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# Weak and Partial Symmetries of Nonlinear PDE in Two Independent Variables

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**Abstract**

Nonclassical infinitesimal weak symmetries introduced by Olver and Rosenau and partial symmetries introduced by the author are analyzed. For a family of nonlinear heat equations of the form $u_t = (k(u)\,u_x)_x + q(u)$, pairs of functions $(k(u), q(u))$ are pointed out such that the corresponding equations admit nontrivial two-dimensional modules of partial symmetries. These modules yield explicit solutions that look like $u(t,x) = F(\theta(t)\,x + \phi(t))$ or $u(t,x) = G(f(x) + g(t))$.

## 1 Introduction

Bluman and Cole ([1]) considered the two-dimensional linear heat equation $u_t = u_{xx}$ with addition of the first order differential equation that was a necessary and sufficient condition of invariance of functions under a certain vector field of infinitesimal point transformations. The vector field was taken a classical infinitesimal symmetry of the system. The symmetries of that type draw a considerable attention (see [2], [3], [4] and references therein). In [5], [6] involutive modules of vector fields of contact infinitesimal symmetries were considered. They were called partial symmetries. It was demonstrated that the modules of the partial symmetries are closely related to the differential substitutions of the Hopf-Cole type, the Bäcklund transformations, functionally invariant solutions of Smirnov and Sobolev, and so on. Fushchych *et al.* proposed to attach additional differential equations which are differential invariants of the classical Lie symmetry group to the differential equations admitting the classical symmetry groups and to find the classical symmetries of the attached system ([2]). It is evident that in general the group thus obtained is an extension of the classical group of the original differential equation. Olver and Rosenau considered a new type of nonclassical symmetries. Their weak symmetries were defined as groups $G$ of transformations such that $G$-invariant solutions could be obtained from the reduced equations in fewer independent variables ([7], [8]).



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## 2 Two-dimensional modules of partial symmetries

Consider the $k$-th order nonlinear differential equation

$$\Delta(t,x,u,p)=0 \tag{1}$$

for the real-valued function $u(t,x)$ and consider a pair of contact vector fields $X_f$ and $X_g$ with functionally independent characteristic functions $f$ and $g$ and try to figure out when the reduction of equation (1) to an algebraic equation is possible for obtaining invariant under $X_f$ and $X_g$ solutions. Let $E_{f,g}$ be a submanifold determined by the pair of equations

$$f(t,x,u,p_t,p_x)=0, \qquad g(t,x,u,p_t,p_x)=0 \tag{2}$$

satisfied by the functions invariant under $X_f$ and $X_g$. It is well known that system (2) is compatible under the relation

$$(f,g)\Big|_{E_{f,g}}=0, \tag{3}$$

where $(f,g)$ is a Lagrangian bracket of the functions $f$ and $g$ defined as a characteristic function of the contact vector field $[X_f,X_g]$, i.e., $[X_f,X_g]=X_{(f,g)}$.

If the relation

$$\operatorname{rank}\left\| \begin{matrix} \partial f/\partial p_t & \partial f/\partial p_x \\ \partial g/\partial p_t & \partial g/\partial p_x \end{matrix} \right\| = 2$$

is satisfied, then the functions $f$ and $g$ may be taken in the form $f=-p_t+a(t,x,u)$, $g=-p_x+b(t,x,u)$.

Let us say that the vector fields $X_f$ and $X_g$ generate the two-dimensional module of partial symmetries of equation (1) if $X_f^{(k)}$ and $X_g^{(k)}$ are tangent to the intersection $E_{f,g}^{(k)}\cap E_\Delta$.

## 3 Weak symmetries

One can treat the infinitesimal weak symmetries corresponding to the one-parameter groups in the following way. Consider the contact vector field $X_f$, the function $\Gamma(t,x,u,p)=X_f^{(k)}(\Delta)(t,x,u,p)$, and the system $W$ of differential equations

$$\Delta=0, \qquad \Gamma=0, \qquad f=0. \tag{4}$$

**Definition.** The vector field $X_f$ is an infinitesimal weak symmetry of equation (1) if

(i) $X_f$ is a classical infinitesimal symmetry of system (4),

(ii) system (4) is compatible.

Since $\Gamma=X_f^{(k)}(\Delta)$ and $X_f(f)=f_u\cdot f$, the criterion that $X_f$ is tangent to $W$ takes the form

$$X(\Gamma)|_W=0. \tag{5}$$

**Theorem.** *Suppose that the vector field $X_f$ is an infinitesimal weak symmetry of equation (1) being neither classical nor partial symmetry of that equation and suppose that its* *characteristic function satisfies the relation rank* $\|f_{p_t}, f_{p_x}\| = 1$. *Suppose also that* $X_f$*-invariant solutions generate at least one-parameter family of solutions. Then there exists a two-dimensional module* $\mathbf{g}$ *of partial symmetries of equation (1) such that each* $X_f$*-invariant solution is a* $\mathbf{g}$*-invariant one, besides, the relation* $E_{\mathbf{g}}^{(k)} \cap E_\Delta = E_{\mathbf{g}}^{(k)}$ *is valid, where* $E_{\mathbf{g}}$ *is a submanifold of* $\mathbf{g}$*-invariant solutions given by the equations of the form (2).*

For the proof see [9]. The theorem means that obtaining vector fields of the weak symmetries is equivalent in general to searching two-dimensional modules of the partial symmetries. The latter problem is essentially simpler than the first one as our calculations show.

## 4 Weak symmetries of the nonlinear heat equation

It is interesting to consider the example of the infinitesimal weak symmetry which is admitted by a unique invariant solution and which does not thereby fall under the Theorem. Consider the equation

$$u_t = u_{xx} + u_x^2 + u^2 \tag{6}$$

and its infinitesimal weak symmetry with the characteristic function $f = -p_x + b(t,x)$, where the function $b(t,x)$ needs to be defined. The following formula is valid: $\Gamma = -b_t + b_{xx} + 2bb_x + 2ub$. Therefore, if the function $b(t,x)$ is fixed, a unique invariant solution $u(t,x)$ is obtained from the equation $\Gamma = 0$:

$$u(t,x) = (b_t - b_{xx} - 2bb_x)/2b. \tag{7}$$

From the above calculations, we can draw a conclusion that system (3) in the case considered takes the form:

$$p_t = b_x + b^2 + u^2, \qquad p_x = b(t,x), \qquad u = (b_t - b_{xx} - 2bb_x)/2b. \tag{8}$$

The compatibility conditions for system (8) are evident:

$$\begin{aligned} \partial/\partial x\left((b_t - b_{xx} - 2bb_x)/2b\right) &= b, \\ \partial/\partial t\left((b_t - b_{xx} - 2\,b\,b_x)/2b\right) &= b_x + b^2 + \left((b_t - b_{xx} - 2\,b\,b_x)/2b\right)^2. \end{aligned} \tag{9}$$

Equations (9) admit separation of variables $b(t,x) = \phi(t)\sin x$ with $\phi(t)$ satisfying the equation

$$d/dt\left(\left(\dot{\phi} + \phi\right)/2\phi\right) = \left(\left(\dot{\phi} + \phi\right)/2\phi\right)^2 + \phi^2. \tag{10}$$

If the function $\phi(t)$ satisfies (10), then

$$u(t,x) = \left(\dot{\phi} + \phi\right)\Big/\,2\phi - \phi\cos x.$$

Galaktionov obtained this solution by applying directly the method of generalized separation of variables in the form $u(t,x) = \theta(t) - \phi(t)\cos x$ to equation (6) in his article [10]. The wide class of exact solutions of nonlinear heat equations was constructed in [11].

## 5 Explicit solutions of a family of nonlinear heat equations

Consider the problem of finding the two-dimensional module $\mathbf{g} = L(X_f, X_g)$ of the partial symmetries with $f = -p_t + a(t,x,u)$, $g = -p_x + b(t,x,u)$ for the family of nonlinear heat equations

$$u_t = (f(u)u_x)_x + g(u). \tag{11}$$

The functions $a(t,x,u)$, $b(t,x,u)$ must satisfy the compatibility condition

$$a_x + a_u b = b_t + b_u a. \tag{12}$$

Besides, in order for (11) and (2) to admit a one-parameter family of solutions, equation (11) must be a differential consequence of (2) which implies the following relation:

$$a = k'(u)\, b^2 + k(u)\,(b_x + b_u\, b) + q(u). \tag{13}$$

If we insert the function $a(t,x,u)$ given by (13) into equation (12), we obtain the equation

$$b_t = k(u)\left(b_{xx} + 2bb_{ux} + b^2 b_{uu}\right) + k(u)'\left(3bb_x + 2b^2 b_u\right) + k''(u)b^3 + bq'(u) - q(u)b_u \tag{14}$$

for the function $b(t,x,u)$.

Equation (14) admits separation of variables $b(t,x,u) = \theta(t)h(u)$. For such solutions equation (14) takes the form:

$$\dot\theta(t) = \theta(t)^3 h(u)(k(u)h(u))'' + \theta(t)h(u)(q(u)/h(u))'$$

which yields the relations

$$h(u)(f(u)h(u))'' = \lambda, \qquad h(u)(g(u)/h(u))' = \mu \qquad \dot\theta = \lambda\,\theta^3 + \mu\,\theta \tag{15}$$

with $\lambda$, $\mu$ constants. Given the function $h(u)$, equations (15) can be solved for $k(u)$ and $q(u)$:

$$\begin{aligned} k(u) &= \left(\int^u dv \int^v \lambda/h(w)\, dw + c_1\, u + c_2\right)/h(u), \\ q(u) &= h(u)\left(\int^u \mu/h(v)\, dv + c_3\right), \end{aligned} \tag{16}$$

where $c_1$, $c_2$, and $c_3$ are arbitrary constants. For obtaining explicit solutions, one must take the function $h(u)$ so that one can explicitly solve the equation

$$\int^u \frac{d\,v}{h(v)} = \theta(t)\,x + \phi(t)$$

arriving to the relation:

$$u(t,x) = F(\theta(t)\,x + \phi(t)). \tag{17}$$

In the last two relations, the function $\phi(t)$ satisfies the ODE obtained by substitution of (17) into equation (11). Let us give one example of explicit solutions:

$$\begin{array}{ll} k(u) = 2\,\lambda\,\frac{u^4}{3} + & u(t,x) = \sqrt{\dfrac{(x+l(t))e^t}{\sqrt{\alpha - e^{2t}}}}, \qquad (\lambda,\mu) = (1,1), \\ c_1\,u^2 + c_2\,u & l(t) = -\dfrac{c_3\,\sqrt{\alpha - e^{2\,t}}}{e^t} - \dfrac{2\,c_3 - c_1}{2}\,\arctan\dfrac{e^t}{\sqrt{\alpha - e^{2\,t}}}, \\ q(u) = \mu\,\frac{u}{2} + \frac{c_3}{2\,u} & u(t,x) = \sqrt{\dfrac{x + l(t)}{\sqrt{1+\alpha\,e^{2\,t}}}}, \qquad (\lambda,\mu) = (1,-1), \\ & l(t) = c_3\,\sqrt{1+\alpha\,e^{2\,t}} - \dfrac{c_1 + 2\,c_3}{2}\,\mathrm{arctanh}\sqrt{1+\alpha\,e^{2\,t}}, \\ & u(t,x) = \sqrt{x/\sqrt{\alpha - 2\,t} + 2\,c_3\,t/3 - (3\,c_1 + 2\,\alpha\,c_3)/6}, \\ & (\lambda,\mu) = (1,0), \\ & u(t,x) = \sqrt{x\,e^t + l(t)}, \\ & l(t) = -c_3 + c_1\,e^{2\,t}/2, \qquad (\lambda,\mu) = (0,1), \end{array} \tag{18}$$

It is clear that the class of equations considered is potentially infinite.

The second kind of solutions of equation (14) consists of solutions $b(t,x,u) = \theta(x)\,h(u)$ with $\theta(x)$, $k(u)$, $q(u)$, and $h(u)$ satisfying the following system of ODE:

$$\begin{aligned} &h(u)\,(h(u)\,k(u))'' = \lambda\,k(u), \quad 3\,h(u)\,k'(u) + 2\,h'(u)\,k(u) = \mu\,k(u), \\ &h(u)\,(q(u)/h(u))' = \nu\,k(u), \quad \theta''(x) + \mu\,\theta'(x)\,\theta(x) + \lambda\,\theta^3(x) + \nu\,\theta(x) = 0, \end{aligned} \tag{19}$$

where $\lambda$, $\mu$, and $\nu$ are constants. Presumably, the case $\lambda = 0$, $\mu = 0$ is the simplest one. Here we have the following particular solutions of system (19)

$$h(u) = u^3, \qquad k(u) = u^{-2}, \qquad q(u) = -\frac{1}{4\,u} + c_1\,u^3, \qquad \theta(x) = -b\,\cos x,$$

where $b$ and $c_1$ are parameters, and the corresponding solutions of (11):

$$u(t,x) = 1/\sqrt{l(t) + b\,\sin x}, \qquad l(t) = -\sqrt{b^2 + 4\,c_1}\tanh\frac{\sqrt{b^2+4\,c_1}\,(t-t_0)}{2}.$$

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