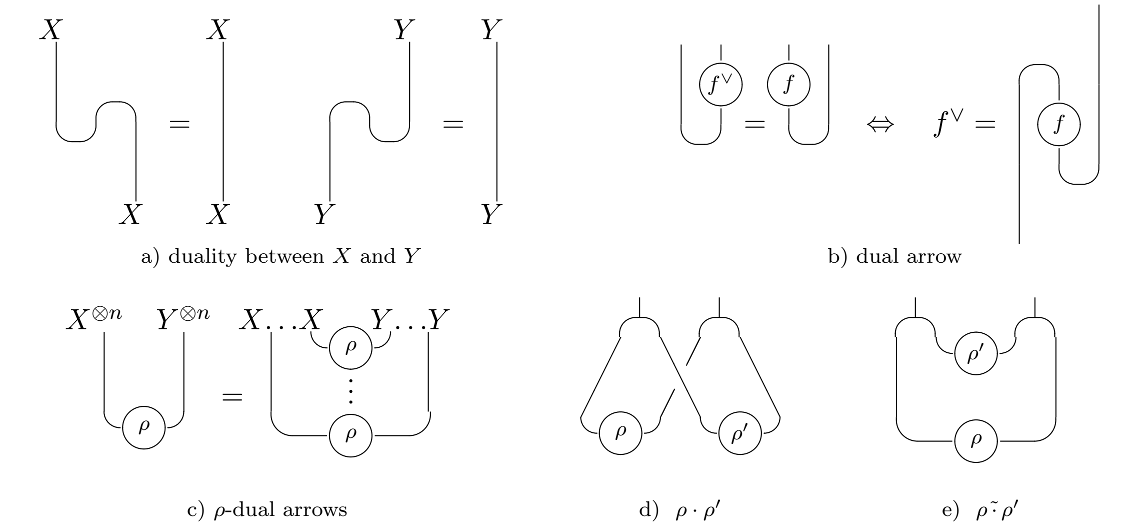
Fig.1 explains our notations: *An algebra* in a monoidal category $\mathcal{C}$ is an object A equipped with unit $\eta = \eta_A : \underline{1} \to A$ and multiplication $\mu = \mu_A : A \otimes A \to A$ obeying the axioms on Fig.1. *A coalgebra* is object C equipped with counit $\epsilon = \epsilon_A : C \to \underline{1}$ and comultiplication $\Delta = \Delta_A : A \to A \otimes A$ obeying the axioms of algebra turned upside-down Finally [13],[14], *a bialgebra A in a braided category $\mathcal{C}$* is an object in $\mathcal{C}$ equipped with algebra and coalgebra structures obeying the compatibility axiom in Fig.1 which means that Δ_A is an algebra homomorphism. *A Hopf algebra A in a braided category $\mathcal{C}$* (*braided group* or *braided Hopf algebra*) is a bialgebra in $\mathcal{C}$ with antipode $S : A \to A$ which is convolution-inverse to the identical map (the last identity in Fig.1). Axioms for (co-)module X over a (co-)algebra A are obtained by "polarization" of the (co-)algebra axioms.

If $\mathcal{C}$ is a braided category, we will denote by $\overline{\mathcal{C}}$ the same category with the same tensor product and with inverse braiding Ψ^{-1}. For any algebra (resp., coalgebra) A in $\mathcal{C}$, we will always consider *the opposite algebra* $(A^{\mathrm{op}}, \mu_{A^{\mathrm{op}}} := \mu_A \circ \Psi^{-1})$ (resp., *the opposite coalgebra* $(A_{\mathrm{op}}, \Delta_{A_{\mathrm{op}}} := \Psi^{-1} \circ \Delta_A)$ as an object of the category $\overline{\mathcal{C}}$. In particular, $(A^{\mathrm{op}})^{\mathrm{op}} = A$. If A is a bialgebra in $\mathcal{C}$, then A^{op} and A_{op} are bialgebras in $\overline{\mathcal{C}}$ (cf. [17]). Antipode S^- for A^{op} (or, the same, for A_{op}) is called *skew antipode* and equals S^{-1} if both S and S^- exist. Majid [17] derived from Hopf algebra axioms that antipode S_A is a bialgebra morphism $(A^{\mathrm{op}})_{\mathrm{op}} \to A$ (or $A \to (A_{\mathrm{op}})^{\mathrm{op}}$) in $\mathcal{C}$.

1.4 For objects X, Y of a monoidal category $\mathcal{C}$ we will call any morphism

$$\cup = \cup^{X,Y} : X \otimes Y \to \underline{1} \qquad (\text{ resp. } \cap = \cap_{Y,X} : \underline{1} \to Y \otimes X \ ) \tag{3}$$

a pairing between X, Y (resp., *copairing between* Y, X). *Duality between* X *and* Y is both pairing and copairing (3) obeying the identities in Fig.2a. In this case, X is called *left*



a) duality between X and Y b) dual arrow

c) ρ-dual arrows d) $\rho \cdot \rho'$ e) $\rho \tilde{\cdot} \rho'$

Figure 2: Duals and pairings.

dual to Y (resp., Y is called *right dual* to X) and we will write $X = {}^\vee Y, Y = X^\vee$. Dual arrow $f^\vee$ is defined by one of the two equivalent conditions in Fig.2b. In this way a braided monoidal functor $(\_)^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}_{\mathrm{op}}$ can be defined if $X^\vee$ exists for each $X \in \mathrm{Obj}(\mathcal{C})$. Without loss of generality by coherence theorem we shall assume that $(\_)^\vee$ is a strict monoidal functor: $(X \otimes Y)^\vee = Y^\vee \otimes X^\vee$, $(f \otimes g)^\vee = g^\vee \otimes f^\vee$. Pairing ρ between X and Y extends to pairing between $X^{\otimes n}$ and $Y^{\otimes n}$ defined by the diagram in Fig.2c. We say that arrows $f : X^{\otimes m} \to X^{\otimes n}$ and $g : Y^{\otimes n} \to Y^{\otimes m}$ are ρ-*dual* if $\rho \circ (f \otimes Y^{\otimes n}) = \rho \circ (X^{\otimes n} \otimes g)$.

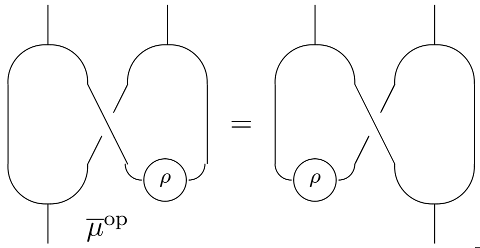
Let A and H be bialgebras in a braided category $\mathcal{C}$. Morphism $\rho : A \otimes H \to \underline{1}$ is called *a bialgebra pairing* if an algebra (resp., coalgebra) structure on A and coalgebra (resp. algebra) structure on H are ρ-dual. Convolution product '$\cdot$' and 'the second' product '$\tilde{\cdot}$' for $\rho, \rho' \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \underline{1})$ are defined in Fig.2d,e. We denote by ρ^-, $\rho^\sim$ corresponding inverses to ρ. Let $\overline{\rho} := \rho^- \circ \Psi^{-1}$. If A or H has (skew) antipode, then $\rho^\sim$ (resp., ρ^-) exists and

$$\rho \circ (S_A \otimes H) = \rho^\sim = \rho \circ (A \otimes S_H) \qquad \rho \circ (S_A^- \otimes H) = \rho^- = \rho \circ (A \otimes S_H^-) \tag{4}$$

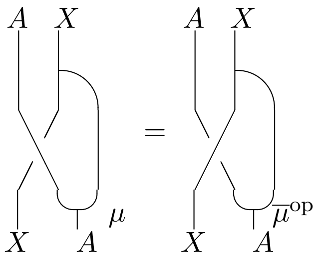
If ρ^- or $\rho^\sim$ exists, then ρ-duality between multiplications and comultiplications implies ρ-duality between units and counits. If (A, H, ρ) is bialgebra pairing in $\mathcal{C}$, then $(A_{\mathrm{op}}, H_{\mathrm{op}}, \rho^-)$, $(A^{\mathrm{op}}, H^{\mathrm{op}}, \rho^\sim)$, $(H^{\mathrm{op}}, A^{\mathrm{op}}, \overline{\rho})$ are bialgebra pairings in $\overline{\mathcal{C}}$.

1.5 Quantum braided groups in a braided category were introduced in [14] and basic theory was developed there. The following are input-output reversed variants of definitions from [14] in a slightly modified form [3] suitable for our use.

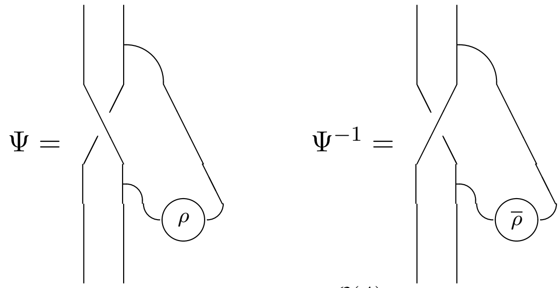
A coquasitriangular bialgebra in a braided category $\mathcal{C}$ is a pair of bialgebras A in $\mathcal{C}$ and $\overline{A}$ in $\overline{\mathcal{C}}$ with the same underlying coalgebra (μ and $\overline{\mu}$ are multiplications in A and $\overline{A}$ respectively), and convolution invertible bialgebra pairing (*coquasitriangular structure*) $\rho : \overline{A}^{\mathrm{op}} \otimes A \to \underline{1}$, satisfying the condition in Fig.3a. (It follows directly from the definition that units for A and for $\overline{A}$ are the same.) *A dual quantum braided group* or *a coquasitriangular*



a) The axiom for a dual quantum braided group $(A, \overline{A}, \rho)$.



b) the condition on comodules from $\mathcal{C}^{\mathcal{O}(A)}$



c) braiding in $\mathcal{C}^{\mathcal{O}(A)}$

Figure 3:

Hopf algebra in $\mathcal{C}$ is a coquasitriangular bialgebra such that A and $\overline{A}$ have antipodes S and $\overline{S}$, respectively. (In this case $\rho^{-} = \rho \circ (\overline{S} \otimes A)$ and $\rho^{\sim} = \rho \circ (A \otimes S)$.)

In particular, for any bialgebra (braided group) A, the pair (A, A^{op}) is a coquasitriangular bialgebra (dual quantum braided group) with the trivial coquasitriangular structure $\rho = \epsilon \otimes \epsilon$.

Category $\mathcal{C}^{\mathcal{O}(A,\overline{A})}$ is a full subcategory of the category $\mathcal{C}^{A}$ of right comodules with objects X satisfying the first identity in Fig.3b. $\mathcal{C}^{\mathcal{O}(A,\overline{A})}$ is a monoidal subcategory of $\mathcal{C}^{A}$ and braided with Ψ and Ψ^{-1} shown in Fig.3c. We use a brief notation $\mathcal{C}^{\mathcal{O}(A)}$ for $\mathcal{C}^{\mathcal{O}(A,A^{\mathrm{op}})}$.

2 On the braided FRT-construction

2.1 Canonical epimorphism $B_n \to S_n$ of the braid group into a permutation group admits a section $S_n \stackrel{\frown}{\to} B_n$ identical on generators and unqueenly defined by the condition that $\widehat{\sigma_1\sigma_2} = \widehat{\sigma_1}\widehat{\sigma_2}$ if $\ell(\sigma_1\sigma_2) = \ell(\sigma_1) + \ell(\sigma_2)$, where $\ell(\sigma)$ is the length (of the minimal decomposition) of σ. For any object X, the obvious action of the braid group B_n on $X^{\otimes n}$ is defined. We will use the same notation $\widehat{\sigma}$ for the image of the braid $\widehat{\sigma} \in B_n$ in $\mathrm{End}_{\mathcal{C}}(X^{\otimes n})$. For $k = 1, \ldots, n$ let us denote by $S_n^k \subset S_n$ the subset of $\dfrac{n!}{k!(n-k)!}$ shuffle permutations which preserve the order of any two elements i and j if $i, j \leq k$ or $i, j > k$. Majid in [16] defined braided binomial coefficient as a sum of $\dfrac{n!}{k!(n-k)!}$ braids in $\mathrm{End}_{\mathcal{C}}(X^{\otimes n})$ and in particular, braided factorial as a sum of $n!$ braids:

$$\begin{bmatrix} n \\ k \end{bmatrix}; X \Bigg] := \sum_{\sigma^{-1} \in S_n^k} \widehat{\sigma}, \qquad [n; X]! := \sum_{\sigma \in S_n} \widehat{\sigma}. \tag{5}$$

2.2 For any object X of a braided category $\mathcal{C}$, the tensor algebra $\mathrm{T}(X) = \{X^{\otimes n}\}_{n\in\mathbf{N}}$ is a graded Hopf algebra with the tensor product as multiplication, comultiplication

$$\Delta_{m,n} := \begin{bmatrix} m+n \\ m \end{bmatrix}; X \Big] :\ X^{\otimes(m+n)} \to X^{\otimes m} \otimes X^{\otimes n} \tag{6}$$

and antipode

$$S|_{X^{\otimes n}} := (-1)^n \circ \widehat{\rho_n} :\ X^{\otimes n} \to X^{\otimes n}, \tag{7}$$

where $S_n \ni \rho_n : (1, 2, \ldots, n) \mapsto (n, n-1, \ldots, 1)$ and $\widehat{\rho_n}$ is a Garside element of B_n. The bialgebra axiom turns into the Newton-Majid binomial formula [16]:

$$(\underline{1} \otimes x + x \otimes \underline{1})^n = (\Delta \circ x)^n = \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix}; X\right] \circ (x^{\otimes k} \otimes x^{\otimes(n-k)}) \tag{8}$$

for any $x :\ Z \to X$.

One can define two graded ideals in $\mathrm{T}(X)$: $I = \{I_n \subset X^{\otimes n}\}$ is an ideal generated by its 'quadratic part' $I_2 := \ker\,[2; X]! = \ker(\Psi_{X,X} + \mathrm{id})$. And let $I^\bullet = \{I^\bullet_n := \ker\,[n; X]\}$.

I is nonzero iff -1 is an eigenvalue of $\Psi_{X,X}$. If we suppose that the multiplicity of this eigenvalue is 1, i.e., we can choose a minimal polynomial of $\Psi_{X,X}$ in the form $p(t) = p_{-1}(t)(t+1)$ with $p_{-1}(-1) = 1$, then $P_{-1} := p_{-1}(\Psi_{X,X})$ is an idempotent, and in this case $I_2 = \ker\,[n; X]! = \mathrm{im}\, P_{-1}$.

It is easy to see that $I \subset I^\bullet$. The following example from [5] shows that, in general, the ideal $I^\bullet$ has 'generators' of the power more than 2.

Example 2.2.1 *Let $\mathrm{T}(X)$ be an algebra generated by the one–dimensional vector space $X = kx$ over a field k with the braiding $\Psi(x \otimes x) = q(x \otimes x)$, $q \in k$. In this case braided integers are 'ordinary' q-integers: $[n]_q := 1 + q + \ldots + q^{n-1}$. And for q a primitive root of 1 of order $n > 2$: $I = \emptyset$ but $I^\bullet = (x^{\otimes n})$.*

Propositon 2.2.1 *Both I and $I^\bullet$ are Hopf ideals in $\mathrm{T}(X)$. We denote by $\mathrm{V}(X)$ and $\mathrm{V}^\bullet(X)$ corresponding factor-algebras.*

For the special case of the category $\mathcal{C}$ built from an arbitrary R-matrix, $\mathrm{V}(X)$ is an algebra of functions on 'quantum vector space'. Majid discovered a Hopf algebra structure on this object (cf. [16] and references therein).

2.3 It is well known that any solution $R = R_{X,X} : X \otimes X \to X \otimes X$, $X \in \mathrm{Obj}(X)$ of the braid equation

$$(X \otimes R) \circ (R \otimes X) \circ (X \otimes R) = (R \otimes X) \circ (X \otimes R) \circ (R \otimes X) \tag{9}$$

with certain invertibility conditions defines a braided structure on the monoidal subcategory of $\mathcal{C}$ generated by the object X and its dual ${}^\vee X$ as described in what follows. Morphisms $R_{X^{\otimes m}, X^{\otimes n}}$ are uniquely defined by the hexagon identities:

$$R_{Y\otimes Y', Z} = (R_{Y,Z} \otimes Y') \circ (Y \otimes R_{Y',Z})\,, \quad R_{Y, Z\otimes Z'} = (Z \otimes R_{Y,Z'}) \circ (R_{Y,Z} \otimes Z') \tag{10}$$

where Y, Y', Z, Z' are powers of X. And let $R_{{}^\vee X^{\otimes m}, {}^\vee X^{\otimes n}} := {}^\vee\left(R_{X^{\otimes m}, X^{\otimes n}}\right)$. We also suppose that there exists $R_{X, {}^\vee X} : X \otimes {}^\vee X \to {}^\vee X \otimes X$ inverse to $(\cup \otimes X \otimes {}^\vee X) \circ ({}^\vee X \otimes$

$R_{X,X} \otimes {}^\vee X) \circ ({}^\vee X \otimes X \otimes \cap)$ Let $R_{X^{\otimes m}, {}^\vee X^{\otimes n}}$ be uniquely defined by the hexagon identities (10) and $R_{{}^\vee X^{\otimes m}, X^{\otimes n}}$ be defined in a dual way. $\mathcal{C}(X, {}^\vee X; R)$ is a subcategory of $\mathcal{C}$ whose objects are tensor products of X and ${}^\vee X$ and morphism $f : Y \to Z$ are those from $\mathcal{C}$ which 'flow' along the braids labeled by $R_{X,\_}$ and $R_{\_,X}$, i.e.,

$$R_{X,Z} \circ (X \otimes f) = (f \otimes X) \circ R_{X,Y}\,, \quad R_{Z,X} \circ (f \otimes X) = (X \otimes f) \circ R_{Y,X}\,. \tag{11}$$

An analog $A(X, R)$ of the FRT-bialgebra [8] can be obtained as a result of some reconstruction for the monoidal functor $\mathcal{C}(X, {}^\vee X; R) \to \mathcal{C}$.

2.4 As the first step, the following lemmas allow us to define a bialgebra $A(X)$.

Lemma 2.4.1 *Let X be an object of $\mathcal{C}$ with (left) dual ${}^\vee X$. Then ${}^\vee X \otimes X$ can be equipped with a coalgebra structure*

$$\Delta_{{}^\vee X \otimes X} := {}^\vee X \otimes \cap_{X, {}^\vee X} \otimes X\,, \quad \epsilon_{{}^\vee X \otimes X} := \cup^{{}^\vee X, X}\,. \tag{12}$$

X (resp. ${}^\vee X$) becomes a right (resp. left) comodule over ${}^\vee X \otimes X$ with coaction

$$\Delta_r^X := \cap_{X, {}^\vee X} \otimes X\,, \quad \Delta_\ell^{{}^\vee X} := {}^\vee X \otimes \cap_{X, {}^\vee X}\,. \tag{13}$$

Lemma 2.4.2 *Let (A, Δ_A), (B, Δ_B) be coalgebras in $\mathcal{C}$ and (X, Δ_r^X), (Y, Δ_r^Y) be right comodules over A and B, respectively. Then $A \otimes B$ is a coalgebra with comultiplication*

$$\Delta_{A \otimes B} := (A \otimes \Psi_{A,B} \otimes B) \circ (\Delta_A \otimes \Delta_B) \tag{14}$$

and $X \otimes Y$ is a right $A \otimes B$-comodule with coaction

$$\Delta_r^{X \otimes Y} := (X \otimes \Psi_{A,Y} \otimes B) \circ (\Delta_r^X \otimes \Delta_r^Y). \tag{15}$$

So, with any two objects X and Y which have left duals, we can connect the following coalgebras: tensor product of two coalgebras $({}^\vee X \otimes X) \otimes ({}^\vee Y \otimes Y)$ and coalgebra $({}^\vee Y \otimes {}^\vee X) \otimes (X \otimes Y)$ related with the object $X \otimes Y$.

Lemma 2.4.3 *Morphism*

$$\mu_{{}^\vee X \otimes X, {}^\vee Y \otimes Y} := \Psi_{{}^\vee X \otimes X, {}^\vee Y} \otimes Y : ({}^\vee X \otimes X) \otimes ({}^\vee Y \otimes Y) \to ({}^\vee Y \otimes {}^\vee X) \otimes (X \otimes Y) \tag{16}$$

is coalgebra isomorphism and interlaces coactions of these coalgebras on $X \otimes Y$.

For objects X, Y, Z with a left dual, the following associativity condition *is true:*

$$\mu_{X \otimes Y, Z} \circ (\mu_{X,Y} \otimes Z) = \mu_{X, Y \otimes Z} \circ (X \otimes \mu_{Y,Z}). \tag{17}$$

Propositon 2.4.4 $A(X) := \{A_n(X) = {}^\vee X^{\otimes n} \otimes X^{\otimes n}\}_{n \in \mathbb{Z}_{\geq 0}}$ *is a graded bialgebra with the following (co)multiplications:*

$$\mu_{m,n} := \mu_{{}^\vee X^{\otimes m} \otimes X^{\otimes m}, {}^\vee X^{\otimes n} \otimes X^{\otimes n}}\,,$$
$$\Delta_n := \Delta_{{}^\vee X^{\otimes n} \otimes X^{\otimes n}} : {}^\vee X^{\otimes n} \otimes X^{\otimes n} \to ({}^\vee X^{\otimes n} \otimes X^{\otimes n}) \otimes ({}^\vee X^{\otimes n} \otimes X^{\otimes n})\,.$$

Graded tensor algebra $\mathrm{T}(X)$ *(resp.,* $\mathrm{T}({}^\vee X)$*) is right (resp., left) $A(X)$-comodule algebra and coalgebra.*

One can carry out the same construction in the category $\overline{\mathcal{C}}$. The result is a bialgebra $\overline{A}(X)$ with the same underlying coalgebra but with new multiplication $\overline{\mu}$, where Ψ is replaced by Ψ^{-1}. The corresponding Hopf algebra $\overline{\mathrm{T}}(X)$ (resp., $\overline{\mathrm{V}}(X), \overline{\mathrm{V}}^\bullet(X)$) coincides with $\mathrm{T}(X)_{\mathrm{op}}$ (resp., $\mathrm{V}(X)_{\mathrm{op}}, \mathrm{V}^\bullet(X)_{\mathrm{op}}$).

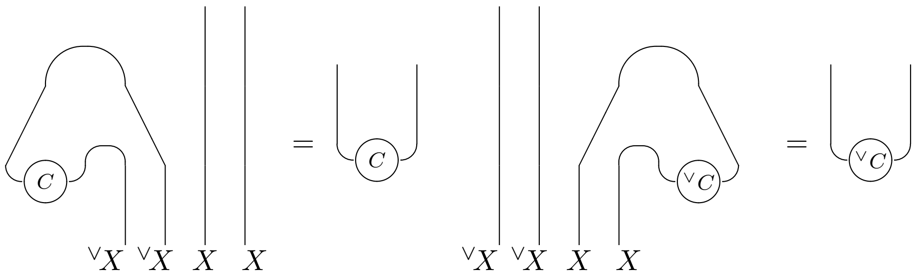


Figure 4: Relations for $A(X;C)$

2.5 Let, moreover, duality C of X with itself be given. Then for each n the pairing and copairing

$$C = C^{X^{\otimes n}, X^{\otimes n}} : X^{\otimes n} \otimes X^{\otimes n} \to \underline{1}, \qquad C = C_{X^{\otimes n}, X^{\otimes n}} : \underline{1} \to X^{\otimes n} \otimes X^{\otimes n}, \tag{18}$$

described by the diagram in Fig.2b and by the input-output reversed diagram, define duality of $X^{\otimes n}$ with itself. Let us define pairing $^\vee C =: {^\vee X^{\otimes n}} \otimes {^\vee X^{\otimes n}} \to \underline{1}$ as morphism left dual to copairing $C_{X^{\otimes n}, X^{\otimes n}}$. We denote by $A(X;C)$ filtered algebra which is a factor-algebra of $A(X)$ by the ideal 'generated by relations' in Fig.4 which means that pairings C and $^\vee C$ are invariant with respect to coactions of $A(X)$ on $\mathrm{T}(X)$ and $\mathrm{T}(^\vee X)$, respectively. And let $\mathrm{T}(X;C)$ be the factor-algebra of $\mathrm{T}(X)$ by relations $C^{^\vee X \otimes X} = \underline{1}$.

Propositon 2.5.1 *$A(X;C)$ is a braided group with antipode and its inverse given by the diagram in Fig5a. $\mathrm{T}(X;C)$ is a right comodule algebra over $A(X;C)$.*

Let $I = \{I_n \in {^\vee X^{\otimes n}} \otimes X^{\otimes n}\}$ be a graded ideal of algebra $A(X)$ generated by relations

$$^\vee R \otimes \mathrm{id}_{X^{\otimes 2}} - \mathrm{id}_{^\vee X^{\otimes 2}} \otimes R : \ {^\vee X^{\otimes 2}} \otimes X^{\otimes 2} \to {^\vee X^{\otimes 2}} \otimes X^{\otimes 2} X^{\otimes 2} \otimes X^{\otimes 2} \tag{19}$$

or explicitly

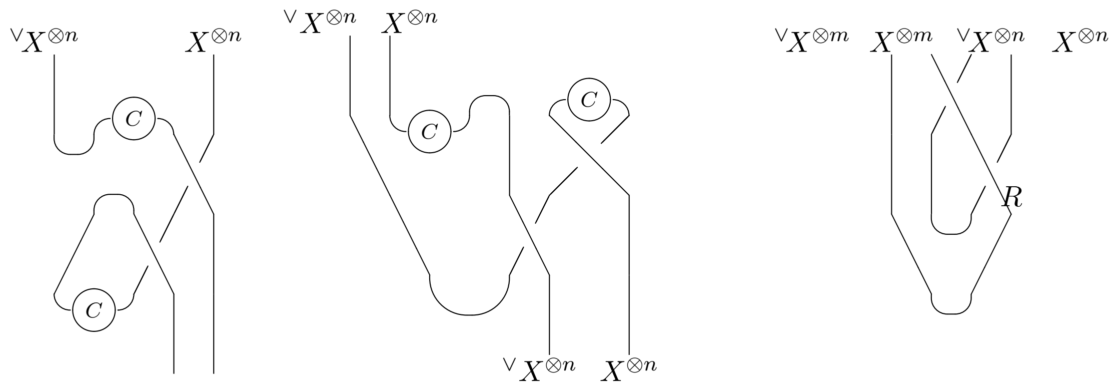
$$I_n = \bigcup_{i=1}^{n-1} ({^\vee X^{\otimes (i-1)}} \otimes {^\vee R} \otimes {^\vee X^{\otimes (n-i-1)}} \otimes X^{\otimes n} \to {^\vee X^{\otimes n}} \otimes X^{\otimes (n-i-1)} \otimes R \otimes X^{\otimes (i-1)}). \tag{20}$$

Lemma 2.5.2 *Ideal I described above is a biideal of $A(X)$. We denote by $A(X,R)$ a corresponding factor-bialgebra.*

Let, moreover, C be morphism in $\mathcal{C}(X, {^\vee X}; R)$, i.e., the pairing C 'flows' along braids labeled by R. Then we define a bialgebra $A(X,R;C)$ which is a factor-algebra of $A(X)$ by an ideal generated by both (19) and the relations given on Fig.4. Similarly, one can define the factor-algebras $\overline{A}(X,R^{-1})$ and $\overline{A}(X,R^{-1};C)$ as the bialgebra $\overline{A}(X)$ in the category $\overline{\mathcal{C}}$.

Lemma 2.5.3 *A family of pairings $\rho_{m,n} := ({^\vee X^{\otimes m}} \otimes X^{\otimes m}) \otimes ({^\vee X^{\otimes m}} \otimes X^{\otimes m}) \to \underline{1}$ described by the diagram on Fig.5b define bialgebra pairings*

$$\begin{aligned} &\rho_{A(X,R)} : \overline{A}(X,R^{-1})^{\mathrm{op}} \otimes A(X,R) \to \underline{1}, \\ &\rho :_{A(X,R;C)} : \overline{A}(X,R^{-1};C)^{\mathrm{op}} \otimes A(X,R;C) \to \underline{1}. \end{aligned} \tag{21}$$



a) antipode and its inverse for $A(X;C)$ b) Coquasitriangular structure on $A(X,R)$

Figure 5:

Theorem 2.5.4 $(A(X,R),\overline{A}(X,R^{-1}),\rho)$ *is a coquasitriangular bialgebra and its factor-algebra* $(A(X,R;C),\overline{A}(X,R^{-1};C),\rho)$ *is a dual quantum braided group in* $\mathcal{C}$ *(or, more precisely, in a certain category of 'graded spaces' over* $\mathcal{C}$*). 'Second inverse'* $\rho^{\sim}=\{\rho^{\sim}_{m,n}\}$ *to quasitriangular structure* ρ *takes the form:*

$$\rho^{\sim}_{m,n}:=(\cup\otimes\cup)\circ({}^\vee X^{\otimes m}\otimes\Psi^{-1}_{X^{\otimes m},{}^\vee X^{\otimes n}}\otimes X^{\otimes n}).$$

Lemma 2.5.5 $X^{\otimes n}$ *equipped with coaction (13) is an object of* $\mathcal{C}^{\mathcal{O}(A(X,R),\overline{A}(X,R^{-1}))}$*. Braiding* $\Psi_{X^{\otimes m},X^{\otimes n}}$ *in this category equals to* $R_{X^{\otimes m},X^{\otimes n}}$*. Corresponding right action (defined by the first diagram in Fig.6b)*

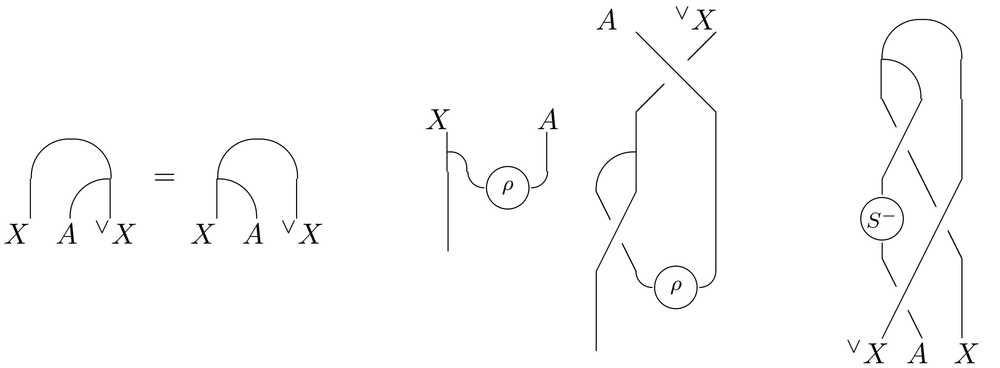
$$\mu_r^{X^{\otimes n}}=\{\mu_{r,m}^{X^{\otimes n}}:X^{\otimes n}\otimes A(X,R)_m\to X^{\otimes n}\}$$

takes the form

$$\mu_{r,m}^{X^{\otimes n}}=(\cup^{{}^\vee X^{\otimes m},X^{\otimes m}}\otimes\mathrm{id}_{X^{\otimes n}})\circ(\mathrm{id}_{{}^\vee X^{\otimes m}}\otimes R_{X^{\otimes n},X^{\otimes m}})\circ(\Psi_{X^{\otimes n},{}^\vee X^{\otimes m}}\otimes\mathrm{id}_{X^{\otimes m}})\,.\quad(22)$$

One can carry out constructions from 2.2 for $X\in\mathrm{Obj}(\mathcal{C}^{\mathcal{O}(A(X,R),\overline{A}(X,R^{-1}))})$ to get the Hopf algebras $\mathrm{T}(X,R),\mathrm{V}(X,R),\mathrm{V}^\bullet(X,R)$ in $\mathcal{C}^{\mathcal{O}(A(X,R),\overline{A}(X,R^{-1}))}$, where the letter '$R$' is added to specify a category. The pair $(\mathrm{V}(X,R),\mathrm{V}(X,R)^{\mathrm{op}})$ is a quantum braided group in $\mathcal{C}^{\mathcal{O}(A(X,R),\overline{A}(X,R^{-1}))}$ with the trivial coquasitriangular structure $\epsilon^{\mathrm{V}(X,R)}\otimes\epsilon_{\mathrm{V}(X,R)}$. The generalized bosonization theorem [3] allows us to define a quantum braided group $(A(X,R)\ltimes\mathrm{V}(X,R),\overline{A}(X,R^{-1})\ltimes\mathrm{V}(X,R)^{\mathrm{op}})$ with the coquasitriangular structure $(\mathrm{id}_{A(X,R)}\otimes\epsilon_{\mathrm{V}(X,R)}\otimes\mathrm{id}_{A(X,R)}\otimes\epsilon_{\mathrm{V}(X,R)})\circ\rho_{A(X,R)}$ in $\mathcal{C}$ which is an analog of the algebra of functions on an inhomogeneous linear group. The same construction performed for algebras $\overline{A}(X,R^{-1})$ and $\overline{\mathrm{V}}(X,R^{-1})=\mathrm{V}(X,R)_{\mathrm{op}}$ produces quantum braided group in $\overline{\mathcal{C}}$. But in this way we obtain another corresponding quantum braided group in $\mathcal{C}$.

2.6 Let $(A,\overline{A},\rho)$ be a dual quantum braided group in $\mathcal{C}$, $(X,\Delta_r^X)\in\mathrm{Obj}(\mathcal{C}^{\mathcal{O}(A,\overline{A})})$, ${}^\vee X$ left dual to X in $\mathcal{C}$ with the left comodule structure $\Delta_\ell^{{}^\vee X}$ defined by the condition in Fig.6a.



a) coaction $\Delta_\ell^{\vee X}$ b) actions μ_r^X, $\mu_\ell^{\vee X}$ c) bi-invariant ω

Figure 6:

Then X (resp., ${}^\vee X$) equipped with the right (resp., left) A-module structure as shown in Fig.6b becomes a right (resp., left) crossed module over A. According to general theory [4], the object $\Gamma := {}^\vee X \otimes A \otimes X$ with actions and coactions

$$\begin{aligned}
\mu_\ell^\Gamma &:= (\mu_\ell^{\vee X} \otimes m_A) \circ (A \otimes \Psi_{A,{}^\vee X} \otimes A) \circ (\Delta_A \otimes {}^\vee X \otimes A) \otimes X\,, \\
\mu_r^\Gamma &:= {}^\vee X \otimes (m_A \otimes \mu_r^X) \circ (A \otimes \Psi_{X,A} \otimes A) \circ (A \otimes X \otimes \Delta_A)\,, \\
\Delta_\ell^\Gamma &:= (m_A \otimes {}^\vee X \otimes A) \circ (A \otimes \Psi_{{}^\vee X,A} \otimes A) \circ (\Delta_\ell^{\vee X} \otimes \Delta_A) \otimes X\,, \\
\Delta_r^\Gamma &:= {}^\vee X \otimes (A \otimes X \otimes m_A) \circ (A \otimes \Psi_{A,X} \otimes A) \circ (\Delta_A \otimes \Delta_\ell^X)
\end{aligned}$$

is a Hopf bimodule over A. Morphism ω defined on Fig.6c is a bicomodule morphism where $\underline{1}$ is equipped with trivial left and right actions equal to ϵ_A. 'Commutant with ω':

$$\mathrm{d} := \mu_r^\Gamma \circ (\omega \otimes A) - \mu_\ell^\Gamma \circ (A \otimes \omega) : A \to \Gamma \tag{23}$$

is a first order bicovariant derivative in the sense of Woronowicz [20] (See [5] for a fully braided context).

In our case $A = A(X, R; C)$ 'biinvariant' ω equals to $\Psi^{-1}_{{}^\vee X,{}^\vee X} \circ {}^\vee C \otimes C : \underline{1} \to {}^\vee X^{\otimes 2} \otimes X^{\otimes 2}$.

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