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A Symmetry Connection Between Hyperbolic and Parabolic Equations

Peter BASARAB-HORWATH

Mathematics Department, University of Linköping, 581 83 Linköping, Sweden
e-mail: pehor@math.liu.se

Abstract

We give ansatzes obtained from Lie symmetries of some hyperbolic equations which reduce these equations to the heat or Schrödinger equations. This enables us to construct new solutions of the hyperbolic equations using the Lie and conditional symmetries of the parabolic equations. Moreover, we note that any equation related to such a hyperbolic equation (for example the Dirac equation) also has solutions constructed from the heat and Schrödinger equations.

1 The real, linear wave equation

The real linear wave equation is

$$\Box u = -m^2 u, \tag{1}$$

where

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}.$$

The symmetry algebra of equation (1) is known:

Theorem 1 *Equation (1) has maximal Lie point-symmetry algebra $< P_\mu, J_{\mu\nu}, D, L >$, where*

$$P_\mu = \frac{\partial}{\partial x_\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad D = x^\mu P_\mu, \quad L = u\partial_u$$

when $m^2 \neq 0$, and $< P_\mu, J_{\mu\nu}, D, K_\mu, L >$ when $m = 0$, where

$$K_\mu = 2x_\mu D - x^2 P_\mu - 2x_\mu L.$$

We have used the usual summation convention and $x^2 = x^\mu x_\mu$ and $\mu, \nu = 0, 1, 2, 3$.

Until 1994, exact solutions of equation (1) were obtained using reductions of the subalgebras $< P_\mu, J_{\mu\nu}, D >$ and $< P_\mu, J_{\mu\nu}, D, K_\mu >$, leaving the symmetry operator L unexploited. One can use L in the following way: let $k \neq 0$ be a real constant and α a constant four-vector. Then it is clear that equation (1) is also invariant under the operator

$$\alpha^\mu P_\mu + kL.$$

This operator gives rise to the invariant-surface condition

$$\alpha^\mu \frac{\partial u}{\partial x^\mu} = ku$$

which yields the Lagrangian system

$$\frac{dx^\mu}{\alpha^\mu} = \frac{du}{ku} \tag{2}$$

or

$$\frac{d(cx)}{c\alpha} = \frac{du}{ku},$$

where $cx = c^\mu x_\mu,\ c\alpha = c^\mu \alpha_\mu$ with c a four-vector. We now choose α to be light-like and $\beta,\ \delta,\ \epsilon$ space-like with

$$\alpha\beta = \alpha\delta = \beta\delta = \beta\epsilon = \delta\epsilon = 0, \quad \beta^2 = \delta^2 = -1, \quad \epsilon^2 = -\frac{m^2}{k^2}, \quad \alpha\epsilon = 1.$$

Then equation (2) becomes

$$\frac{d(\alpha x)}{0} = \frac{d(\beta x)}{0} = \frac{d(\delta x)}{0} = \frac{d(\epsilon x)}{1} = \frac{du}{ku}$$

which has general integral

$$u = e^{k(\epsilon x)} v(\tau, y_1, y_2), \quad \tau = \alpha x,\ y_1 = \beta x,\ y_2 = \delta x, \tag{3}$$

where v is a smooth function of its arguments.

We now use equation (3) as an ansatz for equation (1), and putting $k = 1/2$ we find that the ansatz function v must satisfy the heat equation:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y_1^2} + \frac{\partial^2 v}{\partial y_2^2}. \tag{4}$$

Thus ansatz (3) reduces the hyperbolic equation (1) to the parabolic equation (4). For this reason we call the symmetry operator L a *parabolic* symmetry.

Remark 1 *The "time" $\tau = \alpha x$ in (4) is singular as it is acharacteristic of the wave operator. Thus the Cauchy problem for (4) is related to the Goursat problem of (1), which is discussed in [1].*

It is now clear that exact solutions of (4) will lead to exact solutions of (1). The symmetry algebra of (4) is the algebra $AG_2(2) =< T, P_a, G_a, J_{12}, S, D, M >$ where

$$T = \partial_\tau, \quad P_a = -\partial_{y_a}, \quad G_a = -\tau\partial_{y_a} - \frac{1}{2} y_a v \partial_v, \quad M = -\frac{1}{2} v\partial_v, \quad a = 1, 2,$$

$$J_{12} = y_1\partial_{y_2} - y_2\partial_{y_1}, \quad D = 2\tau\partial_t + y_1\partial_{y_1} + y_2\partial_{y_2} - v\partial_v,$$

$$S = \tau^2\partial_t + \tau y_1\partial_{y_1} + \tau y_2\partial_{y_2} - (\tau + \frac{1}{4}(y_1^2 + y_2^2))v\partial_v.$$

The structure and subalgebra classification of this algebra are given in in [2], where solutions based on this subalgebra analysis are given. Some exact solutions are the following.

The subalgebra $\langle D + (4a+1)M,\, P_2\rangle$ $(a \in \mathbf{R})$ gives the ansatz

$$v = \tau^{-(a+3/4)}\varphi(\omega),\ \ \omega = \frac{y_1^2}{\tau},$$

where φ satisfies

$$4\omega\ddot{\varphi} + (2+\omega)\dot{\varphi} + \left(\frac{3}{4} + a\right)\varphi = 0.$$

The solutions of this equation are given in terms of the Pochhammer-Barnes confluent hypergeometric function (see for example [4])

$$\Phi(a;b;z) = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(b)_n n!}$$

with $b \neq 0$ and where $(a)_n = a(a+1)(a+2)\dots(a+n-1)$, $n \geq 1$. We find then ([4]):

$$\varphi = C_1\,\Phi\left(a+\frac{3}{4};\frac{1}{2};-\frac{\omega}{4}\right) + C_2\left(-\frac{\omega}{4}\right)^{1/2}\Phi\left(a+\frac{5}{4};\frac{3}{2};-\frac{\omega}{4}\right)$$

Thus we find the exact solution

$$v = \tau^{-(a+\frac{3}{4})}\left[C_1\,\Phi\left(a+\frac{3}{4};\frac{1}{2};-\frac{y_1^2}{4\tau}\right) + C_2\left(-\frac{\omega}{4}\right)^{1/2}\Phi\left(a+\frac{5}{4};\frac{3}{2};-\frac{y_1^2}{4\tau}\right)\right].$$

and the corresponding exact solution of (1) is then

$$u = \frac{e^{(\epsilon x)/2}}{(\alpha x)^{(\alpha+\frac{3}{4})}}\left[C_1\,\Phi\left(a+\frac{3}{4};\frac{1}{2};-\frac{(\beta x)^2}{4(\alpha x)}\right) + C_2\left(-\frac{(\beta x)^2}{4(\alpha x)}\right)^{1/2}\Phi\left(a+\frac{5}{4};\frac{3}{2};-\frac{(\beta x)^2}{4(\alpha x)}\right)\right].$$

The subalgebra $< J_{12} + 2aM,\ S + T + 2bM >$, $\quad (a \geq 0,\ b \in \mathbf{R})$ yields the ansatz

$$v = \frac{1}{\sqrt{\tau^2+1}}\exp\left(-b\arctan\tau + a\arctan\frac{y_1}{y_2} - \frac{\tau\omega}{4}\right)\varphi(\omega)$$

with $\omega = \dfrac{y_1^2+y_2^2}{\tau^2+1}$, where φ satisfies

$$\ddot{\varphi} + \frac{1}{\omega}\dot{\varphi} + \frac{1}{16} + \frac{b}{4\omega} + \frac{a^2}{4\omega^2}\varphi = 0.$$

The solution of this equation is given in terms of Whittaker functions [4] so that we obtain for the solution of the heat equation

$$v = \frac{1}{\sqrt{\tau^2+1}}\exp\left(-b\arctan\tau + a\arctan\frac{y_1}{y_2} - \frac{\tau(y_1^2+y_2^2)}{4(\tau^2+1)}\right)W\left(\frac{ib}{8};\frac{ia}{2};\frac{i(y_1^2+y_2^2)}{2(\tau^2+1)}\right)$$

and the corresponding solution of the wave equation is then

$$u = \frac{1}{\sqrt{(\alpha x)^2+1}} \exp\left(\frac{(\epsilon x)}{2} - b\arctan(\alpha x) + a\arctan\frac{(\beta x)}{(\delta x)} - \frac{(\alpha x)((\beta x)^2+(\delta x)^2)}{4((\alpha x)^2+1)}\right)\times$$
$$W\left(\frac{ib}{8};\frac{ia}{2};\frac{i((\beta x)^2+(\delta x)^2)}{2((\alpha x)^2+1)}\right).$$

The above are two solutions obtained from ordinary point symmetries. It is also possible to get solutions from conditional symmetries (see [5])of equation (4). It can be easily shown that one such conditional symmetry is the operator

$$\partial_{y_2} + A(\tau, y_2)v\partial_{y_2}, \tag{5}$$

where the function A satisfies Burgers equation:

$$A_\tau = A_{y_1y_1} + 2AA_{y_2}.$$

One solution of this equation is

$$A = -\frac{y_2}{2\tau} - a\tan\Bigl(a^2 + \frac{ay_2}{\tau}\Bigr)$$

with $a > 0$, and the solution v of equation (4) that the operator (5) defines with this choice of A is

$$v = \frac{1}{\sqrt{\tau}}\cos\left(a^2 + \frac{ay_2}{\tau}\right)\exp\left(-\frac{4a^2+y_2^2}{\tau}\right)\Psi(\tau, y_1),$$

where $\Psi(\tau, y_1)$ is an arbitrary solution of the heat equation. The solution of the wave equation we now find is

$$u = \frac{1}{\sqrt{(\alpha x)}}\cos\left(a^2 + \frac{a(\delta x)}{(\alpha x)}\right)\exp\left(\frac{(\epsilon x)}{2} - \frac{4a^2+(\delta x)^2}{(\alpha x)}\right)\Psi((\alpha x),(\beta x)).$$

2 The complex nonlinear wave equation

The complex nonlinear wave equation

$$\Box\Psi = -F(|\Psi|)\Psi \tag{6}$$

has symmetry algebra as given in the following classification:

(i) when $F(|\Psi|) = \text{ const}\,|\Psi|^2$

$$\partial_\mu,\;\; J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,\;\; K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

$$D = x^\nu\partial_\nu - \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),\quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

where $x^2 = x_\mu x^\mu$.

(ii) when $F(|\Psi|) = \text{ const}\,|\Psi|^k,\;\; k \neq 0, 2$:

$$\partial_\mu,\;\; J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,\;\; D_{(k)} = x^\nu\partial_\nu - \frac{2}{k}\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),\quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iii) when $F(|\Psi|) \neq$ const $|\Psi|^k$ for any k, but $\dot{F} \neq 0$:

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iv) when $F(|\Psi|) =$ const $\neq 0$:

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_\Psi\right),$$

$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}.$$

(v) when $F(|\Psi|) = 0$:

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right)$$

$$D = x^\mu \partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_\Psi\right),$$

$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}.$$

In all these cases we see the operator $M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right)$. Equation (6) is then invariant under

$$\alpha^\mu \partial_\mu + kM, \tag{7}$$

where k is real and α is a light-like constant vector. arguing as in the case for the real wave equation, the operator (7) gives us the ansatz

$$\Psi = e^{ik(\epsilon x)}\, v(\alpha x, \beta x, \delta x), \tag{8}$$

where we now have $\epsilon^2 = \alpha^2 = 0,\ \beta^2 = \delta^2 = -1,\ \epsilon\alpha = 1,\ \alpha\beta = \alpha\delta = \epsilon\beta = \epsilon\delta = 0$. Putting this ansatz into (6), we obtain, with $k = 1/2$,

$$i\partial_\tau = -\Delta v + F(|\Psi|)\Psi, \tag{9}$$

where we have $\tau = \alpha x,\ \ \Delta = \partial^2_{y_1} + \partial_{y_2^2},\ \ y_1 = \beta x, y_2 = \delta x$.

The symmetry algebra of (9) is also classified according to the nonlinearity, as for the symmetry of (6). We find this symmetry algebra to be as follows.

(i) $AG_2(1, 2)$ when $F(|v|) =$ const $|v|^2$:

$$T = \partial_t, \ \ P_a = -\partial_a, \ \ J_{12} = x_1\partial_2 - x_2\partial_1,$$

$$G_a = t\partial_a + \frac{ix_a}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right), \ \ D_2 = 2t\partial_t + x_a\partial_a - \left(v\partial_v + \overline{v}\partial_{\overline{v}}\right),$$

$$S = t^2\partial_t + tx_a\partial_a + \frac{ix_a x_a}{4}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right) - t\left(v\partial_v + \overline{v}\partial_{\overline{v}}\right), \ \ M = -\frac{i}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right).$$

(ii) $AG_1(1, 2)$ when $F(|v|) =$ const $|v|^k, \ \ k \neq 0, 2$:

$$T = \partial_t, \ \ P_a = -\partial_a, \ \ J_{12} = x_1\partial_2 - x_2\partial_1, \ \ G_a = t\partial_a + \frac{ix_a}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right),$$

$$D = 2t\partial_t + x_a\partial_a - \frac{2}{k}\left(v\partial_v + \overline{v}\partial_{\overline{v}}\right), \quad M = -\frac{i}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right).$$

(iii) $AG(1, 2)$ when $F(|v|) \neq \text{const}\,|v|^k$, for any k, but $\dot{F} \neq 0$:

$$T = \partial_t, \;\; P_a = \partial_a, \;\; J_{12} = x_1\partial_2 - x_2\partial_1,$$

$$G_a = t\partial_a + \frac{ix_a}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right), \quad M = -\frac{i}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right).$$

(iv) $AG_2(1, 2)$ when $F = 0$:

$$T = \partial_t, \;\; P_a = \partial_a, \;\; J_{12} = x_1\partial_2 - x_2\partial_1, \;\; G_a = t\partial_a + \frac{ix_a}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right),$$

$$S = t^2\partial_t + tx_a\partial_a + \frac{ix_ax_a}{4}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right) - t\left(v\partial_v + \overline{v}\partial_{\overline{v}}\right), \quad M = -\frac{i}{2}\left(v\partial_v - \overline{v}\partial_{\overline{v}}\right)$$

$$D = 2t\partial_t + x_a\partial_a, \;\; L = \left(v\partial_v + \overline{v}\partial_{\overline{v}}\right).$$

Exact solutions of the nonlinear Schrödinger equation (9) are of course quite difficult to obtain. We mention one type of solution obtained through invariance under the subalgebra $< P_2, T + 2aM >$ with $a \in \mathbf{R}$. The ansatz built from this subalgebra is

$$v = e^{-i\tau}\phi(y_1)$$

and substitution into (8) gives the following equation for ϕ:

$$\ddot{\phi} + a\phi = F(|\phi|)\phi.$$

Specialising to $F(\phi|) = \kappa|\phi|^2$ and writing $\phi = \rho e^{i\theta}$ with $\rho,\ \theta$ functions of y_1, we find that ρ and θ satisfy the equations

$$\ddot{\rho} + a\rho - \rho\dot{\theta}^2 = \kappa\rho^3,$$

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} = 0,$$

from which one easily deduces $\rho^2\dot{\theta} = A$ where A is a constant. Substituting this back into the first equation of the system and integrating, we get

$$\dot{\rho}^2 + a\rho^2 + \frac{A^2}{\rho^2} = \frac{\kappa}{2}\rho^4 + C,$$

where C is another integration constant. Put now $A = 0, a \neq 0$ and we find

$$\dot{\rho}^2 + a\rho^2 - \frac{\kappa}{2}\rho^4 = C$$

an equation which is solvable in terms of Jacobian elliptic functions $E(y_1, r)$, where r is a real number with $|r| < 1$. After some elemntary manipulations using our ansatzes, we find that

$$\Psi = \exp\left(-\frac{i}{2}\left[(\epsilon x) + a(r)(\alpha x)\right]\right) E((\alpha x), r).$$

is a solution of

$$\Box\Psi = -\kappa(r)|\Psi|^2\Psi$$

The elliptic function $E(\omega, r)$ and the parameters $a(r),\ \kappa(r)$ are given in the following table:

$E(\omega, r)$	$a(r)$	κ	$C(r)$
sn	$1+r^2$	$2r^2$	1
cn	$1-2r^2$	$-2r^2$	$1-r^2$
dn	r^2	-2	r^2-1
$ns=1/sn$	$1+r^2$	2	r^2
$nc=1/cn$	$1-2r^2$	$2(1-r^2)$	$-r^2$
$nd=1/dn$	r^2-2	$2(r^2-1)$	-1
$sc=sn/cn$	r^2-2	$2(1-r^2)$	1
$sd=sn/dn$	$1-2r^2$	$2r^2(r^2-1)$	1
$cs=cn/sn$	r^2-2	2	$1-r^2$
$cd=cn/dn$	$1+r^2$	$2r^2$	1
$ds=dn/sn$	$1-2r^2$	2	$r^2(r^2-1)$
$dc=dn/cn$	$1+r^2$	2	r^2

More details of the analysis and other exact solutions are contained in [3].

The ansatz (8) for equation (6) works also for many other types of nonlinearities. For instance the equations

$$\Box\Psi = \lambda \frac{|\Psi|_\mu |\Psi|_\mu}{|\Psi|^2}\Psi \tag{10}$$

is reduced by the ansatz to the equation

$$iv_\tau + \Delta v = \lambda \frac{|v|_a |v|_a}{|v|^2} v \tag{11}$$

which admits the solution

$$v = \operatorname{sech}(\mathbf{a}\cdot(\mathbf{y}-\mathbf{V}\tau))\exp\left(-i\left[\mathbf{a}^2\tau + \frac{\mathbf{V}\cdot\mathbf{y}}{2} - \frac{\mathbf{V}^2\tau}{4}\right]\right)$$

which yields

$$\Psi = \operatorname{sech}\left(a_1(\beta x) + a_2(\delta x) - \mathbf{a}\cdot bfV(\alpha x)\right)\times$$
$$\exp\left(-i\left[-\frac{(\epsilon x)}{2} - \mathbf{a}^2(\alpha x) + \frac{V_1(\beta x)+V_2(\delta x)}{2} - \frac{\mathbf{V}^2(\alpha x)}{4}\right]\right)$$

as a solution of equation (10) . We may think of equation (10) as a relativistic generalisation of (11). Equation (11) was studied in [6] in the context of Galilean-invariant nonlinear Schrödinger equations.

3 Conclusion

We have shown that there is a Lie-algebraic connection between the wave equation (real, linear and complex, nonlinear) and the heat and Schrödinger equations. This allows us to construct solutions of hyperbolic equations from solutions of parabolic ones. Moreover, it is clear that we can use the free Schrödinger equation to construct solutions of the Dirac equation. Indeed, the four components of the Dirac spinor satisfy the complex

Klein-Gordon equation, which, with our ansatz, reduces the four Klein-Gordon equations to four Schrödinger equations. Exact solutions of these give exact solutions of the Dirac equation. If we use the operator L, we find that we can construct solutions of the Dirac equation from the heat equation, or a mixture of the heat equation and Schrödinger equation. A similar connection exists between the heat equation and Maxwell's equations and the Lamé equation (written in potential form). These equations will be studied in future publications. There are other hyperbolic equations which allow a reduction to a parabolic equation. One notable equation is a relativistic equation obtained by Guéret and Vigier ([7]) for de Broglie's double solution. Our ansatz reduces it to a nonlinear Schrödinger equation which was proposed by Guerra and Pusterla ([8]) as an nonlinear equation having the probabilistic interpretation of a linear Schrödinger equation. Thus, it seems that our ansatz has some significance beyond the purely mathematical one, and this connection provides a reason for investigating the symmetry properties and exact solutions of these two equations.

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